MAXIMALLY NON-HAMILTONIAN GRAPHS OF GIRTH 7

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ABSTRACT. We describe a sufficient condition for graphs used in a construction of Thomassen (which yields hypohamiltonian graphs) to produce maximally non-hamiltonian (MNH) graphs as well. Then we show that the Coxeter graph fulfils this sufficient condition, and thus applying the Thomassen's construction to multiple copies of the Coxeter graph yields infinitely many MNH graphs with girth 7. So far, the Coxeter graph was the only known example of a MNH graph of girth 7; also no MNH graph of girth greater than 7 has been found yet. Finally, the Isaacs' flower snarks J_k for odd $k \geq 5$ are shown to fulfil (for certain vertices) this sufficient condition as well.

1. INTRODUCTION

The *hypohamiltonian graphs* are the non-hamiltonian graphs that become hamiltonian after deleting an arbitrary vertex. An amount of known constructions of infinite families of hypohamiltonian graphs exists. We will focus on the one of Thomassen [4] which sometimes produces graphs that are not only hypohamiltonian but maximally non-hamiltonian as well. Recall that a non-hamiltonian graph is called *maximally non-hamiltonian* (MNH) if adding an arbitrary new edge results in a hamiltonian graph.

Using Thomassen's construction, Horák and Širáň [2] showed that for infinitely many values n there exists a family $\mathfrak{T}_n$ of graphs of order n which are both MNH and hypohamiltonian; moreover, $|\mathfrak{T}_n| \to \infty$ exponentially in n. At a first glance, the property of "being MNH" seems to force the graph to contain very short cycles, even though those MNH graphs constructed by Horák and Širáň are triangle free. It is easy to see that the Isaacs' snarks J_k (Fig. 7), which are examples of MNH graphs [1] for odd $k \geq 7$, have girth 6. For many years the Coxeter graph (Fig. 6) provided the only known example of a maximally non-hamiltonian graph with girth 7. Furthermore, no MNH graph of girth greater than 7 is known. The aim of this article is to construct other MNH graphs of girth 7. We describe sufficient conditions for graphs to produce MNH graphs in the Thomassen's construction. Then the Coxeter graph and (for certain vertices) the Isaacs' snarks $J_k, k \geq 5$ are shown to satisfy these conditions. The main result follows immediately: For infinitely many values of

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n there exists an (exponentially growing) family $\mathfrak{F}_n$ of MNH graphs of order n and girth 7.

The problem of the existence of MNH graphs of girth ≥ 8 was explicitly stated in [2].

2. The Thomassen's Construction and MNH Graphs

We first describe the construction of Thomassen [4] in detail. Let G_1 and G_2 be graphs, both containing at least one vertex of degree 3; let $u_1 \in V(G_1)$, $u_2 \in V(G_2)$ be such vertices. Denote the neighbours of u_1 and u_2 in G_1 and G_2 by x_1, y_1, z_1 and x_2, y_2, z_2, respectively. Now, delete the vertices u_1 and u_2 from G_1 and G_2 and then identify the pairs of vertices x_1 and x_2, y_1 and y_2, z_1 and z_2 to new vertices x, y, z, respectively, obtaining thereby a new graph G (see Fig. 1).

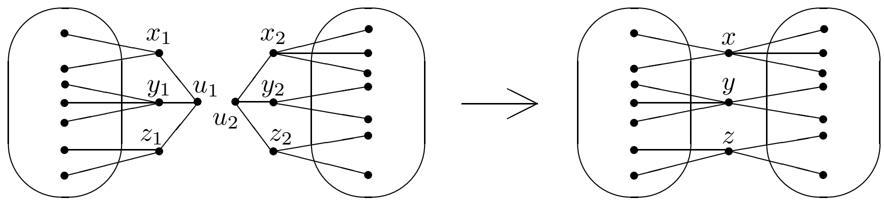


Fig. 1.

We use the notation $G = G_1(u_1; x_1, y_1, z_1) \circ G_2(u_2; x_2, y_2, z_2)$ to specify the way how G is obtained from G_1 and G_2. If it does not cause any confusion, we will use the abbreviated form $G = G_1(u_1) \circ G_2(u_2)$ or simply $G = G_1 \circ G_2$. For the notational convenience, we will refer to the vertices u_1 and u_2 as *pre-contiguous* vertices.

Obviously, the above construction does not result in a decrease of girth. Moreover, it preserves hypohamiltonicity in the following sense.

Theorem 1. [4] *If G_1 and G_2 are hypohamiltonian graphs both containing at least one vertex of degree* 3*, then any graph of the form $G_1 \circ G_2$ is also hypohamiltonian.* □

The *incidence tree* $T = T^G$ of the graph $G = G_1 \circ G_2$ is the graph with the vertex set $V(T) = \{g_1, g_2\}$ and the edge set $E(T) = \{g_1 g_2\}$; so in this case $T \cong K_2$. Let S be a set of vertices of T, such that the graph induced by the set S is connected. If $S = \{g_i\}$, where $i = 1, 2$, then the *S-reduction* $H = H_S^G$ of $G = G_1 \circ G_2$ is the graph isomorphic to G_i, and if $S = \{g_1, g_2\}$, then the S-reduction H is isomorphic to G.

More generally, we now inductively define the graph $G = [[...[G_1(u_1) \circ G_2(u_2)](u_{1,2}) \circ G_3(u_3)](u_{1,2,3}) \circ \cdots \circ G_{n-1}(u_{n-1})](u_{1,2,\ldots n-1}) \circ G_n(u_n)$, together with the incidence tree $T = T^G$ and the S-reductions H_S^G. Assume that $n \geq 3$ and that $G' = [[...[G_1(u_1) \circ G_2(u_2)](u_{1,2}) \circ G_3(u_3)]\ (u_{1,2,3}) \circ \cdots \circ G_{n-2}(u_{n-2})]\ (u_{1,2,\ldots n-2}) \circ G_{n-1}(u_{n-1})$ has already been defined together with the incidence tree $T^{G'}$ and the S-reductions $H_S^{G'}$. Pick a subscript $i, 1 \leq i \leq n-1$, and let

$u_{1,2,\ldots,n-1} \in G_i$ be a vertex of degree 3 in G'. Let $u_n \in G_n$ be a vertex of degree 3. Suppose that the neighbours of $u_{1,2,\ldots,n-1}$ and u_n are x_1, y_1, z_1 and x_2, y_2, z_2, respectively. Then the graph G is defined as $G = G'(u_{1,2,\ldots,n-1}; x_1, y_1, z_1) \circ G_n(u_n; x_2, y_2, z_2)$. Now, the incidence tree $T = T^G$ is the graph with the vertex set $V(T) = V(T^{G'}) + \{g_n\}$ and the edge set $E(T) = E(T^{G'}) + \{g_i g_n\}$.

Let S be a set of vertices of T, such that the subgraph induced by the set S is connected. If $g_n \notin S$, then the S-reduction H_S^G is going to be equal to $H_S^{G'}$. If $g_n \in S$, then we define $H_S^G = H_{S'}^{G'}(u_{1,2,\ldots,n-1}; x_1, y_1, z_1) \circ G_n(u_n; x_2, y_2, z_2)$, where $S' = S - \{g_n\}$. It follows from the definition of the $S-$reduction and from Theorem 1 that H_G^S is hypohamiltonian. Finally, all the vertices $u_1, u_2, \ldots, u_n, u_{1,2,\ldots,p}$ $(p = 2, 3, \ldots, n-1)$ are referred to as pre-contiguous vertices.

Let $G = [[...[G_1 \circ G_2] \circ \ldots] \circ G_n$. We show that if G_i for i=1, 2, …, n has certain properties, then G will be MNH. These properties are described in the following theorem. To be able to state it, we need more definitions and notation.

Let a, b, x, y be distinct vertices of a graph G. Let P be a hamiltonian $a-b$ path with the following properties: the second vertex of P is not x, the last but one vertex of P is not y, and the vertex x is a successor of the vertex y in P. Any hamiltonian $a-b$ path P with the above properties will be referred to as $a-b(y,x)$. Similarly, any hamiltonian $a-b$ path the second vertex of which is x is denoted by $ax-b$. By $N(u)$ we denote the set of neighbours of a vertex u. The subgraph of G obtained by deleting a set of vertices M, together with all the edges incident to at least one of the vertices in M, is denoted by $G - M$.

Lemma 1. *Let $G = [[...[G_1 \circ G_2] \circ \ldots] \circ G_n$, where each G_i $(i = 1, 2, \ldots, n)$ is a hypohamiltonian graph. Let H be an S-reduction of G. Let $a, b \in V(H)$ be two non-adjacent vertices. If there is a hamiltonian $a-b$ path in H, then there is a hamiltonian $a-b$ path in G as well.*

Proof. Let P be a hamiltonian $a-b$ path in H. We prove the Lemma by constructing the desired hamiltonian $a-b$ path in G from the path P. The constructive algorithm follows.

1. (*Termination test*) If $H = G$, then the path P is a hamiltonian $a-b$ path in G. The algorithm is terminated.
2. (*Inflating of H*) Since $H \neq G$, there is an S'-reduction K of G such that $V(H) \cap V(K) = \emptyset$, and $H(u; x_1, x_2, x_3) \circ K(v; y_1, y_2, y_3)$ is an $\{S \cup S'\}$-reduction of G. Put $H := H(u; x_1, x_2, x_3) \circ K(v; y_1, y_2, y_3)$.
3. (*Prolonging the path P*) Suppose that P to be of the form $a \ldots x_1\ u\ x_2 \ldots b$. By Theorem 1, K is a hypohamiltonian graph, and therefore a hamiltonian cycle $y_1\ v\ y_2\ w_1\ w_2 \ldots w_p\ y_1$ exists in $K - y_3$. Put $P := a \ldots x_1 \equiv y_1\ w_p\ w_{p-1} \ldots w_1\ y_2 \equiv x_2 \ldots b$. It is easy to check that P is a hamiltonian $a-b$ path in H.
4. Jump to step 1.

Since $G = [[...[G_1 \circ G_2] \circ \ldots] \circ G_n$, the step 2. guarantees the termination of the above algorithm. Moreover, it follows from the steps 3. and 1. that the resulting path P will be a hamiltonian $a - b$ path in G. □

Theorem 2. *Let $G = [[...[G_1 \circ G_2] \circ \ldots] \circ G_n$, where each G_i $(i = 1, 2, \ldots, n)$ is a hypohamiltonian and MNH graph of girth ≥ 5. Assume that for each $i = 1, 2, \ldots, n$, the following three conditions hold for arbitrary pre-contiguous vertices $u, v \in V(G_i)$, where u is non-adjacent to v:*

- $(1uv)$ *At least for one pair of non-adjacent vertices $u' \in N(u)$ and $v' \in N(v)$ a hamiltonian $u' - v'(v, u)$ path exists in G_i.*
- $(2u)$ *For each x, x' and y such that $x' \in N(x)$, a hamiltonian $xx' - y$ path exists in G_i, where $x = u$ or $x \in N(u)$, and y is not adjacent to x.*
- $(3u)$ *For each pair of different vertices $u', u'' \in N(u)$, the graph $G_i - \{u', x\}$ is hamiltonian, where $x \in N(u'')$ and $x \neq u$.*

Then G is a MNH graph.

Proof. We first prove the following Claim, which is a special case of the Theorem. Note that since G_i $(i = 1, 2, \ldots, n)$ is hypohamiltonian, degree of each vertex of G_i is at least 3.

Claim. *Let $H = [[...[G_1 \circ G_2] \circ \ldots] \circ G_h$, where T^H is a snake (a connected graph in which every vertex has degree 1 or 2). Assume that each G_i $(i = 1, 2, \ldots, h)$ satisfies all the conditions of Theorem 2. Then every pair of non-adjacent vertices a, b such that $a \in V(G_1)$ and $b \in V(G_h)$ can be joined by a hamiltonian path in H.*

Proof of the Claim by induction on h.

For $h = 1$, the graph $H = G_1$ is MNH and the Claim is proved.

For $h = 2$, let $H = G_1(u_1; x_1, y_1, z_1) \circ G_2(u_2; x_2, y_2, z_2)$. First suppose $a \in \{x_1, y_1, z_1\}$ or $b \in \{x_2, y_2, z_2\}$; say $a = x_1$. Since a and b are non-adjacent, b is not adjacent to x_2. Obviously, G_2 is an S-reduction of H, and thus, a hypohamiltonian graph. By $(2u)$, there is a hamiltonian $x_2 - b$ path in G_2. The Lemma 1 guaranties a hamiltonian $a - b$ path in H.

Now, $a \neq x_1, y_1, z_1$ and $b \neq x_2, y_2, z_2$. We consider two cases.

(**A**) Either a is adjacent to one of the vertices x_1, y_1 or z_1 or b is adjacent to one of the vertices x_2, y_2 or z_2; say a is adjacent to z_1. Since the girth of G_i, $i = 1, 2$, is at least 5, a is not adjacent to x_1 or y_1, and z_1, x_1 and u_2, b are pairs of non-adjacent vertices. It follows from $(2u)$, that hamiltonian paths $z_1 a - x_1$ and $u_2 y_2 - b$ exist in G_1 and G_2, respectively. Without loss of generality we suppose that z_2 is a successor of x_2 on the path $u_2 y_2 - b$. We obtain the required hamiltonian $a - b$ path in H as a composition of the paths in G_1 and G_2 (see Fig. 2).

(**B**) Neither a is adjacent to any of the vertices x_1, y_1 and z_1 nor b is adjacent to any of the vertices x_2, y_2 and z_2. Since degree of x_1 is at least 3 in G_1, by $(2u)$, there is a hamiltonian $x_1 c - a$ path such that $c \neq u_1$. Without loss of generality we suppose that z_1 is a successor of y_1 on the path. Similarly, by

(2u), there is a hamiltonian $z_2d - b$ path in G_2 such that $d \neq u_2$. A composition of these paths yields a hamiltonian $a - b$ path in H.

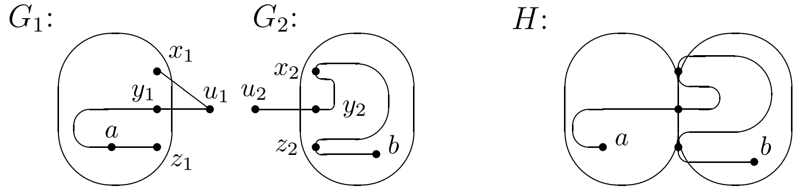


Fig. 2.

Now, let $h \geq 3$. For notational convenience suppose that $H = [G_1(u_1; x_1, y_1, z_1) \circ H''(u_2; x_2, y_2, z_2)](u_3; x_3, y_3, z_3) \circ G_h(u_4; x_4, y_4, z_4)$ and let $H' = G_1(u_1; x_1, y_1, z_1) \circ H''(u_2; x_2, y_2, z_2)$. Since the girth of G_i ($i = 2, 3, \ldots, h-1$) is at least 5, both pre-contiguous vertices of G_i have at most one neighbour in common, and hence it is possible to apply (1uv) to G_i. It is a matter of routine to check that each such path (arisen from (1uv)) decomposes $G_i - \{u, v\}$ (u, v are the two pre-contiguous vertices of G_i) into three disjoint paths, from which at most one can be a single vertex. Furthermore, a composition of the paths in G_2, G_3, ..., G_{h-1} yields three disjoint paths in $H'' - \{u_2, u_3\}$, from which at most one can be a single vertex as well. Suppose without loss of generality that these paths are $x_2 - x_3$, $y_2 - y_3$, $z_2 - z_3$, and z_2 could be identified with z_3. (see Fig. 3).

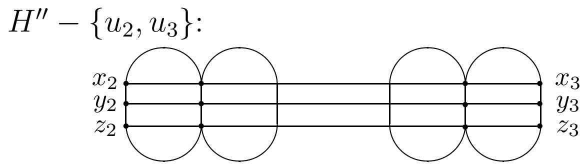


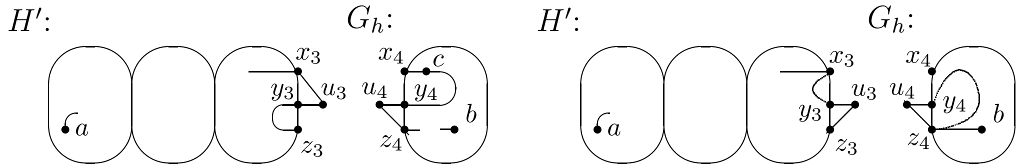
Fig. 3.

First suppose $a \in \{x_1, y_1, z_1\}$ or $b \in \{x_4, y_4, z_4\}$; say $b = x_4$. Since a and b are non-adjacent, a is not adjacent to x_3, and since $a \in V(G_1)$ and $x_3 \in V(G_{h-1})$, and H' is an S-reduction of H (thus $T^{H'}$ is a snake), by induction hypothesis there is a hamiltonian $a - x_3$ path in H'. The Lemma 1 guaranties a hamiltonian $a - b$ path in H.

Now, $a \neq x_1, y_1, z_1$ and $b \neq x_4, y_4, z_4$. We consider two cases.

(**AA**) Either a is adjacent to one of the vertices x_1, y_1 or z_1 or b is adjacent to one of the vertices x_4, y_4 or z_4; say b is adjacent to z_4. Since the girth of G_h is at least 5, the vertex b is not adjacent to x_4 or y_4.

First consider the case when a and z_3 are not adjacent in H'. Since $a \in V(G_1)$, and $z_3 \in V(G_{h-1})$, and since H' is an S-reduction of H (thus $T^{H'}$ is a snake), by induction hypothesis there is a hamiltonian $z_3 - a$ path in H'. Now, the second vertex of the path $z_3 - a$ is u_3 or not; see Fig. 4 (b) and (a). Without loss of generality we suppose that x_3 is a successor of y_3 on the path $z_3 - a$. Since G_h has girth ≥ 5, the vertices x_4 and b are non-adjacent and hence from (2u), a hamiltonian $x_4c - b$ path exists in G_h, where $c \neq u_4$ (Fig. 4 (a)). The vertex z_4 is a successor of y_4 on the path $x_4c - b$, otherwise G_h would be hamiltonian. Further, it follows from $(3u)$ that a hamiltonian cycle exists in $G_h - \{b, x_4\}$ (Fig. 4 (b)). The construction of the resulting hamiltonian $a - b$ path in H is obvious.



(a) (b)

Fig. 4.

Now, consider the case when a and z_3 are adjacent in H'. Obviously, $z_2 \equiv z_3$ in H'' and a is adjacent to z_1. We describe the construction of the hamiltonian $a - b$ path for $h = 3$ (see Fig. 5).

Since z_2 and y_3 are non-adjacent, from $(2u)$, a hamiltonian $z_2u_2 - y_3$ path exists in G_2. There are two equivalent cases depending on the third vertex of the path $z_2u_2 - y_3$. We consider only one of them and let the third vertex of $z_2u_2 - y_3$ be x_2. Since z_1 and y_1 are non-adjacent, from $(2u)$, a hamiltonian $z_1a - y_1$ path exists in G_1, and since u_4 and b are non-adjacent, from $(2u)$, a hamiltonian $u_4x_4 - b$ path exists in G_3. Without loss of generality we suppose that z_4 is a successor of y_4 on the path $u_4x_4 - b$. We obtain the required hamiltonian $a - b$ path in H as a composition of the paths in G_1, G_2 and G_3.

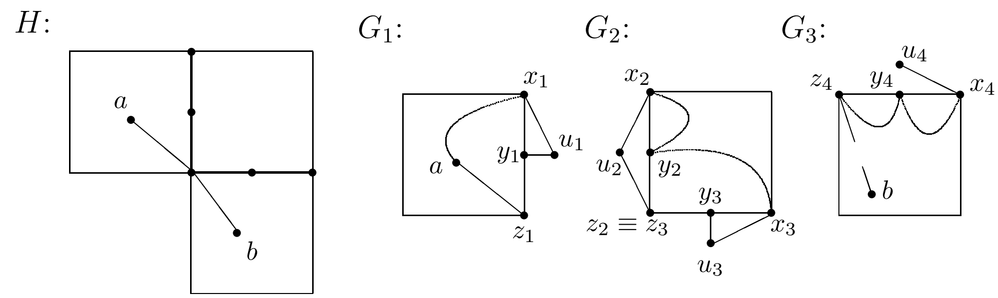


Fig. 5.

The cases for $h \geq 4$ can be handled similarly. The only difference is that we first apply $(2u)$ to G_i $(i = h - 1, h - 2, \ldots, 2)$. It is a matter of routine to observe that in this way we can compose a hamiltonian $z_2u_2 - y_3$ path in H''. Now, these cases are similar to the case $h = 3$.

(**BB**) Neither a is adjacent to any of the vertices x_1, y_1 and z_1 nor b is adjacent to any of the vertices x_4, y_4 and z_4. We construct the hamiltonian $a - b$ path in H using $(2u)$ in G_1 and G_h, and those three disjoint paths $x_2 - x_3$, $y_2 - y_3$ and $z_2 - z_3$ in $H'' - \{u_2, u_3\}$ similarly to the case (**B**). This proves the Claim.

The proof of Theorem 2 continued.

By [4], G is hypohamiltonian and thus does not contain any hamiltonian cycle. We show that any two non-adjacent vertices $x, y \in V(G)$ are ends of a hamiltonian path in G. Consider a minimal (in terms of order, i.e., the number of vertices) S-reduction $H = [[...[G_1 \circ G_2] \circ \ldots] \circ G_h$ of G, which contains the vertices x and y. Obviously, T^H is a snake; moreover, $x \in V(G_1)$ and $y \in V(G_h)$. If this is not the case, H would not be minimal. By the above Claim, there is a hamiltonian $x - y$ path in H, which can be extended to a hamiltonian $x - y$ path in G using constructive proof of Lemma 1. This proves the Theorem. □

Remark 1. *All the conditions* (1uv)*,* (2u)*, and* (3u) *from Theorem 2 are sufficient for G to be MNH. In Remark 2 it is noted that the condition* (3u) *is not necessary. In regard to the conditions* (1uv) *and* (2u) *we do not know whether they are necessary or not. We conjecture that they are not necessary.*

3. Coxeter Graph and Isaacs' Snarks

In this sections we show that if we choose the pre-contiguous vertices from the Coxeter graph in an arbitrary way, and in the Isaacs' snarks in a certain way, then a repetitive use of Thomassen's construction on these graphs will produce MNH graphs. First, we deal with the Coxeter graph.

Definition 1. *The Coxeter graph* (CG) *is the* 28*-vertex cubic graph with vertices* a_i, b_i, c_i, d_i ($i = 1, 2, \ldots, 7$) *which is the edge-disjoint union of three disjoint 7-cycles* $a_1\ a_2\ \ldots\ a_7\ a_1$*,* $b_1\ b_4\ b_7\ b_3\ b_6\ b_2\ b_5\ b_1$*,* $c_1\ c_3\ c_5\ c_7\ c_2\ c_4\ c_6\ c_1$ *and seven disjoint stars* $K_{1,3}$ *of which the* j^{th} *has central vertex* d_j *and hanging vertices* a_j*,* b_j *and* c_j ($j = 1, 2, \ldots, 7$)*.*

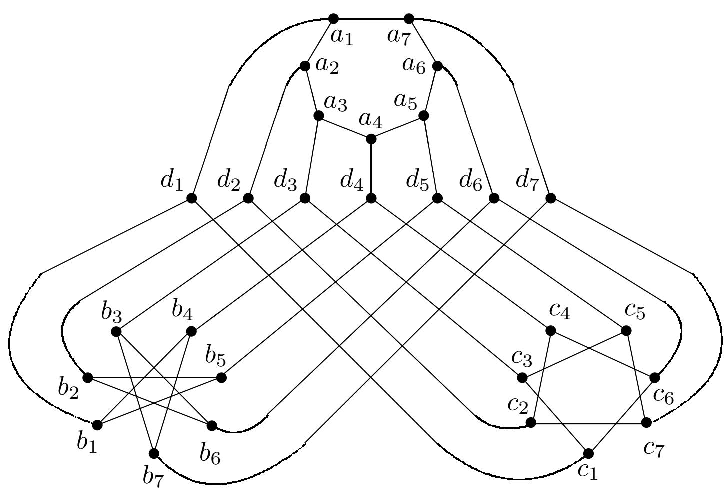


Fig. 6.

The graph CG is vertex transitive as well as distance transitive, i.e., for any two given paths in CG of length i there is an automorphism taking one onto other. The graph CG is cubic, hypohamiltonian and MNH of girth 7 [1].

Lemma 2. *The graph CG fulfils the conditions* (1uv)*,* (2u) *and* (3u) *of the Theorem* 2*.*

Proof. We refer to the fact that the graph CG is distance transitive and has diameter 4.

Condition $(1uv)$.
If $d(u,v)$ =2, we choose the vertices $u = a_4$ and $v = b_4$. One necessary hamiltonian path is:
$a_5\ a_6\ a_7\ a_1\ a_2\ d_2\ b_2\ b_5\ d_5\ c_5\ c_3\ c_1\ d_1\ b_1\ b_4\ d_4\ a_4\ a_3\ d_3\ b_3\ b_6\ d_6\ c_6\ c_4\ c_2\ c_7\ d_7\ b_7$
If $d(u,v)$ =3, we consider the vertices $u = a_4$ and $v = b_3$ and the hamiltonian path:
$a_5\ a_6\ d_6\ c_6\ c_4\ c_2\ d_2\ b_2\ b_6\ b_3\ d_3\ c_3\ c_1\ d_1\ b_1\ b_5\ d_5\ c_5\ c_7\ d_7\ a_7\ a_1\ a_2\ a_3\ a_4\ d_4\ b_4\ b_7$
Finally, let $d(u,v)$ =4. We consider the vertices $u = a_4$ and $v = b_2$ and the hamiltonian path:
$a_5\ a_6\ a_7\ a_1\ a_2\ d_2\ b_2\ b_6\ d_6\ c_6\ c_4\ c_2\ c_7\ d_7\ b_7\ b_3\ d_3\ a_3\ a_4\ d_4\ b_4\ b_1\ d_1\ c_1\ c_3\ c_5\ d_5\ b_5$

Condition $(2u)$. We use the same vertices u, v as above and so it is sufficient to find three hamiltonian paths for each pair of them:
$a_4\ a_5\ a_6\ a_7\ a_1\ d_1\ b_1\ b_5\ d_5\ c_5\ c_3\ c_1\ c_6\ d_6\ b_6\ b_2\ d_2\ a_2\ a_3\ d_3\ b_3\ b_7\ d_7\ c_7\ c_2\ c_4\ d_4\ b_4$
$a_4\ a_3\ a_2\ a_1\ a_7\ d_7\ b_7\ b_3\ d_3\ c_3\ c_5\ c_7\ c_2\ d_2\ b_2\ b_6\ d_6\ a_6\ a_5\ d_5\ b_5\ b_1\ d_1\ c_1\ c_6\ c_4\ d_4\ b_4$
$a_4\ d_4\ c_4\ c_2\ d_2\ a_2\ a_3\ d_3\ c_3\ c_1\ c_6\ d_6\ a_6\ a_5\ d_5\ c_5\ c_7\ d_7\ a_7\ a_1\ d_1\ b_1\ b_5\ b_2\ b_6\ b_3\ b_7\ b_4$
$a_4\ a_5\ a_6\ a_7\ a_1\ d_1\ c_1\ c_6\ d_6\ b_6\ b_2\ d_2\ a_2\ a_3\ d_3\ c_3\ c_5\ d_5\ b_5\ b_1\ b_4\ d_4\ c_4\ c_2\ c_7\ d_7\ b_7\ b_3$
$a_4\ a_3\ a_2\ a_1\ a_7\ a_6\ a_5\ d_5\ b_5\ b_1\ d_1\ c_1\ c_6\ d_6\ b_6\ b_2\ d_2\ c_2\ c_4\ d_4\ b_4\ b_7\ d_7\ c_7\ c_5\ c_3\ d_3\ b_3$
$a_4\ d_4\ c_4\ c_2\ d_2\ b_2\ b_6\ d_6\ c_6\ c_1\ d_1\ a_1\ a_2\ a_3\ d_3\ c_3\ c_5\ c_7\ d_7\ a_7\ a_6\ a_5\ d_5\ b_5\ b_1\ b_4\ b_7\ b_3$
$a_4\ a_5\ a_6\ a_7\ a_1\ d_1\ c_1\ c_6\ d_6\ b_6\ b_3\ b_7\ d_7\ c_7\ c_2\ c_4\ d_4\ b_4\ b_1\ b_5\ d_5\ c_5\ c_3\ d_3\ a_3\ a_2\ d_2\ b_2$
$a_4\ a_3\ a_2\ a_1\ a_7\ a_6\ a_5\ d_5\ b_5\ b_1\ d_1\ c_1\ c_6\ d_6\ b_6\ b_3\ d_3\ c_3\ c_5\ c_7\ d_7\ b_7\ b_4\ d_4\ c_4\ c_2\ d_2\ b_2$
$a_4\ d_4\ c_4\ c_2\ d_2\ a_2\ a_3\ d_3\ c_3\ c_5\ c_7\ d_7\ a_7\ a_1\ d_1\ c_1\ c_6\ d_6\ a_6\ a_5\ d_5\ b_5\ b_1\ b_4\ b_7\ b_3\ b_6\ b_2$

Condition $(3u)$. Recall that CG is vertex transitive as well. Hence we consider the vertex $u = a_4$. Since CG has girth 7, the distance of removed vertices will be 3 in all cases and so it is sufficient to describe a hamiltonian cycle in $CG - \{a_5, c_4\}$:
$a_4\ a_3\ a_2\ a_1\ a_7\ a_6\ d_6\ c_6\ c_1\ d_1\ b_1\ b_5\ d_5\ c_5\ c_3\ d_3\ b_3\ b_6\ b_2\ d_2\ c_2\ c_7\ d_7\ b_7\ b_4\ d_4\ a_4$
This completes the proof. □

The main result of this section follows.

Theorem 3. *There are infinitely many MNH graphs of girth* 7.

Proof. To each positive integer $n \geq 1$ we associate a class $\mathfrak{C}_n$ of graphs by the following inductive definition. Set $\mathfrak{C}_1 = \{CG\}$, and for $n \geq 2$, $\mathfrak{C}_n = \{G; G = H \circ CG, H \in \mathfrak{C}_{n-1}\}$. Let $G \in \mathfrak{C}_n$, $n \geq 1$. According to Lemma 2 and Theorem 2 the graph G is MNH. Since CG has girth 7, the graph G has girth at least 7. Obviously, $|\mathfrak{C}_n| \to \infty$ exponentially in n. □

We now describe the construction of Isaacs' snarks J_k for odd $k \geq 3$.

Definition 2. *For $k \geq 3$, k odd, the graph J_k has the vertex set $\{v_1, v_2, \ldots, v_k, u_1, u_2, \ldots, u_k, x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k\}$ and edge set $E_0 \cup E_1 \cup E_2$*

where

$$\begin{aligned} E_0 &= \cup_{j=1}^{k-1}\{v_jv_{j+1}, y_jx_{j+1}, x_jy_{j+1}\},\\ E_1 &= \cup_{j=1}^{k}\{v_ju_j, u_jx_j, u_jy_j\},\\ E_2 &= \{v_kv_1, y_kx_1, x_ky_1\}. \end{aligned}$$

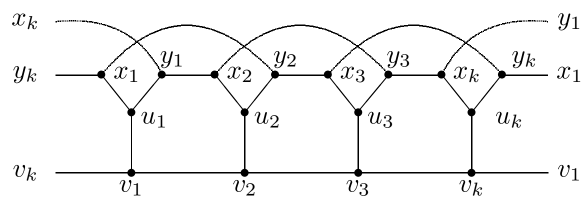


Fig. 7.

It is well known that the graph J_k is cubic, hypohamiltonian and MNH of girth 5 for $k = 5$ and girth 6 for $k \geq 6$ [1].

Lemma 3. *The graph $J_k, k \geq 5$, fulfils the condition* (2u) *of the Theorem* 2 *and the conditions* (1uv) *and* (3u) *of the Theorem* 2 *for $u = u_i$ and $v = u_j$* (*these vertices are as in Definition* 2), *where $1 \leq i \leq k$, $1 \leq j \leq k$ and $i \neq j$.*

Proof. In [1] it was proved that if u and v are non-adjacent vertices of J_k, $k \geq 5$, and u_1, u_2 and u_3 are the neighbours of u, then each edge uu_i, $i = 1, 2, 3$ lies in a hamiltonian $u - v$ path of J_k. Thus, the condition (2u) holds for J_k.
Condition (1uv). Because of the symmetry of J_k it is sufficient to describe a hamiltonian $y_k - y_j(u_j, u_k)$ paths for such j that the number $\xi(k, j)$=$k - j - 1$ is even (k is fixed). A list of such paths follows. Indices throughout are to be taken as modulo k.

The construction of path P_1 $y_k - y_j(u_j, u_k)$ if $\xi(k, j) \geq 4$ and $4 \mid \xi(k, j)$:
Let $Q(i) = u_i\ x_i\ y_{i-1}$.
Let $R(i) = x_i\ u_i\ v_i\ v_{i+1}\ u_{i+1}\ x_{i+1}\ y_i\ x_{i-1}\ y_{i-2}\ u_{i-2}\ v_{i-2}\ v_{i-1}\ u_{i-1}\ y_{i-1}\ x_{i-2}\ y_{i-3}$.
Then, $P_1 = y_k\ x_1\ u_1\ v_1\ v_2\ \ldots\ v_j\ Q(j)\ Q(j-1)\ \ldots\ Q(2)\ x_k\ u_k\ v_k\ v_{k-1}\ v_{k-2}\ u_{k-2}\ x_{k-2}\ y_{k-1}\ u_{k-1}\ x_{k-1}\ y_{k-2}\ x_{k-3}\ y_{k-4}\ u_{k-4}\ v_{k-4}\ v_{k-3}\ u_{k-3}\ y_{k-3}\ x_{k-4}\ y_{k-5}\ R(k-6)\ R(k-10)\ \ldots\ R(j+3)$

The path P_2 $y_k - y_j(u_j, u_k)$ if $\xi(k, j) = 0$:
Let $Q(i) = u_i\ y_i\ x_{i+1}$.
$P_2 = y_k\ x_1\ u_1\ v_1\ v_2\ \ldots\ v_{j-1}\ u_{j-1}\ y_{j-1}\ x_j\ u_j\ v_j\ v_k\ u_k\ x_k\ y_1\ x_2\ Q(2)\ Q(3)\ \ldots\ Q(j-2)\ y_j$

The path P_3 $y_k - y_j(u_j, u_k)$ if $4 \not| \xi(k, j)$:
Let $Q(i) = u_i\ x_i\ y_{i-1}$.
Let $R(i) = x_i\ u_i\ v_i\ v_{i+1}\ u_{i+1}\ x_{i+1}\ y_i\ x_{i-1}\ y_{i-2}\ u_{i-2}\ v_{i-2}\ v_{i-1}\ u_{i-1}\ y_{i-1}\ x_{i-2}\ y_{i-3}$.
$P_3 = y_k\ x_1\ u_1\ v_1\ v_2\ \ldots\ v_j\ Q(j)\ Q(j-1)\ \ldots\ Q(2)\ x_k\ u_k\ v_k\ v_{k-1}\ v_{k-2}\ u_{k-2}\ y_{k-2}\ x_{k-1}\ u_{k-1}\ y_{k-1}\ x_{k-2}\ y_{k-3}\ R(k-4)\ R(k-8)\ \ldots\ R(j+3)$

Condition (3u). Because of the symmetry of J_k it is sufficient to describe the following six hamiltonian cycles.

The hamiltonian cycle C_1 in $J_k - \{x_1, v_k\}$:
Let $Q(i) = u_i\ x_i\ y_{i-1}$.
$C_1 = u_1\ v_1\ v_2\ \dots\ v_{k-1}\ u_{k-1}\ y_{k-1}\ x_k\ u_k\ y_k\ x_{k-1}\ y_{k-2}\ Q(k-2)\ Q(k-3)\ \dots\ Q(2)\ u_1$

The hamiltonian cycle C_2 in $J_k - \{x_1, v_2\}$:
Let $Q(i) = u_i\ x_i\ y_{i+1}$.
$C_2 = u_1\ v_1\ v_k\ v_{k-1}\ \dots\ v_3\ u_3\ y_3\ x_2\ u_2\ y_2\ x_3\ y_4\ Q(4)\ Q(5)\ \dots\ Q(k)\ u_1$

The hamiltonian cycle C_3 in $J_k - \{x_1, x_2\}$:
Let $Q(i) = u_i\ x_i\ y_{i+1}$.
$C_3 = u_1\ v_1\ v_k\ v_{k-1}\ \dots\ v_4\ u_4\ y_4\ x_3\ y_2\ u_2\ v_2\ v_3\ u_3\ y_3\ x_4\ y_5\ Q(5)\ Q(6)\ \dots\ Q(k)\ u_1$

The hamiltonian cycle C_4 in $J_k - \{x_1, x_k\}$:
Let $Q(i) = u_i\ x_i\ y_{i-1}$.
$C_4 = u_1\ v_1\ v_2\ \dots\ v_{k-2}\ u_{k-2}\ y_{k-2}\ x_{k-1}\ y_k\ u_k\ v_k\ v_{k-1}\ u_{k-1}\ y_{k-1}\ x_{k-2}\ y_{k-3}\ Q(k-3)\ Q(k-4)\ \dots\ Q(2)\ u_1$

The hamiltonian cycle C_5 in $J_k - \{v_1, x_2\}$:
Let $Q(i) = y_i\ u_i\ x_i$.
$C_5 = u_1\ y_1\ x_k\ u_k\ v_k\ v_{k-1}\ \dots\ v_4\ u_4\ y_4\ x_3\ y_2\ u_2\ v_2\ v_3\ u_3\ y_3\ x_4\ Q(5)\ Q(6)\ \dots\ Q(k-1)\ y_k\ x_1\ u_1$

The hamiltonian cycle C_6 in $J_k - \{v_1, x_k\}$:
Let $Q(i) = u_i\ x_i\ y_{i-1}$.
$C_6 = u_1\ y_1\ x_2\ u_2\ v_2\ v_3\ \dots\ v_{k-2}\ u_{k-2}\ y_{k-2}\ x_{k-1}\ y_k\ u_k\ v_k\ v_{k-1}\ u_{k-1}\ y_{k-1}\ x_{k-2}\ y_{k-3}\ Q(k-3)\ Q(k-4)\ \dots\ Q(3)\ x_1\ u_1$
This completes the proof. □

Corollary 1. *Let* $G = [[\dots[J_{k_1}(u_{i_1}) \circ J_{k_2}(u_{i_2})](u_{i_{1,2}}) \circ J_{k_3}(u_{i_3})](u_{i_{1,2,3}}) \circ \dots \circ J_{k_n}(u_{i_n})$*, where* $k_i \geq 5$ *for* $i = 1, 2, ..., n$, $n \geq 1$*, and* $u_1, u_2, \dots, u_n, u_{i_{1,2,\dots,p}}$*, where* $p = 2, 3, \dots, n-1$*, are as in Definition* 2*. Then* G *is a MNH graph.* □

Remark 2. *It is easy to observe that the condition* (3u) *does not hold for* $u = x_i\ (v_i)$, $u' = y_{i-1}\ (u_i)$, $u'' = y_{i+1}\ (v_{i-1})$*, and* $x = x_{i+2}\ (v_{i-2})$, $(i = 1, \dots, 5)$*, in the graph* J_5 *(the indices are taken as modulo* 5*). Using computer this was verified also for some greater* J_k*. On the other hand, it can be verified that every graph* $G = J_5 \circ J_5$ *is MNH.*

Since every graph $J_5 \circ J_5$ is a MNH graph, we conjecture that every graph $G = [[\dots[J_{k_1} \circ J_{k_2}] \circ J_{k_3}] \circ \dots] \circ J_{k_n}$, where $k_i \geq 5$, odd, $(i = 1, 2, \dots, n)$, is a MNH graph.

We remark that $|\mathfrak{E}_n| \to \infty$ exponentially in n. Moreover, for each positive integer n we can define another family of MNH graphs as follows. Let $G = [\dots[G_1 \circ G_2] \circ \dots \circ G_{n-1}] \circ G_n$. We say that $x \in G_i$ $(1 \leq i \leq n)$ is *available* if the original graph G_i is either CG or $J_{2k+1}, k \geq 2$ and $x = u_j, 1 \leq j \leq 2k+1$. Set $\mathfrak{M}_1 = \{\ CG,\ J_5,\ J_7,\ \dots,\ J_{2k+1},\ \dots\}$, and for $n \geq 2$, let $\mathfrak{M}_n = \{G;\ G = H(w_1) \circ P(w_2)$, $H \in \mathfrak{M}_{n-1}$ and $P \in \mathfrak{M}_1$, where w_1 and w_2 are available

}. The following Theorem holds; we omit the proof because it is an easy consequence of Theorem 2 and Lemmas 2 and 3 .

Theorem 4. *Every graph in* $\mathfrak{M}_n$, $n \geq 1$ *is a MNH graph.* □

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