On the Complexity of Ordered Colorings

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Revised version – November 2007

Abstract

We introduce two variants of proper colorings with imposed partial ordering on the set of colors. One variant shows very close connections to some fundamental problems in graph theory, e.g., directed graph homomorphism and list colorings. We study the border between tractability and intractability for both variants.

1 Introduction

We introduce two variants of proper colorings with imposed partial ordering on the set of colors. Vertices of all considered graphs G are labeled with integers from 1 to $|V(G)|$, and we will use the normal order of the integers. In particular, the vertices form a completely ordered set. The set of colors forms a partially ordered set. The following ordered colorings are proper colorings satisfying additional requirements.

In the first coloring problem, we require for every two colors A, B for which A is smaller than B in the partial order, that every vertex colored A is smaller than every vertex colored B. We will show that this problem is in P if the set of colors contains at most two incomparable colors by reduction to 2-SAT, and that otherwise it is intractable.

In the second coloring problem, we require for every two colors A, B for which A is smaller than B in the partial order, that for every edge whose end vertices are colored by A and B, that the vertex with color A is smaller than the vertex with color B. Note that a vertex colored A can be greater than a vertex colored B provided they are not adjacent. We show that this problem is

* Research supported in part by NSERC (Natural Science and Engineering Research Council of Canada) grant.
† Research supported in part by PIMS (Pacific Institute for Mathematical Sciences), and by the Science and Technology Assistance Agency grant APVT-20-018902 (Slovak grant agency).
‡ Research supported in part by NSERC (Natural Science and Engineering Research Council of Canada) grant, and VEGA grant No. 2/3164/23 (Slovak grant agency).

NP-complete if the partial ordering on colors contains at least two incomparable pairs. Otherwise, the problem is in P.

2 Basic definitions

Most of our terminology and notation will be standard and can be found in any textbook on graph theory such as [2], and on computational complexity such as [4].

Throughout this paper a graph $G = (V, E)$ is finite, simple and loopless. We will always use n for the number of vertices and assume that the vertices of the input graph G are integers from 1 to n (in other words we set $V = [1, n]$). With this assumption we can consider the vertex set as a linearly ordered set $(V, \leq)$.

The set of available colors is denoted by *Col*. A *coloring* of a graph G is a function $\varphi : V \to Col$ so that for every edge $uv \in E$ we have $\varphi(u) \neq \varphi(v)$. We additionally assume that $\preceq$ is an antisymmetric, reflexive and transitive relation on *Col*, i.e., $\mathcal{C} = (Col, \preceq)$ is a partially ordered set. We sometimes call $(Col, \preceq)$ the *color-poset*. We will study the complexity of two coloring problems in which we will color vertices of the input graph G with colors in *Col* satisfying further requirements. We will refer to a coloring that satisfies these requirements as a feasible coloring.

Let $C \in Col$, the set V_C of vertices in G colored by C will be called a *color class*.

If $(P, \preccurlyeq)$ is a poset, then an *antichain* is a set $S \subseteq P$ so that no two elements from S are comparable (i.e., for all $x, y \in S$ we have $x \not\preccurlyeq y$ and $y \not\preccurlyeq x$).

Given $X, Y \subseteq V$, an edge with one end-vertex in X and the other in Y is called an *X, Y-edge*. Let $E(X, Y)$ be the set of all X, Y-edges.

3 Coloring with ordered color classes

In this section we study the complexity of the following decision problem:

Problem 3.1.
Fix a color-poset $(Col, \preceq)$. Given a graph $G = (V, E)$ with $V = [1, n]$, determine whether G can be colored with Col such that for any two colors A, B with $A \preceq B$ and for any two vertices $u \in V_A$ and $v \in V_B$ we have $u \leq v$.

If a coloring as in Problem 3.1 exists, we say that the graph G can be *feasibly colored with* $(Col, \preceq)$.

Note that if $\preceq$ is the empty relation on *Col*, then Problem 3.1 is the well-known graph coloring problem.

Theorem 3.2.
If the poset $(Col, \preceq)$ does not contain an antichain of size 3, then Problem 3.1 can be solved in polynomial time. Otherwise, the problem is NP-complete.

The proof that Problem 3.1 is in P if the longest antichain in $(Col, \preceq)$ has length at most 2 is quite technical. For this reason we first give the proof for a specific small poset with all the details. The proof of the general case will be done with less detail after that. The fairly straightforward proof that the problem is NP-complete if $(Col, \preceq)$ contains an antichain of length 3 will be the final proof in this section.

Lemma 3.3.
Let $Col = \{A, B, C\}$ and suppose the only relation between the colors is $A \preceq B$. Problem 3.1 can be solved in polynomial time for this poset $(\{A, B, C\}, \preceq)$.

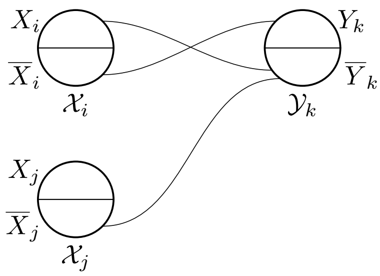
Proof. Suppose a graph G can be feasibly colored with $(\{A, B, C\}, \preceq)$. It follows that there exists a vertex $v \in V$ such that vertices in the set $V_{AC} = [1, v-1]$ are colored A or C, and vertices in the set $V_{BC} = [v, n]$ are colored B or C. Therefore, the subgraphs G_{AC} and G_{BC} of G, induced by V_{AC} and V_{BC}, respectively, are bipartite. Suppose G_{AC} has k components and G_{BC} has ℓ components. Let $\{(X_i, \overline{X_i})\}_{i=1}^{k}$ be the set of bipartitions of connected components $\{\mathcal{X}_i\}_{i=1}^{k}$ of G_{AC}, and let $\{(Y_j, \overline{Y_j})\}_{j=1}^{\ell}$ be the set of bipartitions of connected components $\{\mathcal{Y}_j\}_{j=1}^{\ell}$ of G_{BC}.

Every bipartite component $\mathcal{X}_i$ (respectively $\mathcal{Y}_j$) has exactly two 2-colorings. Thus, we can associate each $\mathcal{X}_i$ with a boolean variable x_i as follows: if the bipartition X_i of $\mathcal{X}_i$ is colored C, then $x_i = 0$; otherwise, $x_i = 1$. Similarly, we associate each $\mathcal{Y}_j$ with a boolean variable y_j: if the bipartition Y_j of $\mathcal{Y}_j$ is colored C, then $y_j = 0$; otherwise, $y_j = 1$.

Every (V_{AC}, V_{BC})-edge of G imposes a constraint on a feasible coloring of G. This constraint can be equivalently expressed in terms of the above boolean variables:

- for every (X_i, Y_j)-edge of G, the clause $x_i \vee y_j$ should be satisfied;
- for every $(X_i, \overline{Y_j})$-edge of G, the clause $x_i \vee \overline{y_j}$ should be satisfied;
- for every $(\overline{X_i}, Y_j)$-edge of G, the clause $\overline{x_i} \vee y_j$ should be satisfied;
- for every $(\overline{X_i}, \overline{Y_j})$-edge of G, the clause $\overline{x_i} \vee \overline{y_j}$ should be satisfied.

Let f_v be the conjunction of the above 2-literal disjunctions for all (V_{AC}, V_{BC})-edges of G, see Figure 1 for an example.



$$\Longleftrightarrow (x_i \vee \overline{y_k}) \wedge (\overline{x_i} \vee y_k) \wedge (\overline{x_j} \vee \overline{y_k})$$

Figure 1: An example of the transformation of a graph G into a 2-CNF formula.

Obviously, there exists a feasible coloring of G if and only if the formula f_v is satisfiable. An assignment of boolean values to variables in f_v can be translated into a coloring of G in linear time. Hence, to find a coloring for G feasible with $(\{A, B, C\}, \preceq)$, we can use the following algorithm

- For each $v \in V$, check if the subgraphs induced on $[1, v-1]$ and on $[v, n]$ are bipartite. If so, construct the formula f_v and use a 2-SAT solver to check the satisfiability of the formula. The graph G has a feasible coloring if and only if for at least one $v \in V$ the subgraphs induced on $[1, v-1]$ and on $[v, n]$ are bipartite and the formula f_v is satisfiable.

In [3], a 2-SAT solver working in linear time in the size of the formula is described. Note that each f_v has $O(n)$ variables and $O(n^2)$ clauses. Since there are n choices for v, Problem 3.1 can be solved in $O(n^3)$ steps. □

We next give the general proofs of the two parts of Theorem 3.2.

Theorem 3.2 A
If the poset $(Col, \preceq)$ *does not contain an antichain of size 3, then Problem 3.1 can be solved in polynomial time.*

Proof. If $\mathcal{P} = (P, \preccurlyeq)$ is a poset and $u \in P$, then by $(P - u, \preccurlyeq)$ we denote the poset with the same ordering $\preccurlyeq$ on the set $P \setminus \{u\}$. For a poset $\mathcal{P}$ with $|P| = p$ elements that does not contain an antichain of size 3, we will construct a collection $\Lambda_{\mathcal{P}}$ of p-tuples $\overline{\lambda} = (\lambda_1, \ldots \lambda_p)$, where each λ_i is a set of one or two elements from P. The *concatenation* of two tuples $\overline{\lambda} = (\lambda_1, \ldots, \lambda_k)$ and $\overline{\mu} = (\mu_1, \ldots, \mu_\ell)$ is $\overline{\lambda} * \overline{\mu} = (\lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_\ell)$.

The construction of $\Lambda_{\mathcal{P}}$ is recursive, as follows:

- If P has one element, say $P = \{a\}$, then $\Lambda_{\mathcal{P}} = \{\, \{a\} \,\}$.
- If $|P| \geq 2$ and P has a maximum element a, then $\Lambda_{\mathcal{P}} = \{\, \overline{\lambda} * (\{a\}) \mid \overline{\lambda} \in \Lambda_{\mathcal{P}-a} \,\}$.
- If $|P| \geq 2$ and P has no maximum element, then, since the longest antichain has length 2, there are two maximal elements a, b (so $a \not\preccurlyeq b$ and $b \not\preccurlyeq a$, while $p \preccurlyeq a$ or $p \preccurlyeq b$ for all $p \in P$). In that case we set $\Lambda_{\mathcal{P}} = \{\, \overline{\lambda} * (\{a, b\}) \mid \overline{\lambda} \in \Lambda_{\mathcal{P}-a} \,\} \cup \{\, \overline{\lambda} * (\{a, b\}) \mid \overline{\lambda} \in \Lambda_{\mathcal{P}-b} \,\}$.

Note that the number of tuples in $\Lambda_{\mathcal{P}}$ is finite.

Now we go back to a color poset $\mathcal{C} = (Col, \preceq)$ with no antichain longer than 3. Set $c = |Col|$. Consider a graph $G = (V, E)$ with $V = [1, n]$. Suppose G can be feasibly colored with $\mathcal{C}$. With any such coloring we can associate a c-tuple $\overline{\lambda} = (\lambda_1, \ldots \lambda_c) \in \Lambda_{\mathcal{C}}$ and a $(c+1)$-tuple of integers $(v_1, v_2 \ldots, v_{c+1})$ so that:

1. We have $1 = v_1 \leq v_2 \leq \cdots \leq v_c \leq v_{c+1} = n + 1$.
2. For all $i = 1, \ldots, c$, the subgraph of G induced by $[v_i, v_{i+1} - 1]$ is an independent set or a bipartite graph.
3. For all $i = 1, \ldots, c$, the colors appearing on the vertices $[v_i, v_{i+1} - 1]$ are exactly the colors in λ_i.

The existence of these tuples follows directly from the fact that each antichain in $\mathcal{C}$ has length 1 or 2, and hence we cannot "mix" three or more colors from *Col* when coloring the graph.

Given a $(c + 1)$-tuple of integers $(v_1, v_2 \ldots, v_{c+1})$, it is easy to check if this tuple satisfies conditions 1 and 2 above. If it does, then we can choose a c-tuple $\overline{\lambda} = (\lambda_1, \ldots \lambda_c) \in \Lambda_{\mathcal{C}}$ and use the same technique as applied in the proof of Lemma 3.3 to transform the problem of the existence of a feasible coloring of G in accordance with condition 3 to the existence of a solution of a certain 2-SAT problem. Hence the question if there exists a coloring of G in accordance with two chosen tuples satisfying conditions $1-3$ can be done in time polynomial in n.

The number of $(c + 1)$-tuples $(v_1, v_2 \ldots, v_{c+1})$ with $1 = v_1 \leq v_2 \leq \cdots \leq v_c \leq v_{c+1} = n + 1$ is $O(n^c)$. And the number of tuples in $\Lambda_{\mathcal{C}}$ is $O(2^c)$. And so, to check if a graph G on n ordered vertices $[1, n]$ can be feasibly colored with $\mathcal{C}$, we need to consider at most $O(n^c)$ 2-SAT problems. Using estimates for the numbers of variables and clauses in each of the 2-SAT problems similar to those in the proof of Lemma 3.3, we can conclude that Problem 3.1 can be solved in polynomial time. □

Theorem 3.2 B
If the poset $(Col, \preceq)$ *contains an antichain of size 3 or more, then Problem 3.1 is NP-complete.*

Proof. Let $\mathcal{C} = (Col, \preceq)$ be a poset containing an antichain of size 3 or more. It is obvious that Problem 3.1 is in NP. To prove the problem is NP-complete we give a reduction from the proper K-coloring problem, which is well known to be NP-complete for any fixed $K \geq 3$.

Let $S \subseteq Col$ be an antichain in $\mathcal{C}$ with $|S| \geq 3$, chosen such that S is *maximal*. I.e., for all $C \in Col \setminus S$ we have that either $C \preceq A$ for some $A \in S$, or $A' \preceq C$ for some $A' \in S$, but not both (since S is an antichain). Let $Col_D \subseteq Col$ be the set of colors $C \in Col \setminus S$ such that $C \preceq A$ for some $A \in S$, and define $Col_U \subseteq Col$ similarly for colors in $Col \setminus S$ which are larger than some color in S. Set $K = |S|$, $n_D = |Col_D|$ and $n_U = |Col_U|$.

Given a graph $G' = (V', E')$ on n' vertices, we construct a graph G with ordered vertex set $[1, n]$ where $n = n_D + n' + n_U$, so that G' has a K-coloring if and only if G can be feasibly colored with $\mathcal{C}$. Let the vertices of G be $[1, n_D] \cup [n_D + 1, n_D + n'] \cup [n_D + n' + 1, n_D + n' + n_U]$. Add edges so that the graph on $[n_D + 1, n_D + n']$ is isomorphic to G', and vertices in $[1, n_D] \cup [n_D + n' + 1, n_D + n' + n_U]$ are universal (joined to all other vertices).

Suppose G' has a K-coloring. Then we can color G in accordance with $\mathcal{C}$ as follows:

- Give each vertex in $[1, n_D]$ its own color from Col_D, using some linear extension of the order imposed by $\mathcal{C}$ on Col_D.
- Color the vertices in $[n_D + 1, n_D + n']$ with colors from S, according to the K-coloring possible on G'.
- Give each vertex in $[n_D + n' + 1, n_D + n' + n_U]$ its own color from Col_U, using some linear extension of the order imposed by $\mathcal{C}$ on Col_U.

It is easy to check that this coloring of G is feasible with $\mathcal{C}$, where the crucial observation is that S is an antichain in $\mathcal{C}$ and hence every proper coloring with colors from S is always in accordance with the poset order.

Next suppose that G has a feasible coloring with $\mathcal{C}$. Such a coloring must have the following properties:

- Each vertex in $[1, n_D]$ has a distinct color from Col_D, and this color is smaller than any color appearing on $[n_D + 1, n_D + n'] \cup [n_D + n' + 1, n_D + n' + n_U]$.
- Each vertex in $[n_D + n' + 1, n_D + n' + n_U]$ has a distinct color from Col_U, and this color is larger than any color appearing on $[1, n_D] \cup [n_D + 1, n_D + n']$.

From this it follows that the vertices in $[n_D + 1, n_D + n']$ are colored with colors from S. Since S is an antichain, the only requirement to color those vertices with S is that is must be a proper coloring. Such a proper coloring immediately gives a K-coloring of G'. $\square$

4 Coloring with ordered edges

In this section we study the complexity of the following decision problem:

Problem 4.1.
Fix a color-poset $(Col, \preceq)$. *Given a graph* $G = (V, E)$ *with* $V = [1, n]$, *determine whether* G *can be colored with* Col *such that for any two colors* A, B *with* $A \preceq B$ *and for any edge* $(u, v) \in E$ *with* $u \in V_A$ *and* $v \in V_B$ *we have* $u \leq v$.

We will transform this problem into a directed graph homomorphism problem. In this section, by (u, v) we mean a directed edge (arc) from u to v. We will use E to denoted the set of directed edges of a directed graph (digraph) as well.

The semicomplete digraphs depicted in Figure 2 play a crucial role in our characterization.

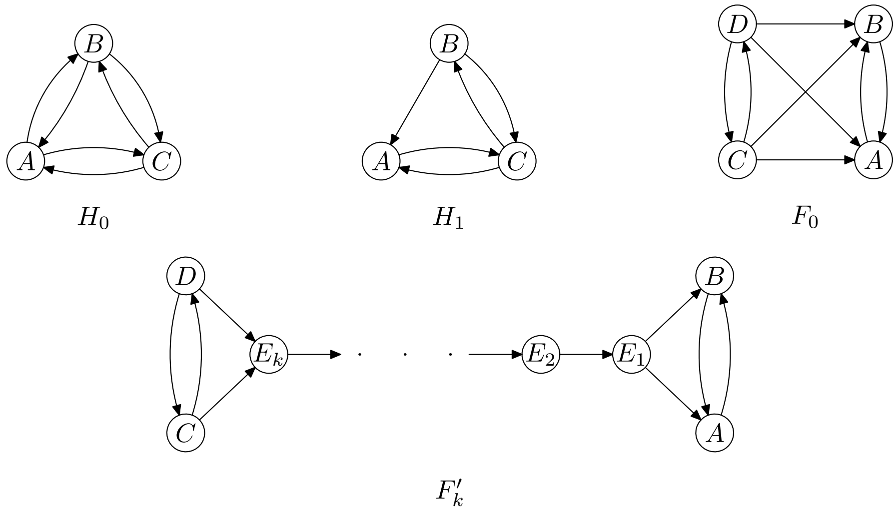


Figure 2: The crucial semicomplete digraphs in the dichotomy characterization of Problem 4.1. F_k is obtained by taking the transitive closure of F'_k.

Definition 4.2.
Let H be a fixed digraph. An H-coloring *of a digraph G is a homomorphism from G to H (a mapping φ from $V(G)$ to $V(H)$ such that if $(x, y) \in E(G)$ then $(\varphi(x), \varphi(y)) \in E(H)$). The* acyclic H-coloring problem *asks whether for a given acyclic digraph G there is an H-coloring of G.*

Definition 4.3.
A semicomplete digraph *is a digraph H which contains a spanning tournament, i.e., between any two vertices there is either one arc or both opposite arcs. We say that a semicomplete digraph is* quasi-acyclic *if after removal of all pairs of oppositely directed arcs we obtain a transitive acyclic subdigraph of H.*

An alternative definition of a quasi-acyclic semicomplete digraph is as a semicomplete digraph in which every directed cycle contains at least two arcs that come from pairs of opposite arcs. It is easy to see that there is a one-to-one correspondence between posets and quasi-acyclic semicomplete digraphs. Given a color-poset $\mathcal{C} = (Col, \preceq)$, to obtain the corresponding digraph take $V = Col$. And for every two colors A and B, $A \neq B$, if $A \prec B$ add an arc (A, B), if $B \prec A$ add an arc (B, A), and otherwise add a pair of opposite arcs (A, B) and (B, A). The resulting semicomplete digraph is quasi-acyclic and will be denoted $H_{\mathcal{C}}$.

Similarly, a graph G with ordered vertex set $[1, n]$ can be considered as a digraph G' if we replace all edges $uv \in E(G)$ with $u < v$ by arcs (u, v).

Without a proof we note that, with this notation, we have that G has a coloring by $\mathcal{C}$ feasible for Problem 4.1 if and only if there is an $H_{\mathcal{C}}$-coloring of G'.

Theorem 4.4.
Let H be a quasi-acyclic semicomplete digraph. The acyclic H-coloring problem can be solved in polynomial time if H contains at most one directed 2-cycle (a pair of oppositely directed arcs). Otherwise, the problem is NP-complete.

This is exactly the same dichotomy as for the general H-coloring problem where H is any semicomplete digraph, cf. [1]. In particular, the first part of the theorem follows from the polynomiality of this problem. But for convenience we give a short proof below as well (Lemma 4.5).

The second part of the theorem follows from the Lemmas 4.6 – 4.11 that make up most of the remainder of this section. In those lemmas we first prove the NP-completeness of the acyclic H-coloring problem for three crucial quasi-acyclic semicomplete digraphs and then for any quasi-acyclic semicomplete digraph containing any of those three as an induced subdigraph.

Lemma 4.5.
Let H be a quasi-acyclic semicomplete digraph. The acyclic H-coloring problem can be solved in polynomial time if H contains at most one directed 2-cycle.

Proof. The problem is trivial if $|V(H)| = 1$. If $|V(H)| = 2$ and there is a 2-cycle in H, then an acyclic digraph G has a homomorphism from G to H if and only if the underlying graph of G (obtained by ignoring the directions on the arcs) is bipartite. Similarly, if $|V(H)| = 2$ and there is no 2-cycle in H, then an acyclic digraph G has a homomorphism from G to H if and only if we can partition $V(G)$ into $G_D \cup G_U$ such that every arc in G has its tail in G_D and its head in G_U.

So assume $|V(H)| \geq 3$. Since H has an acyclic spanning tournament, there are both a universal sink and a universal source in H, i.e., there is a vertex $u \in V(H)$ so that for all $v \in V(H)$, $v \neq u$, we have $(v, u) \in E(H)$; and there is a vertex $u' \in V(H)$ so that for all $v \in V(H)$, $v \neq u'$, we have $(u', v) \in E(H)$. Since H has at most one 2-cycle $\{(p, q), (q, p)\}$ and after removing that 2-cycle we obtain a transitive acyclic subdigraph of H, it is easy to check that we can choose at least one of u, u' different from both p and q.

Suppose that the universal sink u in H is different from p and q. Let G be an acyclic digraph and let G_i be the set of sinks in G. Then there is a homomorphism from G to H if and only if there is a homomorphism from $G - G_i$ to $H - u$. Indeed, homomorphisms from $G - G_i$ to $H - u$ are easily extendable to homomorphisms from G to H. The converse follows since in any homomorphism from G to H, the neighbors of vertices in G_i are not mapped to u, and hence we can remap the vertices in G_i to u if necessary.

A similar observation holds if the universal source in H is different from p and q. We use induction on $|V(H)|$ to get a straightforward polynomial algorithm to check if there is a homomorphism from H to G. ☐

The following follows immediately from the NP-completeness of the ordinary 3-coloring problem.

Lemma 4.6.
Let H_0 be the quasi-acyclic semicomplete digraph depicted in Figure 2. The acyclic H_0-coloring problem is NP-complete.

Lemma 4.7.
Let H_1 be the quasi-acyclic semicomplete digraph depicted in Figure 2. The acyclic H_1-coloring problem is NP-complete.

Proof. The problem is obviously in NP. We will show that given a boolean formula $f(x_1, x_2, \ldots, x_m)$ in conjunctive normal form (CNF), we can construct an acyclic digraph $\widetilde{G}(f)$ which has an H_1-coloring φ if and only if f is satisfiable. For every boolean variable a of f (respectively, every auxiliary variable encoding the boolean value of a subformula of f), we will construct the gadget

$G_a = (V_a, E_a)$ defined as follows: $V_a = \{v_a^1, \ldots, v_a^6\}$ and $E_a = \{\,(v_a^2, v_a^1),\ (v_a^3, v_a^1),\ (v_a^3, v_a^2),\ (v_a^4, v_a^2),\ (v_a^5, v_a^3),\ (v_a^5, v_a^4),\ (v_a^6, v_a^1),\ (v_a^6, v_a^4),\ (v_a^6, v_a^5)\,\}$, see Figure 3.

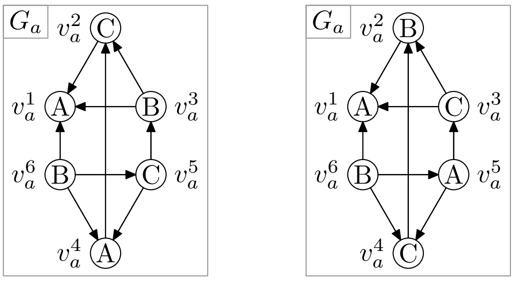


Figure 3: The two possible homomorphisms (coloring patterns) from G_a to H_1: on the left hand side $\mathrm{Pat}(G_a) = P^0$; on the right hand side $\mathrm{Pat}(G_a) = P^1$.

Each gadget G_a is acyclic and has only two H_1-colorings, called *color patterns*, see Figure 3. Let the color pattern P^0 of G_a with $\varphi(v_a^1) = \varphi(v_a^4) = A$, $\varphi(v_a^3) = \varphi(v_a^6) = B$ and $\varphi(v_a^2) = \varphi(v_a^5) = C$, represent the fact that the value of the boolean variable a is 0. Similarly, let the color pattern P^1 of G_a with $\varphi(v_a^1) = \varphi(v_a^5) = A$, $\varphi(v_a^2) = \varphi(v_a^6) = B$ and $\varphi(v_a^3) = \varphi(v_a^4) = C$, represent the fact that $a = 1$.

Now let us give an overview of the main ideas in the construction of $\widetilde{G}(f)$. Firstly, we construct a gadget G_{x_i} for every variable x_i occurring in the formula f. Secondly, for every disjunctive subformula d_i of $f = d_1 \wedge d_2 \wedge \cdots \wedge d_k$, we construct an acyclic digraph $G(d_i)$ containing a gadget G_{d_i} and using the gadgets $G_{x_1}, \ldots, G_{x_m}$ such that for every H_1-coloring of $G(d_i)$, $\mathrm{Pat}(G_{d_i}) = P^1$ if and only if color patterns of $G_{x_1}, \ldots, G_{x_m}$ correspond to a true assignment for d_i. Finally, we extend each $G(d_i)$ to a digraph $\widetilde{G}(d_i)$ such that the gadget G_{d_i} will have the color pattern P^1 in every H_1-coloring of $\widetilde{G}(d_i)$. Since the gadgets $G_{x_1}, \ldots, G_{x_m}$ are common for every $\widetilde{G}(d_i)$, it will follow that there is an H_1-coloring of $\widetilde{G}(f) = \bigcup_{i=1}^{k} \widetilde{G}(d_i)$ if and only if there is a true assignment for f, i.e., if f is satisfiable.

In the following we describe a recursive construction of the digraph $G(d)$, where $d = d_i$. If d has no $\vee$'s, then either $d = x_j$ or $d = \overline{x_j}$, for some $j = 1, \ldots, m$. As described above, add a gadget G_d to $G(d)$. To complete the construction add the arc $(v_d^3, v_{x_j}^2)$ if $d = x_j$, and the arc $(v_d^2, v_{x_j}^2)$ if $d = \overline{x_j}$, see Figure 4. First suppose $d = x_j$. If the color pattern of G_{x_j} is P^0, then because of the arc $(v_d^3, v_{x_j}^2)$, the only possible color pattern for G_d is P^0, and similarly if the color pattern of G_{x_j} is P^1, the only possible color pattern for G_d is P^1. Since every true assignment of d has $x_j = 1$ and the remaining variables can have arbitrary values, for every H_1-coloring of $G(d)$, $\mathrm{Pat}(G_d) = P^1$ if and only if color patterns of $G_{x_1}, \ldots, G_{x_m}$ correspond to such a true assignment. If $d = \overline{x_j}$, the argument is similar.

Assume now that d contains an $\vee$. Then there are disjunctive subformulas d' and d'' in f such that $d = d' \vee d''$. We recursively construct digraphs $G(d')$ and $G(d'')$ (in polytime). Construct $G(d)$ as follows. Take a union of $G(d')$ and $G(d'')$ and add a new variable gadget G_d and a disjunction gadget consisting of three new vertices w_d^1, w_d^2, w_d^3 and the arcs (w_d^2, w_d^1), (w_d^3, w_d^2). Furthermore, add the arcs $(w_d^1, v_{d'}^2)$, $(w_d^1, v_{d'}^5)$, $(w_d^3, v_{d''}^2)$, $(w_d^3, v_{d''}^5)$ and (v_d^4, w_d^2) connecting the disjunction gadget with the variable gadgets, see Figure 5.

It is easy to check that in every H_1-coloring of $G(d')$ (respectively $G(d'')$), the gadget $G_{d'}$

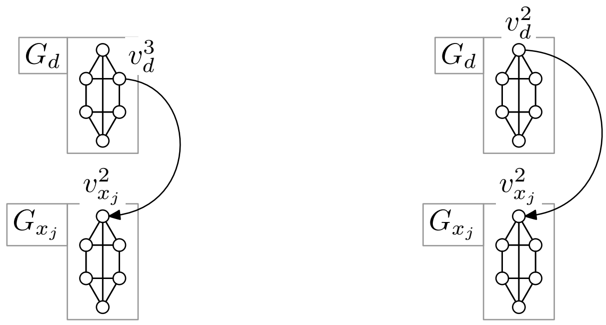


Figure 4: On the left: $G(d)$ for the boolean formula $d = x_j$; on the right: $G(d)$ for the boolean formula $d = \overline{x_j}$.

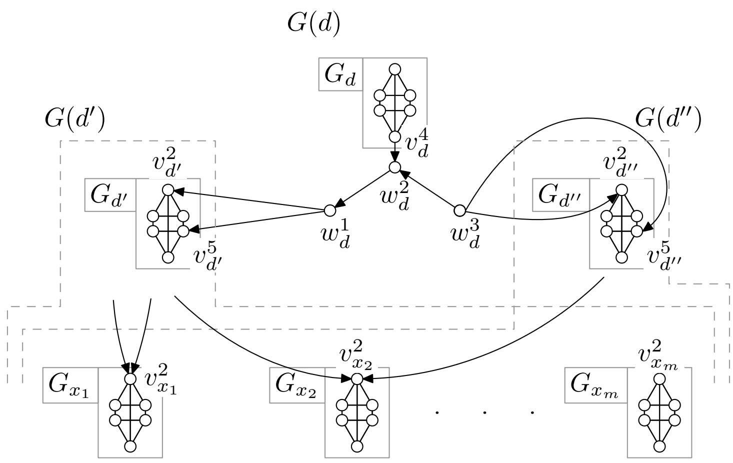


Figure 5: The construction of $G(d)$ for the boolean formula $d = d' \vee d''$ using the union of the digraphs $G(d')$ and $G(d'')$. In the example above we assume that d' contains both x_1 and $\overline{x_1}$ which is indicated by two arcs entering the gadget G_{x_1}. Similarly, the arcs entering the gadget G_{x_2} indicate that variable x_2 (or its complement) occurs in both d' and d''. Finally, no arc entering the gadget G_{x_m} indicates that neither d' nor d'' contain x_m or $\overline{x_m}$.

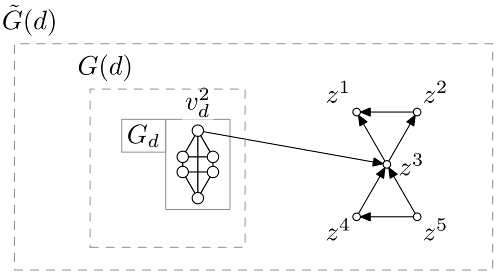


Figure 6: The construction of $\widetilde{G}(d)$.

in $G(d')$ (respectively $G_{d''}$ in $G(d'')$) has color pattern P^1 if and only if the color patterns of gadgets $G_{x_1}, \ldots, G_{x_m}$ represent a true assignment for d' (respectively d''). Otherwise it has color pattern P^0. Since the color patterns of gadgets $G_{x_1}, \ldots, G_{x_m}$ represent a true assignment for d if and only if they represent a true assignment for d' or d'', it is enough to show that in any H_1-coloring of $G(d)$, $\mathrm{Pat}(G_d) = P^1$ if and only if $\mathrm{Pat}(G_{d'}) = P^1$ or $\mathrm{Pat}(G_{d''}) = P^1$.

Consider an H_1-coloring of $G(d)$. If $\mathrm{Pat}(G_{d'}) = \mathrm{Pat}(G_{d''}) = P^1$, then $\varphi(w_d^1) = \varphi(w_d^3) = C$ and $\varphi(w_d^2) = A$ or B, and hence $\mathrm{Pat}(G_d) = P^1$. (Note that this is true even if $\varphi(w_d^2) = B$; the vertex v_d^4 cannot be colored A because of the arc (v_d^4, w_d^2).) If $\mathrm{Pat}(G_{d'}) = P^1$ and $\mathrm{Pat}(G_{d''}) = P^0$, then $\varphi(w_d^1) = C$, and since (w_d^3, w_d^2) is an arc, $\varphi(w_d^2) = A$ and $\varphi(w_d^3) = B$. Hence, $\mathrm{Pat}(G_d) = P^1$. The case $\mathrm{Pat}(G_{d'}) = P^0$ and $\mathrm{Pat}(G_{d''}) = P^1$ is analogous to the previous one. If $\mathrm{Pat}(G_{d'}) = \mathrm{Pat}(G_{d''}) = P^0$, then $\varphi(w_d^1)$ and $\varphi(w_d^3)$ is either A or B. Since $G(d)$ contains the arcs (w_d^2, w_d^1) and (w_d^3, w_d^2), we must have $\varphi(w_d^2) = C$, and hence $\mathrm{Pat}(G_d) = P^0$. This verifies the construction of the digraph $G(d)$.

Finally, we extend each $G(d)$ to a digraph $\widetilde{G}(d)$ such that the gadget G_d will have the color pattern P^1 in every H_1-coloring of $\widetilde{G}(d)$. Then, for every H_1-coloring of $\widetilde{G}(d)$, the color patterns of $G_{x_1}, \ldots, G_{x_m}$ must represent a true assignment for d. It follows that $\widetilde{G}(f)$ has an H_1-coloring if and only if f is satisfiable.

Let Z be the digraph with $V(Z) = \{z^1, \ldots, z^5\}$, and $E(Z) = \{\, (z^2, z^1), (z^3, z^2), (z^3, z^1), (z^4, z^3), (z^5, z^3), (z^5, z^4) \,\}$. Since z^3 has two incoming (outgoing) arcs from vertices connected by an arc, it cannot be colored B (respectively A). Hence, z^3 must be colored C. Consequently, $\varphi(z^1) = \varphi(z^4) = A$ and $\varphi(z^2) = \varphi(z^5) = B$. Now, we are ready to construct $\widetilde{G}(d)$. Take the union of $G(d)$ and Z and add an arc (v_d^2, z^3), see Figure 6. Obviously, in any H_1-coloring of $\widetilde{G}(d)$, $\mathrm{Pat}(G_d) = P^1$.

Finally, let us deduce that the digraph $\widetilde{G}(f)$ is acyclic. Firstly, by induction, we show that for every subformula d used in the construction, $G(d)$ is acyclic. This is certainly true for every $d = x_j$ or $d = \overline{x_j}$. Now, consider the formula $d = d' \vee d''$, see Figure 5. By induction $G(d')$ and $G(d'')$ are acyclic and since all arcs incident with the common part of $G(d')$ and $G(d'')$ (the input variable gadgets) end in the common part, the union $G(d') \cup G(d'')$ is acyclic as well. The digraph $G(d)$ contains this union and two new gadgets and all arcs connecting these two parts start in the new gadgets and end in the union. Hence, $G(d)$ is also acyclic. Thus, for every d_i in f, $G(d_i)$ and obviously also $\widetilde{G}(d_i)$ is acyclic. By the same argument as above their union $\widetilde{G}(f)$ is also acyclic. $\square$

Lemma 4.8.
Let F_0 be the quasi-acyclic semicomplete digraph depicted in Figure 2. The acyclic F_0-coloring problem is NP-complete.

Proof. The problem is obviously in NP. The proof of NP-hardness follows the lines of the proof of Lemma 4.7. We will again show that given a boolean formula $f(x_1, x_2, \ldots, x_m)$ in CNF, we can construct an acyclic digraph $\widetilde{G}(f)$ which has a F_0-coloring if and only if f is satisfiable.

The main differences with the proof of Lemma 4.7 is that the new variable gadgets will share a four-vertex subdigraph Z with $V(Z) = \{z^1, \ldots, z^4\}$ and $E(Z) = \{\, (z_i, z_j) \mid i > j \,\}$, see Figure 7. Note that the digraph Z has a unique F_0-coloring up to swapping A and B, or C and D. Without

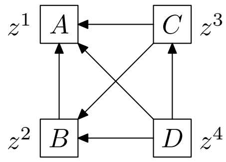
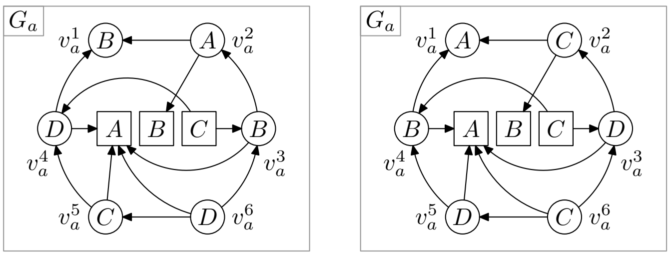


Figure 7: The digraph Z.

loss of generality, it is enough to consider only those F_0-colorings φ of $\widetilde{G}(f)$ which have $\varphi(z^1) = A$, $\varphi(z^2) = B$, $\varphi(z^3) = C$ and $\varphi(z^4) = D$. This assumption will reduce the number of color patterns of variable gadgets to two.

For every boolean variable a of f (input or auxiliary), we will construct the following acyclic gadget $G_a = (V_a, E_a)$ with $V_a = \{v_a^1, \ldots, v_a^6\}$ and $E_a = \{\, (v_a^2, v_a^1), (v_a^3, v_a^2), (v_a^4, v_a^1), (v_a^5, v_a^4), (v_a^6, v_a^3), (v_a^6, v_a^5) \,\}$. As proposed above, we complete the construction of the gadget by connecting it to the common digraph Z by arcs (v_a^3, z^1), (v_a^4, z^1), (v_a^5, z^1), (v_a^6, z^1), (v_a^2, z^2), (z^3, v_a^3) and (z^3, v_a^4), see Figure 8. The figure shows two coloring patterns of G_a: P^0 with $\varphi(v_a^1) = \varphi(v_a^3) = B$, $\varphi(v_a^2) =$



$Pat(G_a) = P^0$ $\quad$ $Pat(G_a) = P^1$

Figure 8: Two coloring patterns of the gadget G_a: $\mathrm{Pat}(G_a) = P^0$ (on the left); and $\mathrm{Pat}(G_a) = P^1$ (on the right). The squares labeled with A, B and C represent vertices z^1, z^2 and z^3 of the digraph Z common to all gadgets, respectively. For simplicity, we omitted the fourth vertex of Z and the arcs inside Z.

$A, \varphi(v_a^4) = \varphi(v_a^6) = D$ and $\varphi(v_a^5) = C$, which will represent the fact that $a = 0$; and P^1 with $\varphi(v_a^1) = A$, $\varphi(v_a^2) = \varphi(v_a^6) = C$, $\varphi(v_a^3) = \varphi(v_a^5) = D$ and $\varphi(v_a^4) = B$ corresponding with $a = 1$.

For each disjunctive subformula $d = d_i$ of $f = d_1 \wedge \cdots \wedge d_k$, the construction of $\widetilde{G}(d)$ is analogous to the construction in the proof of Lemma 4.7. Therefore, we only describe the main ingredients of the construction: the base step $d = x_j$ or $d = \overline{x_j}$, the inductive step $d = d' \vee d''$, and forcing the color pattern of G_d to P^1. In the base step, add the arc $(v_{x_j}^6, v_d^5)$, if $d = x_j$; or the arc $(v_{x_j}^6, v_d^6)$, if $d = \overline{x_j}$, joining the input variable gadget G_{x_j} to variable gadget G_d, see Figure 9.

One can easily check that $G(d)$ is acyclic and has the required property.

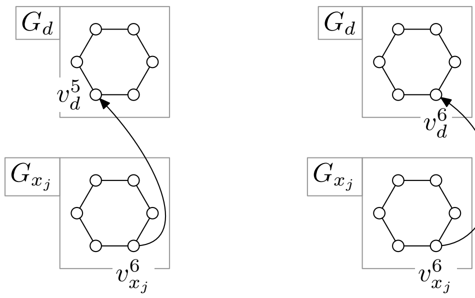


Figure 9: On the left: $G(d)$ for the boolean formula $d = x_j$; on the right: $G(d)$ for the boolean formula $d = \overline{x_j}$.

In the inductive step, construct $G(d)$ as follows. Take the union of $G_{d'}$ and $G_{d''}$, add one new gadget G_d and three new vertices w_d^1, w_d^2, w_d^3. Furthermore, add the arcs $(v_{d'}^2, w_d^1)$, $(v_{d'}^5, w_d^2)$, $(v_{d'}^5, v_d^4)$, $(w_d^3, v_{d''}^5)$, $(v_{d''}^5, v_d^4)$, (w_d^2, w_d^1), (w_d^2, v_d^1), and (w_d^3, w_d^2), see Figure 10.

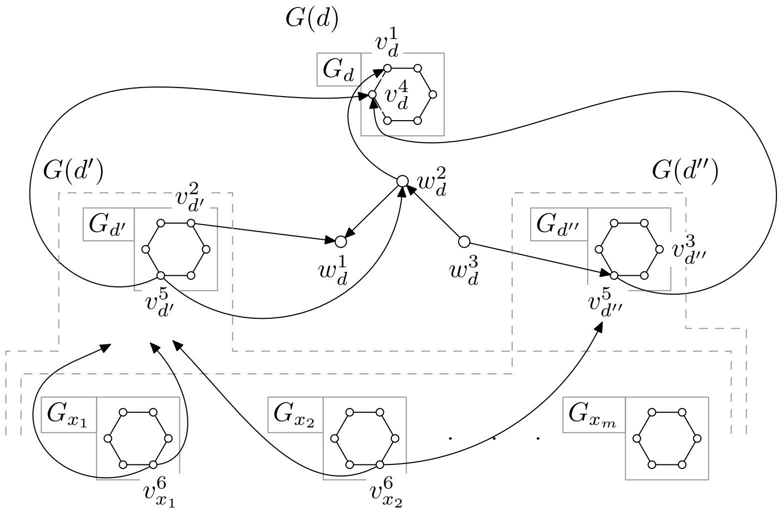


Figure 10: The construction of $G(d)$ for the boolean formula $d = d' \vee d''$ using the union of digraphs $G(d')$ and $G(d'')$.

We show that in any feasible coloring of $G(d)$, $\mathrm{Pat}(G_d) = P^1$ if and only if $\mathrm{Pat}(G_{d'}) = P^1$ or $\mathrm{Pat}(G_{d''}) = P^1$. Notice that the arcs $(v_{d'}^5, v_d^4)$ and $(v_{d''}^5, v_d^4)$ guarantee that if one of the gadgets $G_{d'}$ or $G_{d''}$ has color pattern P^1, gadget G_d must have color pattern P^1. Now, it is enough to show that there is a feasible coloring for the three $w-$vertices in each one of these three cases. Indeed we can color the $w-$vertices as follows: if $\mathrm{Pat}(G_{d'}) = P^1$ and $\mathrm{Pat}(G_{d''}) = P^0$, $\varphi(w^1) = A$, $\varphi(w^2) = B$, $\varphi(w^3) = D$; if $\mathrm{Pat}(G_{d'}) = P^0$ and $\mathrm{Pat}(G_{d''}) = P^1$, $\varphi(w^1) = B$, $\varphi(w^2) = D$, $\varphi(w^3) = C$; and if $\mathrm{Pat}(G_{d'}) = P^1$ and $\mathrm{Pat}(G_{d''}) = P^1$, $\varphi(w^1) = A$, $\varphi(w^2) = B$, $\varphi(w^3) = C$. It remains to consider the case when $\mathrm{Pat}(G_{d'}) = P^0$ and $\mathrm{Pat}(G_{d''}) = P^0$. We have $\varphi(v_{d'}^2) = A$ and $\varphi(v_{d'}^5) = \varphi(v_{d''}^5) = C$. The arc $(v_{d'}^2, w_d^1)$ forces $\varphi(w_d^1) = B$ and the arc $(w_d^3, v_{d''}^5)$ forces $\varphi(w_d^3) = D$. Furthermore, the arcs $(v_{d'}^2, w_d^1)$, $(w_d^3, v_{d''}^5)$ and $(v_{d''}^5, w_d^2)$ force $\varphi(w_d^2) = A$. Because of the arc (w_d^2, v_d^1), $\varphi(v_d^1) = B$. Hence, $\mathrm{Pat}(G_d) = P^0$. This verifies the construction of the digraph $G(d)$.

Finally, to extend $G(d)$ to $\widetilde{G}(d)$ forcing G_d to color pattern P^1, add the arc (v_d^1, z^2). Let us observe that the digraph $\widetilde{G}(f)$ is acyclic. Note that if a digraph contains a sink (only incoming arcs) or a source (only outgoing arcs), then it can be removed from the digraph without affecting its acyclicity. Therefore, it is enough to show that the digraph $\widetilde{G}(f)'$ obtained from $\widetilde{G}(f)$ by removing the vertices of Z and the vertices w_d^1 and w_d^3, for every subformula d used in the construction, is acyclic. That can be easily seen using a similar argument as in the proof of Lemma 4.7. □

Lemma 4.9.
Let $k \geq 1$ and let F_k be the quasi-acyclic semicomplete digraph obtained by forming the transitive closure of the digraph F_k' depicted in Figure 2. The acyclic F_k-coloring is NP-complete.

Proof. The reduction from SAT is similar to the reduction described in the proof of Lemma 4.8. In fact, all the gadgets and connections among them require just slight modifications. The digraph Z in the proof of Lemma 4.8 can be viewed as the transitive closure of the directed path (z^4, z^3, z^2, z^1). In this proof the digraph Z will be the transitive closure of the path $(z^4, z^3, q^k, \dots, q^1, z^2, z^1)$. Again, Z has a unique F^k-coloring up to swapping A and B, or C and D. In any coloring, $\varphi(q^i) = E_i$ and without loss of generality, let $\varphi(z^1) = A$, $\varphi(z^2) = B$, $\varphi(z^3) = C$ and $\varphi(z^4) = D$.

There are $k+4$ colors available to color a variable gadget G_a depicted in Figure 8 which would result in many different color patterns for the gadget. In order to restrict the number of color patterns of G_a to two, we replace the arcs connecting the vertices of the variable gadget with vertices of the digraph Z with the following ten arcs: (q^1, v_a^1), (v_a^2, z^2), (q^2, v_a^2), (v_a^3, q^{k-1}), (z^3, v_a^3), (v_a^4, z^1), (q^2, v_a^4), (v_a^5, q^{k-1}), (z^4, v_a^5), and (v_a^6, q^k), and replace the arcs (v_a^3, v_a^2) and (v_a^5, v_a^4) by two directed paths with $k-1$ internal nodes (all arcs on each path having the same direction as the original arc), see Figure 11(a).

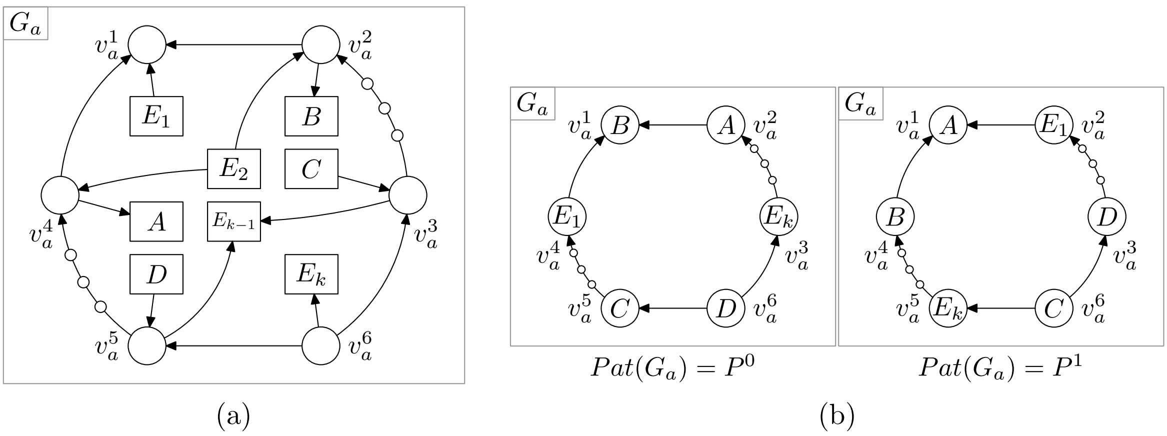


Figure 11: (a) A variable gadget used in the proof that F_k-coloring is NP-complete for $k > 1$. The squares labeled with A, B, C, D, E_1, E_2, E_{k-1} and E_k represent vertices z^1, z^2, z^3, z^4, q^1, q^2, q^{k-1} and q^k of the common digraph Z, respectively. The arrows with circles connecting v_a^3 with v_a^2 and v_a^5 with v_a^4 depict directed paths $(v_a^3, \dots, v_a^2)$ and $(v_a^5, \dots, v_a^4)$ with $k-1$ internal nodes.
(b) Two color patterns of the variable gadget: $\text{Pat}(G_a) = P^0$ (on the left) and $\text{Pat}(G_a) = P^1$ (on the right).

In the case $k = 1$, there are no vertices q^2 and q^{k-1}. Instead of the arcs (q^2, v_a^2), (q^2, v_a^4), (v_a^3, q^{k-1}), and (v_a^5, q^{k-1}), we add arcs (v_a^3, z^1), (v_a^5, z^1), (v_a^3, z^2), (v_a^5, z^2), (z^3, v_a^2), (z^3, v_a^4), (z^4, v_a^2), and (z^4, v_a^4). Note also that in this case the vertices q^1 and q^k are the same (the squares labeled E_1 and E_k in Figure 11(a)), and that the directed paths $(v_a^3, \ldots, v_a^2)$ and $(v_a^5, \ldots, v_a^4)$ contain only one arc each, as in the original variable gadget depicted in Figure 8.

It is easy to see that the arcs connecting vertices of Z to vertices of the gadget force exactly two color choices for each vertex v_a^i: A or B for v_a^1; A or E_1 for v_a^2; E_k or D for v_a^3; B or E_1 for v_a^4; E_k or C for v_a^5; and C or D for v_a^6. Now, assume that $\varphi(v_a^1) = A$. Then $\varphi(v_a^2) = E_1$. Since the path $(v_a^3, \ldots, v_a^2)$ contains $k-1$ internal nodes, $\varphi(v_a^3) \neq E_k$, i.e., $\varphi(v_a^3) = D$. Then $\varphi(v_a^6) = C$ and $\varphi(v_a^5) = E_k$. Similarly, v_a^4 cannot be colored with E_1, i.e., $\varphi(v_a^4) = B$. Note also that $\varphi(v_a^4) = B$ implies that $\varphi(v_a^1) = A$. This defines the color pattern P^1 of the variable gadget G_a. The only other possible color pattern is the complementary one. One can easily check that this complementary color patter P^0 is valid as well, see Figure 11(b).

In the inductive construction of $\widetilde{G}_d$, there are no differences in the base step in the case $d = \overline{x_j}$ (see right picture in Figure 9). In the case $d = x_j$, we add an arc $(v_{x_j}^6, v_d^3)$, see Figure 12; it is easy to see that both gadgets must be in the same state.

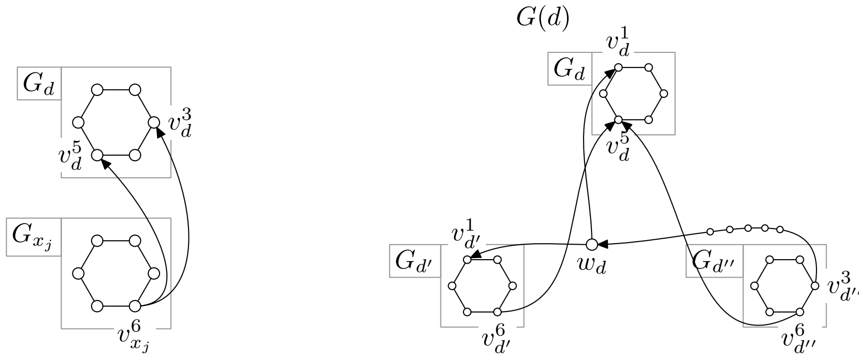


Figure 12: On the left: $G(d)$ for the boolean formula $d = x_j$; on the right: $G(d)$ for the boolean formula $d = d' \vee d''$. The path connecting $v_{d''}^3$ with v_d^5 contains $k-1$ internal nodes.

In the inductive step $d = d' \vee d''$, we construct $G(d)$ as follows. Take the union of $G_{d'}$ and $G_{d''}$, and add one new gadget G_d and one new vertex w_d. Furthermore, add the arcs $(v_{d'}^6, v_d^5)$, $(v_{d''}^6, v_d^5)$, (w_d, v_d^1), $(w_d, v_{d'}^1)$, and a directed path from $v_{d''}^3$ to w_d with $k-1$ internal nodes, see Figure 12. If $\text{Pat}(G_{d'}) = P^1$ (respectively $\text{Pat}(G_{d''}) = P^1$), then the arc $(v_{d'}^6, v_d^5)$ (respectively $(v_{d''}^6, v_d^5)$) forces $\text{Pat}(G_d) = P^1$. On the other hand, if $\text{Pat}(G_{d'}) = P^0$, then $\varphi(w_d)$ cannot be B; and if $\text{Pat}(G_{d''}) = P^0$, then $\varphi(v_{d''}^3) = E_k$ and hence, $\varphi(w_d)$ is either A or B. Thus, if both $\text{Pat}(G_{d'}) = \text{Pat}(G_{d''}) = P^0$, then $\varphi(w_d) = A$ and hence, $\varphi(v_d^1) = B$, i.e., $\text{Pat}(G_d) = P^0$. Note that only $\varphi(w_d) = A$ forces $\text{Pat}(G_d) = P^0$ and that w_d is not forced to have color A if either $\text{Pat}(G_{d'})$ or $\text{Pat}(G_{d''})$ is P^1. Thus, we can see that $\text{Pat}(G_d) = P^1$ if and only if $\text{Pat}(G_{d'}) = P^1$ or $\text{Pat}(G_{d''}) = P^1$. The rest of the proof is analogous to the proof of Lemma 4.8. □

The following lemma gives a simple characterization when a digraph contains at least two directed 2-cycles.

Lemma 4.10.
A quasi-acyclic semicomplete digraph H contains at least two directed 2-cycles, if and only if it contains either H_0, H_1 or F_0 (see Figure 2) as an induced subdigraph.

Proof. Sufficiency is trivial. For the necessity suppose that H does not contain H_0. Now, if there are two 2-cycles in H which share a vertex, then the three vertices on these two 2-cycles induce H_1. Otherwise, take any two 2-cycles. Since they are vertex-disjoint, the four vertices on them induce F_0. □

Therefore, to prove the NP-completeness part of Theorem 3.2, it is enough to show the following lemma.

Lemma 4.11.
Suppose H is a quasi-acyclic semicomplete digraph. If H contains either H_0, H_1 or F_0 (see Figure 2), as an induced subdigraph, then the acyclic H-coloring problem is NP-complete.

Proof. We will distinguish three cases:
Case 1. We first prove that when H contains H_0 as an induced subdigraph, the acyclic H-coloring problem is NP-complete by reduction from the proper 3-coloring problem. In particular, given a graph G, we will construct an acyclic digraph K which has an H-coloring if and only if G is 3-colorable.

Let A, B and C be the vertices of H_0. Find a directed Hamilton path P in H on which vertices A, B and C are consecutive (by using a topological sort on the poset corresponding to H). Let P_ℓ (respectively P_u) denote the subpath of P containing all vertices preceding (respectively following) the three vertices A, B and C on P. Note that P has the property that its transitive closure is H-colorable. This will be used later in the proof.

We construct an acyclic digraph K as follows. Start with subpaths P_ℓ and P_u and vertices in $V(G)$. Add an arc from the last vertex of P_ℓ to every vertex in $V(G)$ and from every vertex in $V(G)$ to the first vertex of P_u, see Figure 13.

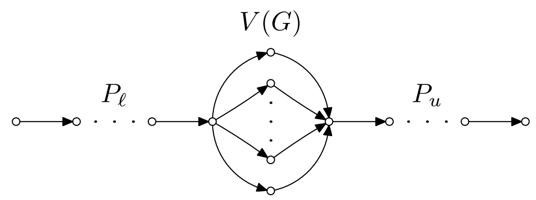


Figure 13: First step in the construction of the acyclic digraph $\vec{K}$.

Next take the transitive closure of this digraph. Observe that the resulting digraph has many H-colorings in which vertices on P_ℓ and P_u are mapped to corresponding vertices in H, and vertices in $V(G)$ are arbitrarily mapped to vertices A, B and C. Consider any acyclic orientation of G and add every arc of this orientation to the constructed digraph joining corresponding vertices in $V(G)$. The resulting acyclic digraph is K.

Firstly, suppose that G is 3-colorable with colors A, B and C. For an H-coloring of K choose the H-coloring from the previous paragraph that agrees on vertices in $V(G)$ with the 3-coloring.

Secondly, suppose that K has an H-coloring. Since $V_P = V(P_\ell) \cup V(P_u)$ induces a transitive tournament in K, any such H-coloring must use all but three vertices of H. Since every vertex of $V(G)$ is incident (is a tail or a head of an arc) with every vertex in V_P, in any H-coloring of K the vertices in $V(G)$ are mapped to the remaining three vertices of H, i.e., G is 3-colorable.

Case 2. Next, we prove that when H contains H_1 but not H_0 as an induced subdigraph, the H-coloring problem is NP-complete by reduction from the acyclic H_1-coloring problem (see Lemma 4.7). Let G be an acyclic digraph for which we want to decide whether it is H_1-colorable. We will construct an acyclic digraph K which is H-colorable if and only if G is H_1-colorable. The construction of K is exactly as in the previous case with the only difference that G already fixes an acyclic orientation used in the construction. However, it is not obvious that there exists a directed Hamilton path in H with three consecutive vertices B, C and A inducing H_1.

Consider the poset $(V(H), \preceq)$ corresponding to H. We say that a triple $[B, A, C]$ of different vertices in H is *nice* if $B \preceq A$, and the pairs A, C and B, C are both incomparable. Obviously, there is a nice triple since H_1 is an induced subdigraph of H. Take a directed Hamilton path P in H on which there is a nice triple $[B, A, C]$ such that the distance between B and A along P is the smallest possible. We will show that the distance between B and A along P is one. Suppose by contradiction that $E \neq B$ is the immediate predecessor of A on P. Note that for any two X, Y such that X precedes Y on P, H contains the arc (X, Y). (This follows since H is semicomplete.) Hence, if H contains both arcs (A, E) and (E, A) (a double arc (A, E)), then by exchanging A and E we obtain another directed Hamilton path in H with smaller distance of A and B, a contradiction. Thus, we may assume that H contains the arc (E, A) but not (A, E), i.e., $E \prec A$. Moreover, either $E \prec C$ or they are incomparable, since otherwise $C \prec A$.

Suppose that $B \prec E$. It follows that C and E are incomparable, and hence the triple $[B, E, C]$ is nice and the distance between B and E is smaller along P. Hence, B and E must be incomparable. Let F be an immediate successor of B on P. By a similar argument as above, it follows that $B \prec F$, and A and F are incomparable. Therefore, $E \neq F$ and they are incomparable. Now, $[B, F, E]$ is a nice triple with distance one between B and F on P, a contradiction.

We say that a nice triple $[B, A, C]$ on a directed Hamilton path is *very nice* if the distance between B and A along P is one. By the above argument, there is a directed Hamilton path with a very nice triple. Take such a path P and a very nice triple $[B, A, C]$ such that the sum of the distances between B and C and between A and C on P is the smallest possible. We will show that this sum of distances is three. Without loss of generality suppose that C follows A on P. Suppose by contradiction that $E \neq C$ is the immediate successor of A on P. If E is incomparable with both A and B then $[B, A, E]$ is a very nice triple with a smaller sum of distances. It follows that C and E must be incomparable (otherwise, by transitivity, at least one of A, C or B, C would be comparable). If A and E are incomparable as well then A, C, E induce H_0 in H, a contradiction. Hence, $[A, E, C]$ is a very nice triple with a smaller sum of distances on P, a contradiction.

Obviously, if G is H_1-colorable, we can use this coloring to construct an H-coloring of K as above. Conversely, suppose now that K has an H-coloring. As before, it follows that $V(G)$ is mapped to some three vertices of H. Since H_0 is not an induced subdigraph of H, the three vertices induce a subdigraph of H_1. Therefore, G is H_1-colorable.

Case 3. Finally, we prove that when H contains F_0 but neither H_0 nor H_1 as an induced subdigraph, the H-coloring problem is NP-complete by reduction from one of the following problems: the acyclic F_0-coloring problem (Lemma 4.8) or the acyclic F_k-coloring problem for some $k \geq 1$ (Lemma 4.9).

Take a directed Hamilton path P in H. Since H_0 and H_1 are not induced subdigraphs of H, it is easy to see that we can make sure that any two incomparable vertices in the corresponding poset $(V(H), \preceq)$ are consecutive on P. Since F_0 is an induced subdigraph of H there are at least two incomparable pairs. Take two such pairs D, C and B, A which are in this order on P and are closest to each other. Similarly, it is easy to see that for any vertex E in between C and B on P, $D \prec E$, $C \prec E$, $E \prec B$ and $E \prec A$. Therefore, vertices of the subpath P_m of P from D to A induce F_k where k is the number of vertices between C and D.

We will show that the H-coloring problem is NP-complete by reduction from the acyclic F_k-coloring problem. Let P_ℓ (respectively P_u) denote the subpath of P containing all vertices preceding D (respectively following A) on P. Given G, construct the acyclic digraph K similarly as in the previous cases. Obviously, if G is F_k-colorable, we can use this coloring to construct an H-coloring of K as in the previous cases. Conversely, suppose now that K has an H-coloring. We will show that vertices of $V(G)$ in K are colored with colors on P_m in H. By contradiction, suppose $x \in V(G)$ is colored by Y not on P_m. Without loss of generality let $Y \in P_\ell$. Since for every vertex z on P_ℓ, there is an arc (z, x) in K, the color of z must be either incomparable with Y or a predecessor of Y on P. Since there is at most one incomparable vertex with Y and it must lie on P_ℓ of H, we conclude that the color of z is on P_ℓ in H and different from Y. Since each vertex on P_ℓ of K must have a different color, there are not enough colors for them, a contradiction. Therefore, G is F_k-colorable. $\square$

Acknowledgment

We like to thank the anonymous referees for their careful reading and extremely helpful comments.

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