LOCALLY PANCYCLIC GRAPHS

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Abstract. We prove the following theorem. Let G be a graph of order n and let $W \subseteq V(G)$. If $|W| \geq 3$ and $d_G(x) + d_G(y) \geq n$ for every pair of non-adjacent vertices $x, y \in W$, then either G contains cycles $C^3, C^4, \ldots, C^{|W|}$ such that C^i contains exactly i vertices from W ($i = 3, 4, \ldots, |W|$), or $|W| = n$ and $G = K_{n/2,n/2}$, or else $|W| = 4$, $G[W] = K_{2,2}$, and the structure of G is depicted in Figure 1. This generalizes a result of Bondy [3] who proved the above for $|W| = n$, and also a recent result of Bollobás and Brightwell [2], ensuring the existence of $C^{|W|}$ only.

1   Introduction

The characterization of hamiltonian graphs is apparently a very hard problem, though various sufficient conditions are known (cf. [6] for a survey). Many such conditions were given in terms of vertex degrees; the following two theorems are probably the best known representatives.

Theorem A. (Ore, [10]) *Let G be a graph of order $n \geq 3$. If $d(x) + d(y) \geq n$ for every pair of non-adjacent vertices $x, y \in V(G)$, then G is hamiltonian.*

Theorem B. (Chvátal, [9]) *Let G be a graph of order $n \geq 3$ with vertex-degree sequence $d_1 \leq d_2 \leq \ldots \leq d_n$. If $d_k \leq k < n/2$ implies $d_{n-k} \geq n - k$, then G is hamiltonian.*

Going a step further towards the cycle structure, a graph of order n is said to be *pancyclic* if it contains cycles of every length k, $3 \leq k \leq n$. Even though pancyclicity is a much stronger requirement on graphs than hamiltonicity, Bondy in [4] proposed an interesting metaconjecture according to which almost any non-trivial hamiltonicity condition also implies pancyclicity (there may be a simple family of exceptional graphs).

Bondy's conjecture is still in question for many hamiltonicity conditions and there are even some conditions for which it is false. However, for several interesting hamiltonicity results Bondy's conjecture turned out to be true. For example, Theorems A and B have the following pancyclicity extensions.

Theorem A°. (Bondy, [3]) *Let G be a graph of order $n \geq 3$. If $d(x) + d(y) \geq n$ for every pair of non-adjacent vertices $x, y \in V(G)$, then G is either pancyclic or $G = K_{n/2,n/2}$.*

Theorem B°. (Bauer-Schmeichel, Schmeichel-Hakimi, [1,11]) *Let G be a graph of order $n \geq 3$ with vertex-degree sequence $d_1 \leq d_2 \leq \ldots \leq d_n$. If $d_k \leq k < n/2$ implies $d_{n-k} \geq n - k$, then G is either pancyclic or bipartite.*

Hamiltonicity conditions were generalized in several other directions, for example, to sufficient conditions guaranteeing the existence of cycles through prescribed vertices. The following two theorems are examples of such results; they are again generalizations of Theorems A and B, respectively.

Theorem A•. (Bollobás-Brightwell, [2]) *Let G be a graph of order n and let $W \subseteq V(G)$. If $|W| \geq 3$ and $d_G(x) + d_G(y) \geq n$ for every pair of non-adjacent vertices $x, y \in W$, then G has a cycle containing all the vertices of W.*

Theorem B•. (Stacho, [14]) *Let G be a graph of order n with vertices ordered according to their degrees $d(x_1) \leq d(x_2) \leq \ldots \leq d(x_n)$. Let $W = \{x_{n-w+1}, x_{n-w+2}, \ldots, x_n\}$ such that $3 \leq w \leq n$ and $d(x_{n-w+1}) > n - w$. If for any $n - w < k < n/2$ $d(x_k) \leq k$ implies $d(x_{n-k}) \geq n - k$, then G has a cycle containing all the vertices of W.*

In other words, Bondy's metaconjecture suggests that almost any hamiltonicity condition guarantees besides the required hamiltonian cycle also many other cycles (it is much stronger than is necessary).

It is natural to ask whether conditions for cycles through prescribed vertices are not unnecessarily strong like hamiltonicity conditions. Following this line of thought, Bondy's metaconjecture was extended in [14] as follows.

Let G be a graph and let $W \subseteq V(G)$, $|W| \geq 3$. The graph G is called *W-locally-pancyclic* if G contains cycles C^i for every i, $3 \leq i \leq |W|$, such that C^i contains exactly i vertices from W. The new metaconjecture we propose is:

Let G be a graph and let $W \subseteq V(G)$, where $|W| \geq 3$. Almost any non-trivial condition which implies the existence of a cycle through all vertices from W also implies that the graph G is W-locally-pancyclic (there may be a simple family of exceptional graphs).

The aim of this paper is to show that the new metaconjecture is true for Theorem A•. Our main result is the following theorem, the proof of which occupies the rest of the paper.

Theorem 1. *Let G be a graph of order n and let $W \subseteq V(G)$. If $|W| \geq 3$ and $d_G(x) + d_G(y) \geq n$ for every pair of non-adjacent vertices $x, y \in W$, then either G is W-locally-pancyclic, or $|W| = n$ and $G = K_{n/2,n/2}$, or else $|W| = 4$, $G[W] = K_{2,2}$, and the structure of G is depicted in Figure 1.*

Figure 1. The filled vertices form the set W

Theorem 1 shows that the assumptions of Theorem A• are sufficient not only for the existence of a cycle through all vertices of W but also for the existence of cycles containing a specified number of vertices of W. Obviously, Theorem 1 is a common generalization of Theorems A° and A•. Indeed, Theorem A° is a special case of Theorem 1 (when $|W| = n$) and Theorem A• guarantees the existence of a cycle $C^{|W|}$ only. We conclude by remarking that it would be very interesting to try extending Theorem B• in the sense of the new metaconjecture.

2 Definitions and Notation

Throughout we consider only finite undirected graphs without loops or multiple edges. Our terminology and notation will be standard except as indicated. All undefined terms can be found in [8]. If G is a graph of order n, then by $d_G(x)$ we denote the degree of the vertex x in G and by $N_G(x)$ the neighborhood of x. We shall omit the subscript "G" if no confusion can arise. If $W \subseteq V(G)$, then by $G[W]$ we denote the subgraph of G induced by W. Similarly, by $N_W(x)$ we denote the set $N(x) \cap W$ and by $d_W(x)$ we denote $|N_W(x)|$.

Let $K \subseteq V(G)$ and let $L \subseteq (V(G) \setminus K) \times (V(G) \setminus K)$. Then by $G - K + L$ we denote the graph obtained from G by deleting all vertices in K and adding edges between all pairs of vertices in L.

3 Auxiliary Results

3.1 Hamiltonicity Results

Lemma 1. [13] *Let G be a connected non-hamiltonian graph with a longest path $P = (1, 2, \ldots, m)$. If there exist two vertices $i, j \in V(P)$ such that $(1, j), (m, i) \in E(G)$ and $i < j$, then a vertex $k \in V(P)$, $i < k < j$, must exist such that $(1, k), (m, k) \notin E(G)$.*

Lemma 2. *Let G be a graph with a hamiltonian cycle $(1, 2, \ldots, n, 1)$, where n is even, $n \geq 6$. Let $\{k : k \text{ is even}\} \subseteq N(1)$ and $\{k : k \text{ is odd}\} \subseteq N(2), N(n)$. Then for each $r \in \{2, 3, \ldots, n - 1\}$ G contains a hamiltonian cycle missing the edge $(r, r + 1)$.*

Proof. The statement of lemma can be easily verified for $n = 6$. Hence we may assume that $n \geq 8$. If $r = 2$, then $(1, 2, 5, 4, 3, n, n - 1, \ldots, 6, 1)$ is the required cycle. If $r = n - 1$, then $(1, n, 3, 4, \ldots, n - 1, 2, 1)$ is the required cycle. Consequently $3 \leq r \leq n - 2$. Now if r is even (odd), then $(1, r, r - 1, \ldots, 2, r + 1, r + 2, \ldots, 1)$ $((1, r + 1, r + 2, \ldots, n, r, r - 1, \ldots, 1))$ is again the required cycle.

Lemma 3. *Let G be a graph with a hamiltonian path $(1, 2, \ldots, n)$. Let $S = \{i \in V(G) : (1, i + 1), (n, i) \notin E(G), i \neq n\}$. If $d(1) + d(n) \geq n - |S|$, then G is hamiltonian.*

Proof. Assume G is not hamiltonian. Let $R = \{i \in V(G) : (1, i+1) \in E(G)\}$. Obviously, $S \cap R = \emptyset$ and $N(n) \cap (S \cup R) = \emptyset$. Moreover, $n \notin S \cup R$. Hence

$$d(n) \leq n - |\{n\}| - |S| - |R| \leq n - 1 - d(1) - |S|.$$

3.2 Pancyclicity Results

Lemma 4. [7] *Let G be a graph of order n. Suppose G contains an $(n - 1)$-cycle C_{n-1}. If $d(x) \geq n/2$ for $x \notin V(C_{n-1})$, then G is pancyclic.*

Lemma 5. [15] *Let G be a graph of order n containing a hamiltonian cycle C. If x, y, and z are three consecutive vertices on C with $d(x) + d(z) > n$, then G is pancyclic.*

Lemma 6. [12] *Let G be a graph of order $n \geq 3$ with a hamiltonian cycle $(1, 2, \ldots n, 1)$.*

(i) *If $d(1) + d(n) \geq n$, then G is either pancyclic, bipartite, or missing only an $(n-1)$-cycle.*

(ii) *If $d(1) + d(n) > n$, then G is pancyclic.*

Lemma 7. *Let G be a graph with a hamiltonian cycle $C = (1, 2, \ldots, n, 1)$, $n \geq 4$. Let $d(1) \geq n/2$ and let $N(n) \cup N(1) \cup N(2) = V(G)$. Then G is either pancyclic, missing only an $(n-1)$-cycle, or $(n, 4), (2, n-2) \notin E(G)$ and there are x and y, with $x \leq y$ such that*

(i) *if $(1, 4), (1, n-2) \in E(G)$, then $x \geq 4$, $y \leq n-2$, and*
   $$N(1) = \{k, l, n-m \ : \ 2 \leq k \leq x-1, \ k \text{ even}; \ x \leq l \leq y; \ 0 \leq m \leq n-y-1, \ m \text{ even}\},$$

(ii) *if $(1, 4) \notin E(G)$ and $(1, n-2) \in E(G)$, then $x \geq 5$, $y \leq n-2$, and*
   $$N(1) = \{2, k, l, n-m \ : \ 5 \leq k \leq x-1, \ k \text{ odd}, \ x \leq l \leq y, \ 0 \leq m \leq n-y-1, \ m \text{ even}\},$$

(iii) *if $(1, 4) \in E(G)$ and $(1, n-2) \notin E(G)$, then $x \geq 4$, $y \leq n-3$, and*
   $$N(1) = \{k, l, n-m, n \ : \ 2 \leq k \leq x-1, \ k \text{ even}, \ x \leq l \leq y, \ 3 \leq m \leq n-y-1, \ m \text{ odd}\},$$

(iv) *if $(1, 4), (1, n-2) \notin E(G)$, then $x \geq 5$, $y \leq n-3$, and*
   $$N(1) = \{2, k, l, n-m, n \ : \ 5 \leq k \leq x-1, \ k \text{ odd}, \ x \leq l \leq y, \ 3 \leq m \leq n-y-1, \ m \text{ odd}\}.$$

The structure of G can be seen in Figure 2 (i)–(iv).

Figure 2.

Proof. Assume G is neither pancyclic nor missing only an $(n-1)$-cycle and fulfils the assumptions of the lemma.

Fact 1. *The edges $(1, 3), (1, n-1)$ do not belong to $E(G)$.*

First assume that $(1, 3) \in E(G)$. Then $(1, 2, 3, 1)$ is a 3-cycle and $(1, 3, 4, \ldots, n, 1)$ is an $(n-1)$-cycle. It follows that $n \geq 6$ and there must be a t, $4 \leq t \leq n-2$, such that there is no t-cycle in G. Then the edges $(1, t), (2, t)$, and (n, t) cannot be in $E(G)$ because of paths $(1, 2, \ldots, t), (2, 1, 3, 4, \ldots, t)$, and $(n, 1, 3, 4, \ldots, t)$, respectively. But this contradicts the assumptions of the lemma. The case $(1, n-1) \in E(G)$ is analogous.

Using Fact 1, it is an easy but time consuming exercise to verify the lemma for $n \leq 9$. Consequently we may assume that $n \geq 10$. In what follows we prove (i). Note that G is neither pancyclic nor missing only an $(n-1)$-cycle.

Fact 2. *The edges $(n, 4), (2, n-2)$ do not belong to $E(G)$.*

First assume that $(n, 4) \in E(G)$. Then since $n \geq 10$, $(n, 4, 5, \ldots, n)$ is an $(n-3)$-cycle, say C'. By Fact 1, $d_{C'}(1) \geq (n-2)/2$ and thus, by Lemma 4, the graph $G[V(C') \cup \{1\}]$ is pancyclic. Now G is either pancyclic or missing only an $(n-1)$-cycle, a contradiction. The case $(2, n-2) \in E(G)$ is analogous.

Fact 3. *If for some l, $4 \leq l \leq n-2$, $(1, l), (1, l+1) \in E(G)$, then $(2, l+2), (n, l-1) \notin E(G)$.*

Let $(2, l+2) \in E(G)$. There must be a t, $3 \leq t \leq n-2$, such that G contains no t-cycle. First assume that $t \leq l+1$. Then the edges $(1, l-t+3), (2, l-t+3)$, and $(n, l-t+3)$ cannot be in $E(G)$ because of paths $(1, l+1, l, \ldots, l-t+3), (2, 1, l, l-1, \ldots, l-t+3)$, and $(n, 1, l, l-1, \ldots, l-t+3)$, respectively, a contradiction with the assumptions of the lemma. Now suppose that $t > l+1$. Then the edges $(1, t), (2, t)$, and (n, t) cannot be in $E(G)$ because of paths $(1, 2, \ldots, t), (2, 3, \ldots, l, 1, l+1, l+2, \ldots, t)$, and $(n, 1, l, l-1, \ldots, 2, l+2, l+3, \ldots, t)$, respectively. But this again contradicts the assumptions of the lemma. The case $(n, l-1) \in E(G)$ is analogous.

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Fact 4. *The edges* $(1,5), (1, n-3)$ *do not belong to* $E(G)$.

Assume $(1,5) \in E(G)$. Then $(1,4,5,1)$ is a 3-cycle, $(1,2,3,4,1)$ is a 4-cycle, $(1,2,3,4,5,1)$ is a 5-cycle, and $(1,2,3,\ldots,n-2,1)$ is an $(n-2)$-cycle. Since $n \geq 10$, G cannot contain any t-cycle for some t, where $6 \leq t \leq n-3$. Thus the edges $(1,t), (2,t)$, and (n,t) cannot be in $E(G)$ because of paths $(1,2,\ldots,t), (2,3,4,1,5,6,\ldots,t)$, and $(n,n-1,n-2,1,5,6,\ldots,t)$, respectively. But this is a contradiction with the assumptions of the lemma. The case $(1,n-3) \in E(G)$ can be handled similarly.

Fact 5. *If* $(2,4) \in E(G)$, *then* $(2,5) \notin E(G)$ *and similarly if* $(n,n-2) \in E(G)$, *then* $(n,n-3) \notin E(G)$.

Suppose $(2,4), (2,5) \in E(G)$. Then $(2,3,4,2)$ is a 3-cycle. Assume for some t there is no t-cycle, where $4 \leq t \leq n-2$. Then $(1,t+1), (2,t+1), (n,t+1) \notin E(G)$ because of paths $(1,2,4,5,\ldots,t+1), (2,3,\ldots,t+1)$, and $(n,1,2,5,6,\ldots,t+1)$, respectively. But this contradicts the assumptions of the lemma. The case $(n,n-2), (n,n-3) \in E(G)$ is analogous.

Fact 6. *For any* l, $3 \leq l \leq n$, *at most one of edges* $(2,l)$ *and* $(n,l-1)$ *belongs to* $E(G)$.

Since $(2,3,\ldots,n)$ is a hamiltonian path in $G - \{1\}$ and since $d(1) \geq n/2$, it follows from Lemma 4 that $(2,n) \notin E(G)$. Hence we may assume that $4 \leq l \leq n-1$. But now $(2,l,l+1,\ldots,n,l-1,l-2,\ldots,2)$ is an $(n-1)$-cycle missing the vertex 1, and so G is pancyclic by Lemma 4, a contradiction.

Obviously, if there is no vertex $c \in V(C)$ such that $(1,c), (1,c+1) \in E(G)$, then $N(1)$ is as in (i) with $x = y$. Thus we assume there is such a vertex. Now there are vertices $x, y \in V(C)$ such that $x < y$ and for all $x \leq t \leq y$ we have $(1,t) \in E(G)$ and $(1,x-1), (1,y+1) \notin E(G)$. By Fact 4, $x > 5$ and $y < n-3$. Moreover, we may assume that

$$n - y + 1 \leq x - 1. \tag{1}$$

Using Fact 3, the edges $(2,x-1), (n,y+1)$ belong to $E(G)$. Let $p \in V(C)$ be the vertex with the property that for $p \leq t < x$ the edge $(2,t) \in E(G)$ and $(2,p-1) \notin E(G)$ (note that, possibly, $p = x-1$). Using Fact 5, $5 \leq p \leq x-1$. According to Fact 6, we have $(1,p-1) \in E(G)$. Similarly, let $q \in V(C)$ be the vertex for which $(n,q+1) \notin E(G)$ and $(n,t) \in E(G)$ for any t, $y < t \leq q$. Using Fact 5, $y+1 \leq q \leq n-3$. By Fact 6, $(1,q+1) \in E(G)$. The structure of G is depicted in Figure 3.

Figure 3.

Obviously, $(1,x,x+1,1), (1,2,3,4,1), (2,x-1,x,x+1,1,2)$, and $(1,2,\ldots,n-2,1)$ are $3, 4, 5$, and $(n-2)$-cycles, respectively.

Fact 7. *For* $t = 6,7,\ldots,p-1$ *there is a* t-cycle in G.

We have $p \leq x-1$. First assume that $p = x-1$. Consider the vertex $p-t+4$. At least one of the vertices $n, 1$, or 2 must be adjacent to $p-t+4$. The following three paths guarantee the existence of a t-cycle: $(n,1,x,p,p-1,\ldots,p-t+4), (1,x+1,x,p,p-1,\ldots,p-t+4)$, and $(2,1,x,p,p-1,\ldots,p-t+4)$.

Now, let $p < x-1$. Because $(1,x,x-1,2,3,4,1)$ is a 6-cycle, we may suppose that $t > 6$. Consider the vertex $p-t+6$. At least one of the vertices $n, 1$, or 2 must be

adjacent to $p - t + 6$. The following three paths guarantee the existence of a t-cycle: $(n, 1, x, x - 1, 2, p, p - 1, \ldots, p - t + 6)$, $(1, x + 1, x, x - 1, 2, p, p - 1, \ldots, p - t + 6)$, and $(2, x - 2, x - 1, x, x + 1, 1, p - 1, p - 2, \ldots, p - t + 6)$.

Fact 8. *For $t = p, p + 1, \ldots, y$ there is a t-cycle in G.*

Because for $t = x, x + 1, \ldots, y$ we have $(1, t) \in E(G)$, there is a t-cycle, where $t \in \{x, x + 1, \ldots, y\}$. Let $t \in \{p, p + 1, \ldots, x - 1\}$. If $p \le t < x - 1$, then $(2, 3, \ldots, t + 1, 2)$ is a t-cycle. If $t = x - 1$, then the cycle $(1, 4, 5, \ldots, x + 1, 1)$ is the required one.

Fact 9. *If for some t, $y < t < n - 2$, there is no t-cycle in G, then $(1, t), (2, t) \notin E(G)$.*

Obviously, $(1, t) \notin E(G)$. If $(2, t) \in E(G)$, then $(2, 3, \ldots, x, 1, x + 1, x + 2, \ldots, t, 2)$ is a t-cycle, a contradiction.

Fact 10. *If for some t, $y < t < n - 2$, there is no t-cycle in G, then $(n, k) \notin E(G)$, where $k = 4, 5, \ldots, y - 2$.*

Suppose $(n, k) \in E(G)$, where $k \in \{4, 5, \ldots, y - 2\}$; take k as large as possible. If $k = y - 2$ or $y - 3$, then $(n, k, k - 1, \ldots, 1, k + 2, k + 3, \ldots, t)$ is a path of length $t - 1$ joining vertices n and t, hence $(n, t) \notin E(G)$. Using Fact 9, $(1, t), (2, t) \notin E(G)$, a contradiction. Consequently $k \le y - 4$. Now, by the maximality of k, either $(2, k + 2) \in E(G)$ or $(1, k + 2) \in E(G)$. Thus if $(1, k + 2) \in E(G)$ $((2, k + 2) \in E(G))$, then $(n, k, k - 1, \ldots, 1, k + 2, k + 3, \ldots, t)$ $((n, k, k - 1, \ldots, 2, k + 2, k + 3, \ldots, y - 1, 1, y, y + 1, \ldots, t))$ is a path of length $t - 1$ joining vertices n and t, again a contradiction.

Fact 11. *If for some t, $y < t < n - 2$, there is no t-cycle in G, then for each $k = 5, 6, \ldots, x - 2$ it holds: if $(2, k) \in E(G)$, then $(2, k + 1) \notin E(G)$.*

Assume $(2, k), (2, k + 1) \in E(G)$ for some $5 \le k \le x - 2$; take k as small as possible. By Fact 10, we have $(1, k - 1) \in E(G)$. But now $(n, 1, k - 1, k - 2, \ldots, 2, k + 1, k + 2, \ldots, t)$ is a path of length $t - 1$ joining vertices n and t. Hence the edge (n, t) is not in $E(G)$. According to Fact 9, $(1, t), (2, t) \notin E(G)$, a contradiction.

Fact 12. *If for some t, $y < t < n - 2$, there is no t-cycle in G, then for $k = 5, 6, \ldots, x - 2$ it holds: if $(1, k) \in E(G)$, then $(1, k - 1) \notin E(G)$.*

Suppose by way of contradiction that $(1, k), (1, k - 1) \in E(G)$ for some $5 \le k \le x - 2$; take k as large as possible. Using Fact 10, we have $(2, k + 1) \in E(G)$. But now the vertices n and t are joined by the path $(n, 1, k - 1, k - 2, \ldots, 2, k + 1, k + 2, \ldots, t)$ of length $t - 1$. Thus $(n, t) \notin E(G)$. By Fact 9, $(1, t), (2, t) \notin E(G)$, a contradiction.

By Facts 7 and 8, there must exist a t, $y < t < n - 2$, such that G contains no t-cycle. Consequently using previous Facts, we conclude that for $k = 2, 3, \ldots, x - 1$, k odd (even), $(1, k) \notin E(G)$ $((1, k) \in E(G))$. Note that it follows that x is even and $p = x - 1$.

Fact 13. *If for some t, $y < t < n - 2$, there is no t-cycle in G, then for $k = y + 1, y + 2, \ldots, n - 3$, $(2, k) \notin E(G)$.*

Suppose that $(2, k) \in E(G)$. According to Fact 9, $k \ne t$ and $(n, t) \in E(G)$. First assume that $k > t$. Consider the vertex $x - k + t - 1$. Since $k - t \ge 1$, it follows that $x - k + t - 1 \le x - 2$. We have $t > y$ and $k \le n - 3$, consequently $k - t \le n - y - 4 \le x - 6$, and so $x - k + t - 1 \ge 5$. Now, if $x - k + t - 1$ is even (odd), then $(1, x - k + t - 1, x - k + t - 2, \ldots, 2, k, k - 1, \ldots, x, 1)$

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$((1, x - k + t, x - k + t - 1, \ldots, 2, k, k - 1, \ldots, x + 1, 1))$ is a t-cycle, a contradiction.

Now, let $k < t$ and let k be as small as possible. By Fact 3, $k > y + 1$. If $k = y + 2$, then the cycle $(n, 1, y, y - 1, \ldots, 2, y + 2, y + 3, \ldots, t, n)$ is a t-cycle, a contradiction. Consequently $k > y + 2$. Now, by the minimality of k, either $(1, k - 2) \in E(G)$ or $(n, k - 2) \in E(G)$. It follows that either $(n, 1, k - 2, k - 3, \ldots, 2, k, k + 1, \ldots, t, n)$ or $(n, k - 2, k - 3, \ldots, y, 1, y - 1, y - 2, \ldots, 2, k, k + 1, \ldots, t, n)$ is a t-cycle, a contradiction.

Fact 14. *If for some t, $y < t < n - 2$, there is no t-cycle in G, then for $k = y + 1, y + 2, \ldots, n - 3$ the following holds: if $(1, k) \in E(G)$, then $(1, k + 1) \notin E(G)$.*

Assume that for some k both $(1, k), (1, k + 1) \in E(G)$. Using Fact 9, $k \neq t$ and $(n, t) \in E(G)$. First suppose that $k > t$. Let $k = n - l$. By (1), it holds that $6 \le l + 3 \le n - t + 2 < n - y + 2 \le x$. Thus either $(1, l + 3) \in E(G)$ or $(2, l + 3) \in E(G)$. But then either $(n, n - 1, \ldots, k, 1, l + 3, l + 4, \ldots, t, n)$ or $(n, n - 1, \ldots, k + 1, 1, 2, l + 3, l + 4, \ldots, t, n)$ is a t-cycle, a contradiction.

Now assume that $k < t$ and k is as small as possible. By assumptions, $k > y + 1$. Hence according to Fact 3, we have $(2, k - 1) \in E(G)$. But this contradicts Fact 13.

Fact 15. *If for some t, $y < t < n - 2$, there is no t-cycle in G, then for $k = y + 1, y + 2, \ldots, n - 3$ it holds that: if $(n, k) \in E(G)$, then $(n, k + 1) \notin E(G)$.*

Obviously, the edge $(1, n - t + 2) \notin E(G)$. Because $6 \le n - t + 2 < x$, we have $(2, n - t + 2) \in E(G)$. But now $(2, 1, n - 2, n - 3, \ldots, k + 1, n, k, k - 1, \ldots, n - t + 2, 2)$ is a t-cycle, a contradiction.

It follows from Facts 7 and 8 that there must exist a t, $y < t < n - 2$, such that G contains no t-cycle. Hence by previous Facts, we conclude that for $k = 0, 1, \ldots, n - y - 1$, k odd (even), $(1, n - k) \notin E(G)$ $((1, n - k) \in E(G))$. This proves (i). In what follows we prove (ii), (iii), and (iv). We need to observe the folowing fact.

Fact 16. *If $(1, 4) \notin E(G)$, then $(n, 5) \notin E(G)$; similarly, if $(1, n - 2) \notin E(G)$, then $(2, n - 3) \notin E(G)$.*

Suppose $(1, 4) \notin E(G)$ and $(n, 5) \in E(G)$. Then since $n \ge 10$, $(n, 5, 6, \ldots, n)$ is an $(n - 4)$-cycle, say C'. According to Fact 1, $d_{C'}(1) \ge (n - 2)/2$ and thus, by Lemma 4, the graph $G[V(C') \cup \{1\}]$ is pancyclic. Since $(1, 4) \notin E(G)$, it follows from Fact 2 that $(2, 4) \in E(G)$. Similarly, by Fact 2, either $(1, n - 2) \in E(G)$ or $(n, n - 2) \in E(G)$. Consequently either $(1, n - 2, n - 3, \ldots, 1)$ or $(1, 2, 4, 5, \ldots, n - 2, n, 1)$ is an $(n - 2)$-cycle. Because G contains an $(n - 1)$-cycle, it is pancyclic, a contradiction. The second case can be proved similarly.

To prove (ii), assume that $(1, n - 2) \in E(G)$ and $(1, 4) \notin E(G)$. It follows from Fact 2 that $(2, 4) \in E(G)$. Now according to Facts 5 and 16, $(1, 5)$ belongs to $E(G)$. Let $G' = G - \{3\}$. Denote the order of G' by n'; obviously, $n' = n - 1$. The graph G' fulfils the assumptions of Lemma 7. If G' was pancyclic or missing only an $(n' - 1)$-cycle, then since $(1, 2, \ldots, n - 2, 1)$ is an $(n - 2)$-cycle, G would be pancyclic, a contradiction. Since $(1, n - 2), (1, 5) \in E(G)$, $N_{G'}(1)$ has to be as in (i). Now, one can observe that $N_G(1)$ is as in (ii). The cases (iii) and (iv) can be proved similarly to the case (i). This completes the proof of Lemma 7.

3.3 Local-Pancyclicity Results

Let G be a graph. Let $W \subseteq V(G)$, let $S = V(G) \setminus W$, and let $x, y \in W$. We say that the vertices x and y are S-*adjacent* if there is an $x - y$ path with all internal vertices in S. Note that any adjacent vertices are S-adjacent as well. Moreover, we say that a graph H is a $(W; x, y)$-*restriction of* G if x and y are S-adjacent and H is obtained from G by deleting all internal vertices of P and adding the edge (x, y) (iff $(x, y) \notin E(G)$), where P is an $x - y$ path with all internal vertices in S. A graph G' obtained from G by successive $(W; u, v)$-restrictions, where $u, v \in W$, is called a W-*restriction of* G. If, moreover, any S-adjacent vertices are also adjacent in G', we refer to G' as a W-*shrinking of* G. Note that W is a subset of the vertex set of any W-restriction of G. The following two observations are obvious.

Observation 1. *Let G be a graph and let $W \subseteq V(G)$. Let G' be a W-restriction of G. If G' is W-locally-pancyclic, then G is W-locally-pancyclic as well.*

Observation 2. *Let C be a cycle and let $W \subseteq V(C)$. Then the W-shrinking of C is a $|W|$-cycle $C_{|W|}$ with the vertex set W.*

Let G be a graph and let $W \subseteq V(G)$, $|W| \geq 3$. We say that a cycle is a $\{{}^W_t\}$-*cycle* if it contains exactly t vertices from W, where $3 \leq t \leq |W|$.

Lemma 8. *Let G be a graph with a hamiltonian cycle $(1, 2, \ldots, n, 1)$, $n \geq 4$. Let $d(1) \geq n/2$ and let $N(n) \cup N(1) \cup N(2) = V(G)$. Let $r, s \in \{2, 3, \ldots, n - 1\}$, with $r \neq s$. Let H be the graph obtained from G by deleting the edge $(\epsilon, \epsilon + 1)$ and adding a new vertex ϵ' and all the edges $(\epsilon, \epsilon'), (\epsilon', \epsilon + 1), (2, \epsilon')$, and (n, ϵ'), where $\epsilon = r, s$. Then the graph H is either $V(G)$-locally-pancyclic or missing only a $\{{}^{V(G)}_{n-1}\}$-cycle.*

Proof. Assume not. If G is pancyclic or missing only an $(n - 1)$-cycle, then since G is a $V(G)$-shrinking of H, by Observation 1, H will be either $V(G)$-locally-pancyclic or missing only a $\{{}^{V(G)}_{n-1}\}$-cycle. Thus it follows from Lemma 7 that there are x and y with $x \leq y$ such that

(a) if $(1, 4), (1, n - 2) \in E(G)$, then $N_G(1)$ is as in (i) of Lemma 7,

(b) if $(1, 4) \notin E(G)$ and $(1, n - 2) \in E(G)$, then $N_G(1)$ is as in (ii) of Lemma 7,

(c) if $(1, 4) \in E(G)$ and $(1, n - 2) \notin E(G)$, then $N_G(1)$ is as in (iii) of Lemma 7,

(d) if $(1, 4), (1, n - 2) \notin E(G)$, then $N_G(1)$ is as in (iv) of Lemma 7.

We may assume the notation is chosen in such way that $n - y + 1 \leq x - 1$. Let G' be the graph obtained from G by deleting the edge $(r, r + 1)$ and adding a new vertex g and all the edges $(r, g), (g, r + 1), (2, g)$, and (n, g). Obviously, G' is a $(V(G); s, s + 1)$-restriction of H. Consequently for any $\{{}^{V(G)}_t\}$-cycle in G' there exists a $\{{}^{V(G)}_t\}$-cycle in H. This is the reason why we will describe all the required $\{{}^{V(G)}_t\}$-cycles in G' instead of H.

We begin with part (a). Let r or s be in $\{2, n - 1\}$, say $r = n - 1$ (the other cases are analogous). For $t = 3, 4, \ldots, n - 1$ if $(1, t) \in E(G)$ $((1, t) \notin E(G))$ then $(1, 2, \ldots, t, 1)$ $((1, t - 1, t - 2, \ldots, 2, g, n, 1))$ is a $\{{}^{V(G)}_t\}$-cycle. We conclude that H is $V(G)$-locally-pancyclic.

Let r or s be in $\{3, n - 2\}$, say $r = 3$ (the remaining cases are analogous). Consider the cycle $C^{n-3} = (n, g, 4, 5, \ldots, n)$ and its $(V(G) \setminus \{1, 2, 3\})$-shrinking, say C_{n-3}. Since

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$d_{C_{n-3}}(1) \geq (n-2)/2$, by Lemma 4, the graph $F = G' - \{g, 2, 3\} + \{(n, 4)\}$ is pancyclic. Because F is a $(V(G) \setminus \{2, 3\})$-shrinking of $G' - \{2, 3\}$, the graph H is either $V(G)$-locally-pancyclic or missing only a $\{\begin{smallmatrix} V(G) \\ n-1 \end{smallmatrix}\}$-cycle. Thus we may assume that $r, s \in \{4, 5, \ldots, n-3\}$.

First consider the case when n is even. We have $(1, r) \in E(G')$ (for otherwise it is enough to relabel vertices). For $t = 3, 4, \ldots, r$ if $(1, t) \in E(G')$ $((1, t) \notin E(G'))$, then $(1, t, t-1, \ldots, 1)$ $((1, t-1, t-2, \ldots, 2, g, n, 1))$ is a $\{\begin{smallmatrix} V(G) \\ t \end{smallmatrix}\}$-cycle. For $t = r+1, r+2, \ldots, n-1$ if $(1, t) \in E(G')$ $((1, t) \notin E(G'))$, then $(1, t, t-1, \ldots, 1)$ $((1, r, r-1, \ldots, 2, g, n, n-1, n-2, \ldots, n-t+r+1, 1))$ is a $\{\begin{smallmatrix} V(G) \\ t \end{smallmatrix}\}$-cycle. Hence G' and thus H is $V(G)$-locally-pancyclic.

Next assume that n is odd. It follows that $x < y$. Now G satisfies all the assumptions of Lemma 7 (with x and y chosen in a similar nature as in the proof of Lemma 7), hence all Facts 1–15 hold for G (recall G is neither pancyclic nor missing only an $(n-1)$-cycle).

Fact 17. *H contains a $\{\begin{smallmatrix} V(G) \\ t \end{smallmatrix}\}$-cycle, where $t = 3, 4, 5$, and $n-2$.*

Obviously, $(2, g, n, 1, 2)$ (in G'), $(1, 2, 3, 4, 1)$ (in G), $(1, 4, 3, 2, g, n, 1)$ (in G'), and $(1, 4, 5, \ldots, 1)$ (in G) are $\{\begin{smallmatrix} V(G) \\ t \end{smallmatrix}\}$-cycles, where $t = 3, 4, 5$, and $n-2$, respectively. Consequently H contains a $\{\begin{smallmatrix} V(G) \\ t \end{smallmatrix}\}$-cycle, where $t = 3, 4, 5$, and $n-2$.

By Facts 7 and 8, there is a t-cycle (in G) for $t = 6, 7, \ldots, y$. It follows that there are $\{\begin{smallmatrix} V(G) \\ t \end{smallmatrix}\}$-cycles, $6 \leq t \leq y$, in H. Thus for some t, $y < t < n-2$, there is no $\{\begin{smallmatrix} V(G) \\ t \end{smallmatrix}\}$-cycle in G'.

Fact 18. *It holds that $r, s \notin \{4, 5, \ldots, y-2\}$.*

Using Fact 9, $(1, t), (2, t) \notin E(G)$. Hence $(n, t) \in E(G)$. First assume that $r = y-2$. Then $(n, g, y-2, y-3, \ldots, 1, y, y+1, \ldots, t, n)$ is a $\{\begin{smallmatrix} V(G) \\ t \end{smallmatrix}\}$-cycle, a contradiction. Thus suppose $r \in \{4, 5, \ldots, y-3\}$. Since $r+2 \leq y-1$, either $(1, r+2) \in E(G)$ or $(2, r+2) \in E(G)$. Now, if $(1, r+2) \in E(G)$ $((2, r+2) \in E(G))$, then $(n, g, r, r-1, \ldots, 1, r+2, r+3, \ldots, t, n)$ $((n, g, r, r-1, \ldots, 2, r+2, r+3, \ldots, y-1, 1, y, y+1, \ldots, t, n))$ is a $\{\begin{smallmatrix} V(G) \\ t \end{smallmatrix}\}$-cycle, a contradiction. The case $s \in \{4, 5, \ldots, y-2\}$ is analogous.

Fact 19. *It holds that $r, s \notin \{y, y+1, \ldots, n-3\}$.*

Suppose $r \in \{y, y+1, \ldots, n-3\}$. The case $s \in \{y, y+1, \ldots, n-3\}$ is analogous. First we show that $r \neq t$. If not, then $(2, 3, \ldots, x, 1, x+1, x+2, \ldots, t, g, 2)$ is a $\{\begin{smallmatrix} V(G) \\ t \end{smallmatrix}\}$-cycle, a contradiction. Consequently either $r < t$ or $r > t$.

First, let $r > t$. Consider the vertex $x - r + t - 1$. Because $r - t \geq 1$, we have that $x - r + t - 1 \leq x - 2$. Since $t > y$ and $r \leq n-3$, it follows that $r - t \leq n - y - 4 \leq x - 6$, consequently $x - r + t - 1 \geq 5$. Now, if $x - r + t - 1$ is even (odd), then $(1, x - r + t - 1, x - r + t - 2, \ldots, 2, g, r, r-1, \ldots, x, 1)$ $((1, x - r + t, x - r + t - 1, \ldots, 2, g, r, r-1, \ldots, x+1, 1))$ is a $\{\begin{smallmatrix} V(G) \\ t \end{smallmatrix}\}$-cycle, a contradiction.

Second let $r < t$. If $(1, n - t + 2) \in E(G)$, then $(1, n, n-1 \ldots, n-t+2, 1)$ is a t-cycle (in G), a contradiction. Since $5 \leq n - t + 2 \leq x - 1$, we have $(2, n - t + 2) \in E(G)$. By Fact 13, $(2, k) \notin E(G)$, where $y < k < n-2$. Obviously, $(1, t) \notin E(G)$, and so $(n, t) \in E(G)$. Moreover, either $(1, r-1) \in E(G)$ or $(n, r-1) \in E(G)$. If $(1, r-1) \in E(G)$, then $(2, g, r+1, r+2, \ldots, 1, r-1, r-2, \ldots, n-t+2, 2)$ is a $\{\begin{smallmatrix} V(G) \\ t \end{smallmatrix}\}$-cycle, a contradiction. Consequently $(n, r-1) \in E(G)$. It follows that $r > y+1$. But then $(n, r-1, r-2, \ldots, y, 1, y-1, y-2, \ldots, 2, g, r+1, r+2, \ldots, t, n)$ is a $\{\begin{smallmatrix} V(G) \\ t \end{smallmatrix}\}$-cycle, a contradiction.

Using previous facts we have $r = s$, a contradiction. This proves part (a).

Now we prove part (b). Let one of r, s be 2, say $r = 2$ (the other case is analogous). Then for $t = 3, 4, \ldots, n - 2$, if $(1, n - t + 2) \in E(G)$ $((1, n - t + 2) \notin E(G))$ then $(1, n, n - 1, \ldots, n - t + 2, 1)$ $((1, n - t + 3, n - t + 4, \ldots, n, g, 2, 1))$ is a $\{^{V(G)}_{t}\}$-cycle, a contradiction. Let one of r, s be 3; a contradiction in this case can be obtained in a way similar to the corresponding case in part (a). Hence we may assume $r, s \in \{4, 5, \ldots, n-1\}$.

Now, the graph $H' = H - \{3\}$ is either $(V(G) \setminus \{3\})$-locally-pancyclic or missing only a $\{^{V(G)\setminus\{3\}}_{n-2}\}$-cycle, by part (a). Because $(1, 2, \ldots, n - 2, 1)$ is a $\{^{V(G)}_{n-2}\}$-cycle, the graph H is either $V(G)$-locally-pancyclic or missing only a $\{^{V(G)}_{n-1}\}$-cycle, a contradiction. The proof of part (c) is analogous to the proof of part (b).

Finally, we prove part (d). Let r or s be in $\{2, n - 1\}$, say $r = n - 1$ (the remaining cases are analogous). Then $(2, g, n, 1, 2)$ is a $\{^{V(G)}_{3}\}$-cycle, and $(1, 2, 4, 5, 1)$ is a $\{^{V(G)}_{4}\}$-cycle. For $t = 5, 6, \ldots, n - 2$, if $(1, t) \in E(G)$ $((1, t) \notin E(G))$ then $(1, 2, \ldots, t, 1)$ $((1, t - 1, t - 2, \ldots, 2, g, n, 1))$ is a $\{^{V(G)}_{t}\}$-cycle, a contradiction. Let r or s be in $\{3, n - 2\}$; a contradiction can be obtained similarly to the corresponding case in part (a). Consequently we may assume $r, s \in \{4, 5, \ldots, n - 3\}$.

Thus the graph $H' = H - \{3, n - 1\}$ is either $(V(G) \setminus \{3, n - 1\})$-locally-pancyclic or missing only a $\{^{V(G)\setminus\{3,n-1\}}_{n-3}\}$-cycle, by part (a). Since $(1, 5, 6, \ldots, 1)$ is a $\{^{V(G)}_{n-3}\}$-cycle and $(1, 2, 4, 5, \ldots, 1)$ is a $\{^{V(G)}_{(}\}$-cyclen-1), it follows that H is $V(G)$-locally-pancyclic, a contradiction. This proves Lemma 8.

Lemma 9. *Let G be a graph with a hamiltonian cycle $(1, 2, \ldots, n, 1)$, where n is even, $n \geq 4$. Let $\{k \; : \; k \text{ is even}\} \subseteq N(1)$. Let $r \in \{2, 3, \ldots, n - 1\}$. Let G' be the graph obtained from G by deleting the edge $(r, r + 1)$ and adding a new vertex g and all the edges $(r, g), (g, r+1), (2, g)$, and (n, g). Then the graph G' is either $V(G)$-locally-pancyclic or missing only a $\{^{V(G)}_{n-1}\}$-cycle.*

Proof. The proof is analogous to the proof of Lemma 8 (part (a) with even n).

4 Proof of Theorem 1

Proof. (of Theorem 1) By Theorem A$^\circ$, we may assume $W \neq V(G)$ and according to Theorem A$^\bullet$ we may assume $|W| \geq 4$. Set $|W| = w$ and $H = V(G) \setminus W$. Obviously, if we delete all edges from $G[H]$, the new graph will satisfy the assumptions of Theorem 1, consequently we may assume

(Φ) $E(G[H]) = \emptyset$.

Now, we distinguish two cases.

Case 1. For all $c \in W$, $d_W(c) < w/2$.

Fact 20. *Every pair of non-adjacent vertices in W has a common neighbor in H.*

If $x, y \in W$, then $d_W(x) + d_W(y) < w$. Moreover, if $(x, y) \notin E(G)$, then $d_H(x) + d_H(y) > n - w$. Hence $|N_H(x) \cap N_H(y)| \geq 1$.

Suppose by way of contradiction there is an integer t such that G does not contain any $\{^W_t\}$-cycle. Take t as large as possible. According to Theorem A$^\bullet$, $t < w$ and hence there is a $\{^W_{t+1}\}$-cycle in G, say C^{t+1}. Take C^{t+1} in such way that

(Ψ) $|V(C^{t+1}) \cap H|$ is as small as possible.

There are two possibilities to consider:

Case 1.1. C^{t+1} contains no vertex from H. Take three consecutive vertices on C^{t+1}, say x, y, and z. Because G contains no $\{^W_t\}$-cycle, $(x, z) \notin E(G)$. But since $x, z \in W$, according to Fact 20, there is a vertex from H adjacent to both of them. By the choice of C^{t+1}, the vertex is not on C^{t+1}, contradicting the non-existence of a $\{^W_t\}$-cycle.

Case 1.2. C^{t+1} contains a vertex from H. Let $C^{t+1} = (1, 2, \ldots, m, 1)$. We may assume $1 \in W$ and $2 \in H$. It follows from (Φ) that $3 \in W$. Let q be as large as possible such that $3 < q \le m$ and $q \in W \cap V(C^{t+1})$. Since $|W| \ge 4$, q always exists. Let $R = G[\{3, 4, \ldots, q\}]$ and let $S = V(G) \setminus V(R)$. By the choice of q, R contains t vertices from W, and so there is no hamiltonian cycle in R. Because $(3, 4, \ldots, q)$ is a hamiltonian path in R, there is no common neighbor of 3 and q in $H \cap S$ and $(3, q) \notin E(G)$. If there was a common neighbor of 3 and q in $W \cap S$, we would have a contradiction with (Ψ). Consequently, $d_S(3) + d_S(q) \le |S|$ and $d_R(3) + d_R(q) < |V(R)|$. Observing that $|V(R)| + |S| = n$ we have $d_G(3) + d_G(q) = d_S(3) + d_S(q) + d_R(3) + d_R(q) < n$, a contradiction, since $(3, q) \notin E(G)$ and $3, q \in W$.

Case 2. There is a vertex $c \in W$ with $d_W(c) \ge w/2$.

Assume the contrary and let G be not W-locally-pancyclic. By Theorem A$^\bullet$, there is a $\{^W_w\}$-cycle, say C^w. Assume C^w is chosen in such way that

(Ω) $|V(C^w) \cap H|$ is as small as possible.

Let $C^w = (1, 2, \ldots, m, 1)$. We may assume $c = 1$. There are two possibilities.

Case 2.1. C^w contains no vertex from H. Hence $m = w$ and $W = \{1, 2, \ldots, w\}$.

Fact 21. *The edge $(2, w)$ does not belong to $E(G)$.*

Assume $(2, w) \in E(G)$. Since $d_W(1) \ge w/2$ and since $(2, 3, \ldots, w, 2)$ is a hamiltonian cycle in $G[W \setminus \{1\}]$, it follows from Lemma 4 that $G[W]$ is pancyclic, a contradiction.

Fact 22. *We have $N_H(2) \cap N_H(w) = \emptyset$.*

Assume not; let $x \in N_H(2) \cap N_H(w)$. Let $G' = G - \{x\} + \{(2, w)\}$. Obviously, G' is a $(W; 2, w)$-restriction of G and $(2, 3, \ldots, w, 2)$ is a hamiltonian cycle in $G'[W \setminus \{1\}]$. According to Lemma 4, it holds that $G'[W]$ is pancyclic and, by Observation 1, it follows that G is W-locally-pancyclic, a contradiction.

According to Fact 21, we have $d_G(2) + d_G(w) \ge n$ and hence, by Fact 22, we have $d_W(2) + d_W(w) \ge w$. It follows from Lemma 5 that $d_W(2) + d_W(w) = w$. Hence

$$d_G(2) + d_G(w) = n \qquad \text{and} \qquad H = N_H(2) \cup N_H(w). \qquad (2)$$

Fact 23. *For any vertex $x \in W$ $(x \ne 2, w)$ either $(2, x) \in E(G)$ or $(w, x) \in E(G)$.*

If not, then there exists a vertex, say $i \in W$ $(2 < i < w)$, with $(2, i), (w, i) \notin E(G)$. It follows from (2) that at least for one of vertices 2 or w, say 2, it holds that $d_G(2) \le n/2$. Consequently $d_G(i) \ge n/2$. We claim that $(i, y) \notin E(G)$ for $y \in H$. Indeed, if $(i, y) \in E(G)$, $y \in H$, then, by (2), $y \in N_H(2) \cup N_H(w)$, say $(2, y) \in E(G)$. Let $G' = G - \{y\} + \{(2, i)\}$. Now, $d_{G'[W]}(2) + d_{G'[W]}(w) > w$ and $G'[W]$ is pancyclic by Lemma

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5. Because G' is a $(W; 2, i)$-restriction of G, by Observation 1, it holds that G is W-locally-pancyclic, a contradiction. Consequently, $d_W(i) \geq n/2 > w/2$. One can prove that $d_W(i-1) + d_W(i+1) = w$, (it is sufficient to set $c = i$ at the beginning of Case 2). Hence at least for one of vertices $i-1$ or $i+1$, say $i+1$, it holds that $d_W(i+1) \geq w/2$. But now $d_W(i) + d_W(i+1) > w$ and $G[W]$ is pancyclic according to Lemma 6, a contradiction.

Let $R = G[\{2, 3, \ldots, w\}]$. By Lemma 4, R is non-hamiltonian. The graph R contains a hamiltonian path $(2, 3, \ldots, w)$. Now, by Lemma 1 applied to R and Fact 23, it follows that there is k $(2 < k < w)$ such that 2 is adjacent to $3, 4, \ldots, k$ and w is adjacent to $k, k+1, \ldots, w-1$, respectively. We assert that $G[W]$ is pancyclic if $|W| \geq 5$. Indeed, by an appropriate relabeling, $k \geq 4$. Now for $4 \leq t \leq k$, $(2, 3, \ldots, t, 2)$ is a $(t-1)$-cycle, $(w, 1, 2, 4, 5, \ldots, k, w)$ is a k-cycle, and for $k \leq t \leq w-1$, $(w, 1, 2, \ldots, t, w)$ is a $(t+1)$-cycle, respectively. If $|W| = 4$, then $G[W] = K_{2,2}$ and $|H| = n-4$. Consider the vertices i and $i+2$, where $i = 1$ and 2. The vertices i and $i+2$ cannot be H-adjacent. Since $d_G(i) + d_G(i+2) \geq n$, it follows that $d_H(i) + d_H(i+2) = n-4$. One can easily observe that every vertex in H must be adjacent to exactly two adjacent vertices of $G[W]$. This determines a partition of $G[H]$ into four subgraphs G_{12}, G_{23}, G_{34}, and G_{41} (null graphs are also allowed), such that every vertex of G_{pq} is adjacent to both vertices p and q. The structure of G is depicted in Figure 1.

Case 2.2. C^w contains a vertex from H. Let $g \in H \cap V(C^w)$, where $2 \leq g \leq m$. Now, at least one of paths $(1, 2, \ldots, g)$ or $(g, g+1, \ldots, 1)$ has at least two internal vertices in W, say it is $(g, g+1, \ldots, 1)$. The second case is analogous. It follows from (Φ) that $g-1, g+1 \in W$. Let p be as small as possible such that $g+1 < p \leq m$ and $p \in W$; p always exists. Let $R = G[\{p, p+1, \ldots, g-1\}]$ and let $S = V(G) \setminus V(R)$.

Fact 24. *G contains a $\{^W_{w-1}\}$-cycle, say C^{w-1}, which is missing the vertex $g+1$.*

Assume not. By the choice of p, it follows that any hamiltonian cycle in R is a $\{^W_{w-1}\}$-cycle missing $g+1$. So we may assume R is non-hamiltonian, consequently $d_R(p) + d_R(g-1) < |V(R)|$ and $(p, g-1) \notin E(G)$. Similarly, there is no common neighbor of p and $g-1$ in $H \cap S$ and, using (Ω), we have $(g-1, g+1) \notin E(G)$. Consequently, $d_S(p) + d_S(g-1) \leq |S|$. Because $|V(R)| + |S| = n$, we have that $d_G(p) + d_G(g-1) < n$, a contradiction since $p, g-1 \in W$.

It follows from the previous fact that the theorem is true for $w = 4$. Thus we may assume that $w \geq 5$.

Fact 25. *G cannot contain any $\{^W_{w-1}\}$-cycle missing the vertex 1 in G.*

Assume, there is a $\{^W_{w-1}\}$-cycle, say C^{w-1}, missing the vertex 1. Let $W' = W \setminus \{1\}$. By Observation 2, the W'-shrinking of C^{w-1} is a $(w-1)$-cycle, say C_{w-1}. Since $V(C_{w-1}) = W'$, it holds that $d_{W'}(1) = d_W(1) \geq w/2$. Let $K = V(C^{w-1}) \setminus V(C_{w-1})$ and let $L = E(C_{w-1}) \setminus E(C^{w-1})$. Let $G' = G - K + L$. Obviously, G' is a W-restriction of G. It follows from Lemma 4 that the graph $G'[W]$ is pancyclic; and so G is W-locally-pancyclic, by Observation 1, a contradiction.

Fact 26. *We have $2, m \in W$.*

If for example $2 \in H$, then, by (Φ), $3 \in W$. Let p be as large as possible such that $3 < p \leq m$ and $p \in W$. Because $|W| \geq 5$, p always exists. Now, it can be proved

(analogously to the proof of Fact 24) that there is a $\{^W_{w-1}\}$-cycle missing the vertex 1, contradicting Fact 25.

Fact 27. *It holds that* $W \subseteq N(1) \cup N(2) \cup N(m)$.

Assume there is a vertex $x \in W$ such that $x \notin N(1) \cup N(2) \cup N(m)$. Using Fact 26, vertices $m, 1$, and 2 are in W and are consecutive on C^w. Hence $x \neq m-1, m, 1, 2$, and 3. We claim that $(2, x+1), (m, x-1) \in E(G)$. Assume not and let $(2, x+1) \notin E(G)$ (the second case is analogous). Let $R = G[\{2, 3, \ldots, m\}]$ and $S = V(G) \setminus (V(C^w)$. By Fact 25, R is non-hamiltonian, and so $(2, m) \notin E(G)$. Obviously, $P = (2, 3, \ldots, m)$ is a hamiltonian path in R. Let $F = \{i \in V(P) : (2, i+1), (m, i) \notin E(G), i \neq m\}$. It follows from Lemma 3 that $d_R(2) + d_R(m) < |V(R)| - |F|$. Since $(m, x), (2, x+1) \notin E(G)$, we have $|F| \geq 1$. Similarly, $d_S(2) + d_S(m) \leq |S|$. Consequently, $d_G(2) + d_G(m) = d_S(2) + d_S(m) + d_R(2) + d_R(m) + |\{(2, 1), (m, 1)\}| < |S| + |V(R)| - 1 + 2 = n$, a contradiction because $2, m \in W$.

Thus $(2, x+1), (m, x-1) \in E(G)$. Now, $(2, x+1, x+2, \ldots, m, x-1, x-2, \ldots, 2)$ is a $\{^W_{w-2}\}$-cycle, say C^{w-2}. Let $W' = W \setminus \{1, x\}$. By Observation 2, the W'-shrinking of C^{w-2} is a $(w-2)$-cycle, say C_{w-2}. Since $V(C_{w-2}) = W'$ and since $(1, x) \notin E(G)$, it holds that $d_{W'}(1) = d_W(1) \geq w/2$. Let $K = V(C^{w-2}) \setminus V(C_{w-2})$ and let $L = E(C_{w-2}) \setminus E(C^{w-2})$. Let $G' = G - K + L$. Obviously, G' is a W-restriction of G. It follows from Lemma 4 that the graph $G'[W \setminus \{x\}]$ is pancyclic, and so by Observation 1, G is W-locally-pancyclic, a contradiction.

Let C_w be the W-shrinking of C^w. Let $K = V(C^w) \setminus V(C_w)$ and let $L = E(C_w) \setminus E(C^w)$. Let $G' = G - K + L$. Obviously, G' is a W-restriction of G. Now, C_w is a hamiltonian cycle in $G'[W]$. It follows from Fact 26 that the vertices $m, 1$, and 2 are consecutive on C_w. By Fact 27, it holds that $N_{G'[W]}(m) \cup N_{G'[W]}(1) \cup N_{G'[W]}(2) = V(G'[W])$. Moreover, $d_{G'[W]}(1) = d_W(1) \geq w/2$.

Fact 28. *There are at least two vertices* $x, y \in K$ *such that* $(2, a), (m, a) \in E(G)$, *where* $a = x, y$.

Using Fact 25, any vertex in $H \cap N_G(2) \cap N_G(m)$ (a common vertex to 2 and m from H) is on C^w, and so in K. If there are l vertices in $H \cap N_G(2) \cap N_G(m)$, then, since $|N_H(m) \cup N_H(2)| \leq n - w$, it holds that $n \leq d_G(2) + d_G(m) \leq d_W(2) + d_W(m) + n - w + l$. Hence

$$d_W(2) + d_W(m) \geq w - l. \tag{3}$$

Thus at least for one of vertices m or 2, say 2, it holds that $d_W(2) \geq (w-l)/2$. Assume that $l < 2$. Hence $d_W(2) \geq (w-1)/2$ and thus $d_W(1) + d_W(2) \geq w$. Now by Lemma 6, $G'[W]$ is pancyclic, bipartite, or missing a $(w-1)$-cycle only. If it were pancyclic or missing only a $(w-1)$-cycle, then G would be W-locally-pancyclic, a contradiction. Consequently we may assume $G'[W]$ is bipartite, thus w is even and $w \geq 6$. If $l = 0$, then, by (3) and by the fact that $G'[W]$ is bipartite, it follows that $G'[W]$ satisfies the assumptions of Lemma 2. By Fact 26, it follows that $3 \leq g \leq m-1$ (note that g is a vertex from $H \cap V(G^w)$; see Case 2.2). It follows from (Φ) that $g-1, g+1 \in W$, hence $(g-1, g+1) \in E(C_w)$. Because $2 \leq g-1 \leq m-2$, by Lemma 2, the graph $G'[W]$ contains a hamiltonian cycle missing the edge $(g-1, g+1)$. But this cycle induces a $\{^W_w\}$-cycle in G with fewer vertices in H than C^w, a contradiction with (Ω).

Consequently $l = 1$. By Fact 26, it holds that $g \in \{3, 4, \ldots, m-1\}$ and thus by Lemma 9, the graph $G'[W]$ with the vertex g is either W-locally-pancyclic or missing only a $\{\begin{smallmatrix} W \\ w-1 \end{smallmatrix}\}$-cycle. Since G contains a $\{\begin{smallmatrix} W \\ w-1 \end{smallmatrix}\}$-cycle (Fact 24), according to the Observation 1, G would be W-locally-pancyclic, a contradiction. This proves Fact 28.

It follows from (Φ) that $x \pm 1, y \pm 1 \in W$, and so from C_w, and $x - 1 \neq y - 1$, by Fact 28. Thus the graph $G'[W]$ with vertices x and y satisfies the assumptions of Lemma 8. Hence the graph G is either W-locally-pancyclic or missing only a $\{\begin{smallmatrix} W \\ w-1 \end{smallmatrix}\}$-cycle. By Fact 24, G is W-locally-pancyclic, a contradiction. This proves the theorem.

Last Note. Independently, Theorem 1 was proved in [5].

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