Transient Heat Conduction

In general, temperature of a body varies with time as well as position.

Lumped System Analysis

Interior temperatures of some bodies remain essentially uniform at all times during a heat transfer process. The temperature of such bodies are only a function of time, $T = T(t)$. The heat transfer analysis based on this idealization is called *lumped system analysis*.

Consider a body of arbitrary shape of mass m, volume V, surface area A, density ρ and specific heat C_p initially at a uniform temperature T_i.

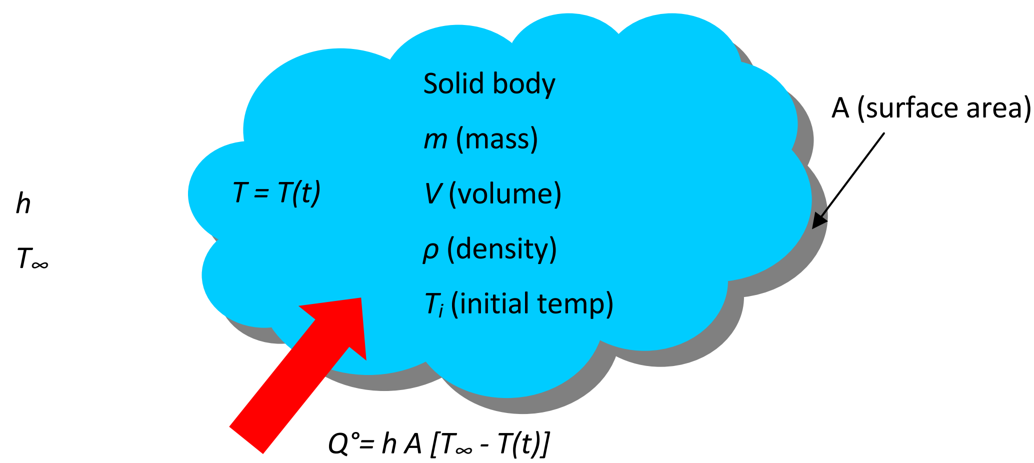


Fig. 1: Lumped system analysis.

At time $t = 0$, the body is placed into a medium at temperature T_∞ ($T_\infty > T_i$) with a heat transfer coefficient h. An energy balance of the solid for a time interval dt can be expressed as:

heat transfer into the body during dt	=	the increase in the energy of the body during dt

$$h\,A\,(T_\infty - T)\,dt = m\,C_p\,dT$$

With $m = \rho V$ and change of variable $dT = d(T - T_\infty)$, we find:

$$\frac{d(T - T_\infty)}{T - T_\infty} = -\frac{hA}{\rho V C_p}dt$$

Integrating from $t = 0$ to $T = T_i$

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-bt}$$

$$b = \frac{hA}{\rho V C_p} \qquad (1/s)$$

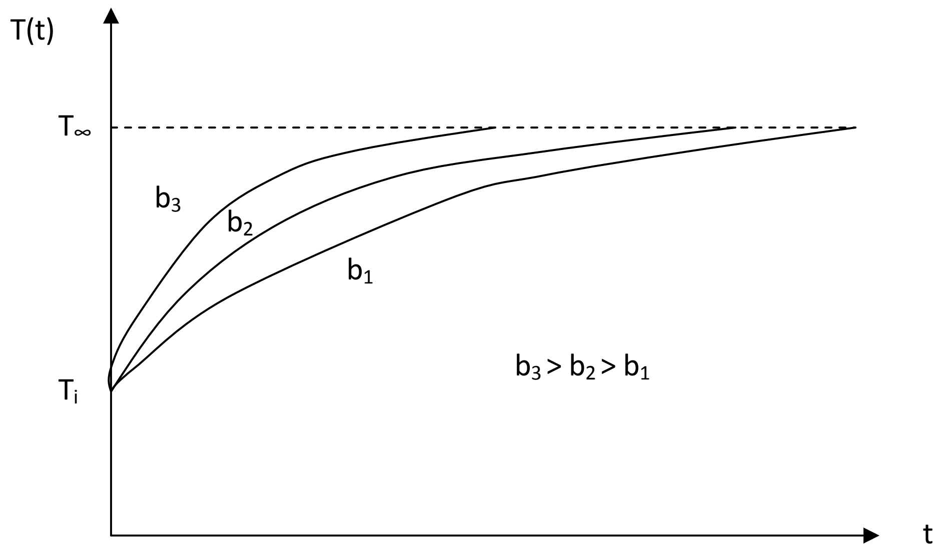


Fig. 2: Temperature of a lump system.

Using above equation, we can determine the temperature *T(t)* of a body at time *t*, or alternatively, the time *t* required for the temperature to reach a specified value *T(t)*.

Note that the temperature of a body approaches the ambient temperature T_∞ exponentially.

A large value of *b* indicates that the body will approach the environment temperature in a short time.

b is proportional to the surface area, but inversely proportional to the mass and the specific heat of the body.

The total amount of heat transfer between a body and its surroundings over a time interval is:

$$Q = m\, C_p\, [T(t) - T_i]$$

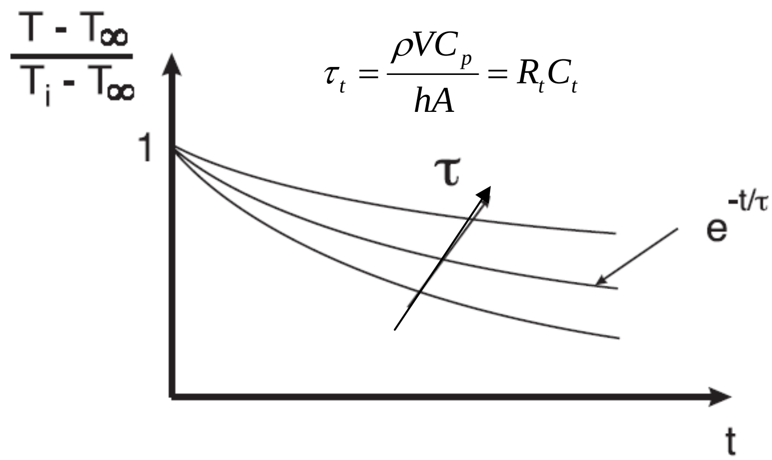
Electrical Analogy

The behavior of lumped systems, shown in Fig. 2 can be interpreted as a thermal time constant

$$\tau_t = \left(\frac{1}{hA}\right)\rho V C_p = R_t C_t$$

$$\tau_t = \frac{1}{b}$$

where R_t is the resistance to convection heat transfer and C_t is the lumped thermal capacitance of the solid. Any increase in R_t or C_t will cause a solid to respond more slowly to changes in its thermal environment and will increase the time respond required to reach thermal equilibrium.



$$\tau_t = \frac{\rho V C_p}{hA} = R_t C_t$$

Fig. 3: Thermal time constant.

Criterion for Lumped System Analysis

Lumped system approximation provides a great convenience in heat transfer analysis. We want to establish a criterion for the applicability of the lumped system analysis.

A characteristic length scale is defined as:

$$L_c = \frac{V}{A}$$

A non-dimensional parameter, the Biot number, is defined:

$$Bi = \frac{hL_c}{k}$$

$$Bi = \frac{h\Delta T}{\frac{k}{L_c}\Delta T} = \frac{\text{convection at the surface of the body}}{\text{conduction within the body}}$$

$$Bi = \frac{L_c / k}{1/h} = \frac{\text{conduction resistance within the body}}{\text{convection resistance at the surface of the body}}$$

The Biot number is the ratio of the internal resistance (conduction) to the external resistance to heat convection.

Lumped system analysis assumes a uniform temperature distribution throughout the body, which implies that the conduction heat resistance is zero. Thus, the lumped system analysis is exact when $Bi = 0$.

It is generally accepted that the lumped system analysis is applicable if

$$Bi \leq 0.1$$

Therefore, small bodies with high thermal conductivity are good candidates for lumped system analysis.

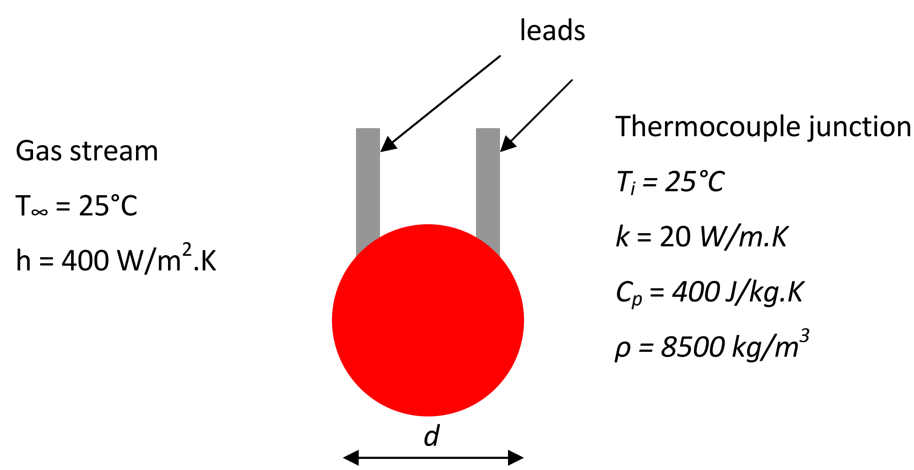
Note that assuming *h* to be constant and uniform is an approximation.

Example 1

A thermocouple junction, which may be approximated by a sphere, is to be used for temperature measurement in a gas stream. The convection heat transfer coefficient between the junction surface and the gas is known to be $h = 400\ W/m^2.K$, and the junction thermophysical properties are $k = 20\ W/m.K$, $C_p = 400\ J/kg.K$, and $\rho = 8500\ kg/m^3$. Determine the junction diameter needed for the thermocouple to have a time constant of 1 s. If the junction is at 25°C and is placed in a gas stream that is at 200°C, how long will it take for the junction to reach 199°C?

Assumptions:

1. Temperature of the junction is uniform at any instant.
2. Radiation is negligible.
3. Losses through the leads, by conduction, are negligible.
4. Constant properties.



Solution:

To find the diameter of the junction, we can use the time constant:

$$\tau_t = \frac{1}{hA}\rho V C_p = \frac{1}{h\pi D^2} \times \frac{\rho \pi D^3}{6} C_p$$

Rearranging and substituting numerical values, one finds, $D = 0.706$ mm.

Now, we can check the validity of the lumped system analysis. With $L_c = r_0 / 3$

$$Bi = \frac{hL_c}{k} = 2.35 \times 10^{-4} \le 0.1 \rightarrow \text{Lumped analysis is OK.}$$

Bi << 0.1, therefore, the lumped approximation is an excellent approximation.

The time required for the junction to reach *T = 199°C* is

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-bt}$$

$$b = \frac{hA}{\rho V C_p}$$

$$t = \frac{1}{b} \ln \frac{T_i - T_\infty}{T(t) - T_\infty}$$

$$t = 5.2\, s$$

Transient Conduction in Large Plane Walls, Long Cylinders, and Spheres

The lumped system approximation can be used for small bodies of highly conductive materials. But, in general, temperature is a function of position as well as time.

Consider a plane wall of thickness *2L*, a long cylinder of radius r_0, and a sphere of radius r_0 initially at a uniform temperature T_i.

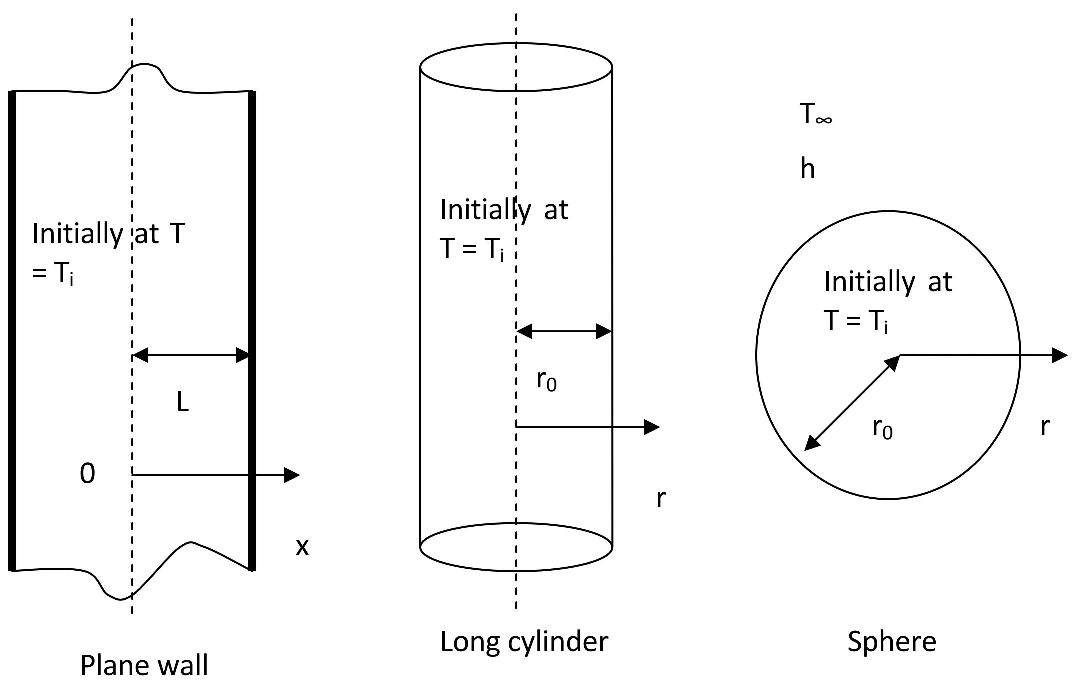


Fig. 4: Schematic for simple geometries in which heat transfer is one-dimensional.

We also assume a constant heat transfer coefficient *h* and neglect radiation. The formulation of the one-dimensional transient temperature distribution T(x,t) results in a partial differential equation (PDE), which can be solved using advanced mathematical methods. For plane wall, the solution involves several parameters:

$$T = T\,(x, L, k, \alpha, h, T_i, T_\infty)$$

where $\alpha = k/\rho C_p$. By using dimensional groups, we can reduce the number of parameters.

$$\theta = \theta(x, Bi, \tau)$$

To find the temperature solution for plane wall, i.e. Cartesian coordinate, we should solve the Laplace's equation with boundary and initial conditions:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad (1)$$

Boundary conditions: $\frac{\partial T(0,t)}{\partial t} = 0, \quad -k\frac{\partial T(L,t)}{\partial x} = h[T(L,t) - T_\infty]$ (2a)

Initial condition: $T(x,0) = T_i$ (2b)

So, one can write:

$$\frac{\partial^2 \theta}{\partial X^2} = \frac{\partial \theta}{\partial \tau}$$

where,

$$\theta(x,t) = \frac{T(x,t) - T_\infty}{T_i - T_\infty} \qquad \text{dimensionless temperature}$$

$$X = \frac{x}{L} \qquad \text{dimensionless distance}$$

$$Bi = \frac{hL}{k} \qquad \text{Biot number}$$

$$\tau = \frac{\alpha t}{L^2} \qquad \text{Fourier number}$$

The general solution, to the PDE in Eq. (1) with the boundary conditions and initial conditions stated in Eqs. (2), is in the form of an infinite series:

$$\theta = \sum_{n=1}^{\infty} A_n e^{-\lambda_n^2 \tau} \cos(\lambda_n X)$$

Table 11-1 , Cengle's book, lists solutions for plane wall, cylinder, and sphere.

There are two approaches:

1. Use the first term of the infinite series solution. This method is only valid for Fourier number > 0.2

2. Use the *Heisler charts* for each geometry as shown in Figs. 11-15, 11-16 and 11-17.

Using the First Term Solution

The maximum error associated with method is less than 2%. For different geometries we have:

$$\theta(x,t)_{wall} = \frac{T(x,t)-T_\infty}{T_i - T_\infty} = A_1 \exp\left(-\lambda_1^2 \tau\right)\cos(\lambda_1 x / L)$$

$$\theta(x,t)_{cylinder} = \frac{T(r,t)-T_\infty}{T_i - T_\infty} = A_1 \exp\left(-\lambda_1^2 \tau\right) J_0(\lambda_1 r / r_0)$$

$$\theta(x,t)_{sphere} = \frac{T(r,t)-T_\infty}{T_i - T_\infty} = A_1 \exp\left(-\lambda_1^2 \tau\right) \frac{\sin(\lambda_1 r / r_0)}{(\lambda_1 r / r_0)}$$

$$\text{where} \qquad \tau > 0.2$$

where A_1 and λ_1 can be found from Table 11-2 Cengel book.

Using Heisler Charts

There are three charts, Figs. 11-15 to 11-17, one associated with each geometry:

1. The first chart is to determine the temperature at the center T_0 at a given time.
2. The second chart is to determine the temperature at other locations at the same time in terms of T_0.
3. The third chart is to determine the total amount of heat transfer up to the time t.