# 7 Sumsets in Abelian Groups (Kneser's Theorem)

Throughout this section we shall assume that $G$ is an additive abelian group. Our goal here is to prove a theorem of Kneser which gives a natural lower bound on the size of a sumset in an abelian group. This result may be viewed as a generalization of the Cauchy-Davenport theorem. To state it, we need just one added piece of notation. If $A \subseteq G$, we define the *stabilizer* of $A$ to be $\mathcal{S}(A) = \{g \in G \mid A + g = A\}$. Note that $\mathcal{S}(A) \leq G$.

**Theorem 7.1 (Kneser)** *If $A, B \subseteq G$ are finite and nonempty and $K = \mathcal{S}(A + B)$, then $|A + B| \geq |A + K| + |B + K| - |K|$.*

**Proof.** We proceed by induction on $|A + B| + |A|$. Suppose that $K \neq \{0\}$ and let $\phi : G \to G/K$ be the canonical homomorphism. Then $\mathcal{S}(\phi(A+B))$ is trivial, so by applying induction to $\phi(A), \phi(B)$ we have that $|A + B| = |K|(|\phi(A) + \phi(B)|) \geq |K|(|\phi(A)| + |\phi(B)| - 1) = |A + K| + |B + K| - |K|$. Thus, we may assume that $K = \{0\}$. If $|A| = 1$, then the result is trivial, so we may assume $|A| > 1$ and choose distinct $a, a' \in A$. Since $a' - a \notin \mathcal{S}(B) \subseteq \mathcal{S}(A + B) = \{0\}$, we may choose $b \in B$ so that $b + a' - a \notin B$. Now by replacing $B$ by $B - b + a$ we may assume that $\emptyset \neq A \cap B \neq A$.

Let $C \subseteq A + B$ and let $H = \mathcal{S}(C)$. We say that $C$ is a *portion* if

$$|C| + |H| \geq |A \cap B| + |(A \cup B) + H|.$$

Set $C_0 = (A \cap B) + (A \cup B)$ and observe that $C_0 \subseteq A + B$. Since $0 < |A \cap B| < |A|$, we may apply induction to $A \cap B$ and $A \cup B$ to conclude that $C_0$ is a portion. Thus a portion exists, and we may now choose a portion $C$ with $H = \mathcal{S}(C)$ minimal. If $H = \{0\}$ then $|A + B| \geq |C| \geq |A \cap B| + |A \cup B| - |\{0\}| = |A| + |B| - 1$ and we are finished. Therefore, we may assume (for a contradiction) that $H \neq \{0\}$. Since $\mathcal{S}(A + B) = \{0\}$ and $\mathcal{S}(C) = H$, we may choose $x \in A$ and $y \in B$ so that $(x + y + H) \not\subseteq A + B$. Let $A_1 = A \cap (x + H)$, $A_2 = A \cap (y + H)$, $B_1 = B \cap (y + H)$, and $B_2 = B \cap (x + H)$ and note that $A_1, B_1 \neq \emptyset$. For $i = 1, 2$ let $C_i = C \cup (A_i + B_i)$ and let $H_i = \mathcal{S}(A_i + B_i)$. Observe that if $A_i, B_i \neq \emptyset$, then $H_i = \mathcal{S}(C_i) < H$. The following equation holds for $i = 1$, and it also holds for $i = 2$ if $A_2, B_2 \neq \emptyset$. It follows from the fact that $C_i$ is not a portion (by the minimaity of $H$), and induction applied to $A_i, B_i$.

$$\begin{aligned}
|(A \cup B) + H| - |(A \cup B) + H_i| &< (|C| + |H| - |A \cap B|) - (|C_i| + |H_i| - |A \cap B|) \\
&= |H| - |A_i + B_i| - |H_i| \\
&\leq |H| - |A_i + H_i| - |B_i + H_i| \qquad (1)
\end{aligned}$$

If $B_2 = \emptyset$, then $|(A \cup B) + H| - |(A \cup B) + H_1| \geq |H| - |A_1 + H_1|$ which contradicts equation 1 for $i = 1$. We get a similar contradiction under the assumption that $A_2 = \emptyset$. Thus $A_2, B_2 \neq \emptyset$ and equation 1 holds for $i = 1, 2$. If $x + H = y + H$, then $A_1 = A_2$ and $B_1 = B_2$ and we find that $|(A \cup B) + H| - |(A \cup B) + H_1| \geq |H| - |(A_1 \cup B_1) + H_1| \geq |H| - |A_1 + H_1| - |B_1 + H_1|$ which contradicts equation 1. Therefore, $x + H \neq y + H$. The following inequality follows from the observation that the left hand side of equation 1 is nonnegative, and all terms on the right hand side are multiples of $|H_i|$.

$$|H| \geq |A_i| + |B_i| + |H_i| \qquad (2)$$

Let $X = (x + H) \setminus (A_1 \cup B_2)$ and let $Y = (y + H) \setminus (A_2 \cup B_1)$. Note that $X$ and $Y$ are disjoint. The following equation follows from the fact that $A + B$ is not a portion (by the minimality of $H$), and induction applied to $A_i, B_i$.

$$\begin{aligned}
|H| &\geq |(A \cup B) + H| + |A \cap B| - |C| \\
&\geq |X| + |Y| + |A \cup B| + |A \cap B| - |A + B| + |A_i + B_i| \\
&> |X| + |Y| + |A_i| + |B_i| - |H_i| \qquad (3)
\end{aligned}$$

Summing the four inequalites obtained by taking equations 2 and 3 for $i = 1, 2$ and then dividing by two yields $2|H| > |A_1| + |A_2| + |X| + |B_2| + |B_1| + |Y|$. However, $x + H = X \cup A_1 \cup B_2$ and $y + H = Y \cup A_2 \cup B_1$. This final contradiction completes the proof. □