8 Bases of many weights (Schrijver-Seymour Theorem)

In this section we will turn our attention to a different type of structure sumset problem. Namely, we will look at graphs - and more generally matroids, labelled with elements of an abelian group G, and study the set of all weights of spanning trees - or bases. The main theorem we will prove is a lovely result due to Schrijver and Seymour which gives a natural lower bound on the number of distinct weights of trees or bases when $G \cong \mathbb{Z}_p$ for p prime. These authors have also conjectured a fascinating generalization of their result to all abelian groups. We begin with a quick introduction to matroids and a little bit of notation.

A matroid is an algebraic structure which gives a natural abstraction of linear independence in vector spaces. Formally, a *matroid* M consists of a finite ground set E and a distinguished collection of subsets of E called *independent* sets which satisfy the following properties

- $\emptyset$ is independent.
- If $X \subseteq E$ is independent and $Y \subseteq X$, then Y is independent.
- If $X, Y \subseteq E$ are independent and $|X| > |Y|$ then there exists $x \in X \setminus Y$ so that $Y \cup \{x\}$ is independent.

If E is a set of vectors in a vector space, then it is immediate that taking our independent sets to be those subsets of E which are linearly independent gives a matroid. Another important matroid comes from graphs. If F is a graph with edge set E, then the *cycle matroid* of F, denoted M_F is the matroid on E where a set of edges is independent if it does not contain a cycle. The *rank* of a set $X \subseteq E$, denoted $rk_M(X)$ or $rk(X)$ is the size of the largest independent set in X (for the cycle matroid of a graph M_F, the rank of a set $X \subseteq E$ is equal to $|V(F)| - comp(V(F), X)$ - here $comp(\cdot)$ is the number of connected components). A set $X \subseteq E$ *spans* an element $y \in E$ if $rk(X \cup \{y\}) = rk(X)$.

A *base* of a matroid is a maximal (or equivalently maximum size) independent set. In the cycle matroid of a connected graph, the bases are precisely the edge sets of spanning trees. If M is a matroid on E, then there is a matroid called the *dual* of M and denoted M^* with the property that a subset of E is a base in M^* if and only if it is the complement of a base in M (it is immediate from this that $M^{**} = M$ hence the name duality). If F is a graph on

E then the dual of the cycle matroid M_F is called the *bond matroid.* A set of edges in the cocycle matroid is independent if it does not contain a bond.

Numerous operations in graphs have natural genearlizations to the world of matroids. The one we will require is contraction. If x is an element of E, then contracting x gives us a matroid denoted M/x on the ground set $E \setminus \{x\}$ with the property that $X \subseteq E \setminus \{x\}$ is independent in M/x if and only if $X \cup \{x\}$ is independent in M. If F is a graph and e is an edge of F, then it is easily verified that the cycle matroid of F/e is the same as the matroid obtained from the cycle matroid of F by contracting e.

Our interest here will be in matroids equipped with a weight function. If M is a matroid on E, G is an additive abelian group and $w : E \to G$ is a map, then for every $X \subseteq E$ we let $w(X) = \sum_{x \in X} w(x)$ and we define $w(M) = \{w(D) : D \text{ is a base}\}$. Our main theorem is the following lower bound on $|w(M)|$.

Theorem 8.1 (Schrijver-Seymour) *Let M be a matroid with ground set E and let $w : E \to \mathbb{Z}_p$. Then $|w(M)| \geq \min\{p, \sum_{g \in \mathbb{Z}_p} rk(w^{-1}(g)) - rk(M) + 1\}$*

Proof: We proceed by induction on $rk(M)$. As a base, observe that the result is trivial if $rk(M) = 0$. For the inductive step we may now assume that the theorem holds for every matroid with smaller rank. Now, choose $x \in E$ which is not a loop, let $A = \{g \in \mathbb{Z}_p : w^{-1}(g) \text{ spans } x\}$ and let $B = w(M/x)$. Note that both A and B are nonempty. Since $b \in B = w(M/x)$, there exists a base D of M/x with $w(D) = b$. Since $w^{-1}(a)$ spans x, we may choose $y \in w^{-1}(a)$ so that $D' = D \cup \{y\}$ is a base of M. Now $w(D') = a + b$ and we conclude that $A + B \subseteq w(M)$. By applying induction to $w(M/x)$ and Cauchy-Davenport to $A + B$ we have the following (here we use the fact that $rk_M(w^{-1}(g)) - rk_{M/x}(w^{-1}(g))$ is 1 if $g \in A$ and 0 otherwise).

$$\begin{aligned}
|w(M)| &\geq |A + B| \\
&\geq \min\{p, |A| + (\sum_{g \in \mathbb{Z}_p} rk_{M/x}(w^{-1}(g)) - rk(M/x) + 1) - 1\} \\
&= \min\{p, \sum_{g \in \mathbb{Z}_p} rk_M(w^{-1}(g)) - rk(M) + 1\}
\end{aligned}$$

which completes the proof. $\square$

Before stepping to a few corollaries of this theorem, let us remark that it may be viewed as a structural generalization of the Cauchy-Davenport theorem (though the proof uses this as a tool). To see this, let $A, B \subseteq \mathbb{Z}_p$ and let F be a weighted graph obtained from a two edge path by replacing the first edge by $|A|$ copies of itself, one labelled with each element of A and replacing the second edge by $|B|$ copies of itself, one labelled with each element of B. Now we have $A + B = w(M_F)$, so by the above theorem we have

$$\begin{aligned} |A+B| &= |w(M_F)| \\ &\geq \min\{p, \sum_{g \in \mathbb{Z}_p} rk(w^{-1}(g)) - rk(M_F) + 1\} \\ &= \min\{p, |A| + |B| - 1\}. \end{aligned}$$

Next we give an interesting corollary of the Schrijver-Seymour theorem which we will apply to two rather different looking problems.

Corollary 8.2 *Let M be a matroid on E and let $w : E \to \mathbb{Z}_p$. If every cocyle of M includes a base, then one of the following holds*

- *There is a base of M in which all elements have the same weight.*
- $|w(M)| \geq \min\{p, rk(M) + 1\}$

In particular, if $rk(M) = p$, then M has a base of weight 0.

Proof: First observe that $w(M^*) = w(E) - w(M)$, so in particular the set of weights of bases and the set of weights of cobases have the same size. If no basis of M has all elements of the same weight, then by assumption, every cocycle has at least two elements of distinct weight, so $w^{-1}(g)$ is independent in M^* for every $g \in \mathbb{Z}_p$. Then applying the above theorem to the dual we have

$$\begin{aligned} |w(M)| &= |w(M^*)| \\ &\geq \min\{p, \sum_{g \in \mathbb{Z}_p} rk_{M^*}(w^{-1}(g)) - rk(M^*) + 1\} \\ &= \min\{p, |E| - rk(M^*) + 1\} \\ &= \min\{p, rk(M) + 1\} \end{aligned}$$

which completes the proof. $\square$

Before we apply this corollary, let us recall that for a sequence α of group elements, we have defined $\Sigma_k(\alpha)$ to be the set of all sums of length k subsequences of α, and we have defined $\rho(\alpha)$ to be the maximum multiplicity of an element of α. Next we give a second proof of Proposition 5.3, this time using the above corollary.

Proposition 5.3 *If α is a sequence in $\mathbb{Z}_p$ of length $2p-1$, then one of the following holds.*

- $\rho(\alpha) \geq p$
- $\Sigma_p(\alpha) = \mathbb{Z}_p$.

Proof 2: Let α be given by $a_1, a_2, \ldots, a_{2p-1}$, let M be the uniform matroid on $E = \{1, 2, \ldots, 2p-1\}$ of rank p (so the bases of M are precisely the subsets of E with size p), and let $w : E \to \mathbb{Z}_p$ be given by $w(i) = a_i$. Then $\Sigma_p(\alpha) = w(M)$. Observe that the cocycles of this matroid are also all sets of size p, so every cocycle includes a base. Now the result follows from Corollary 8.2. □

Theorem 8.3 (Bialostocki Dierker) *If $w : E(K_{p+1}) \to \mathbb{Z}_p$, then there is a spanning tree with weight* 0.

Proof: Let M be the cycle matroid of K_{p+1}. Then every cocycle includes a base, and $rk(M) = p$, so the result follows immediately from Corollary 8.2 □

The following is a very intersting conjecture due to Schrijver and Seymour which offers a generalization of their result to all abelian groups.

Conjecture 8.4 (Schrijver-Seymour) *Let M be a matroid, let G be an additive abelian group, and let $w : E \to G$ be a weight function. If $H = \mathcal{S}(w(M))$, then $|w(M)| \geq |H| \cdot (\sum_{Q \in R/H} rk(w^{-1}(Q)) - rk(M) + 1)$.*

Our next result is a proof of the above conjecture for uniform matroids. We will state and prove this result in the context of subsequence sums, but will require a bit of added notation. Let α be a sequence in G given by $a_1, a_2, \ldots, a_n$. If $I \subseteq \{1, 2, \ldots, n\}$ we let $\alpha|_I$ denote the subsequence consisting of only those terms with index in I. Similarly, if $R \subseteq G$ we let $\alpha|^R$ denote the subsequence of α consisting of only those terms which lie in R. If $k \leq n$, we let $\Sigma_k(\alpha) = \{a_{i_1} + a_{i_2} + \ldots + a_{i_k} : 1 \leq i_1 < i_2 \ldots < i_k \leq n\}$. If

$Q \subseteq G$ we let $\rho_Q(\alpha) = |\{1 \le i \le n : a_i \in Q\}|$ and if j is a nonnegative integer, we let $\rho_Q(\alpha, j) = \min\{\rho_Q(\alpha), j\}$. If $H \le G$ and $R \subseteq G$ we define

$$\Xi_H^R(\alpha, j) = |H| \cdot \sum_{\substack{Q \in G/H \\ Q \subseteq R}} \rho_Q(\alpha, j).$$

Further, we let $\Xi_H(\alpha, j) = \Xi_H^G(\alpha, j)$ and we let $\Xi(\alpha, j) = \Xi_{\{0\}}(\alpha, j)$. With this, we are ready to state the theorem.

Theorem 8.5 (DeVos, Goddyn, Mohar) *If α is a sequence in G and $J = \mathcal{S}(\Sigma_k(\alpha))$, then $|\Sigma_k(\alpha)| \ge \Xi_J(\alpha, k) - |J|(k-1)$*

Proof: We proceed by induction on $k + |\Sigma_k(\alpha)|$. The theorem holds trivially if $k = 1$ so we may assume that $k > 1$. Let α be given by $a_1, a_2, \ldots, a_n$. If $J \ne \{0\}$ then let $\phi : G \to G/J$ be the canonical homomorphism and let α_ϕ be the sequence $\phi(a_1), \phi(a_2), \ldots, \phi(a_n)$. Then $\Sigma_k(\alpha_\phi)$ has trivial stabilizer, so by induction we have $|\Sigma_k(\alpha)| = |J| \cdot |\Sigma_k(\alpha_\phi)| \ge |J|(\Xi(\alpha_\phi, k) - (k-1)) = \Xi_J(\alpha, k) - |J|(k-1)$ as required. Thus, we may assume that $J = \{0\}$. Let $X = \{a_1\} \cup \{g \in G : \rho_g(\alpha) \ge k\}$ and let α' denote the sequence $a_2, a_3, \ldots, a_n$. Set $C_0 = X + \Sigma_{k-1}(\alpha')$, let $H_0 = \mathcal{S}(C_0)$ and observe that $C_0 \subseteq \Sigma_k(\alpha)$.

We define a set C with $C_0 \subseteq C \subseteq \Sigma_k(\alpha)$ to be a *portion* if setting $H = \mathcal{S}(C)$ we have the following inequality

$$|C| \ge |X| + \Xi_H(\alpha', k-1) - |H|(k-1)$$

We claim that C_0 is a portion. To see this, let $\psi : G \to G/H_0$ be the canonical homomorphism and let α'_ψ be the sequence $\psi(a_2), \psi(a_3), \ldots, \psi(a_n)$. Now $\mathcal{S}(\psi(C_0))$ is trivial, so $\mathcal{S}(\Sigma_{k-1}(\alpha'_\psi))$ is also trivial. Thus, by applying induction to $\Sigma_{k-1}(\alpha'_\psi)$ and Kneser's theorem for the sumset $\psi(X) + \Sigma_{k-1}(\alpha'_\psi)$ we find

$$\begin{aligned} |C_0| &= |H_0| \cdot |\psi(X) + \Sigma_{k-1}(\alpha'_\psi)| \\ &\ge |H_0| \cdot \big(|X|/|H_0| + \Xi(\alpha'_\psi, k-1) - (k-2) - 1\big) \\ &= |X| + \Xi_{H_0}(\alpha', k-1) - |H_0|(k-1) \end{aligned}$$

as desired. Thus, a portion exists, and we now choose a portion C with $H = \mathcal{S}(C)$ minimal. Suppose $H = \{0\}$. Then we have

$$|\Sigma_k(\alpha)| \ge |C| \ge |X| + \Xi(\alpha', k-1) - (k-1) = \Xi(\alpha, k) - (k-1)$$

and we are finished. Thus, we may assume (for a contradiction) that $H \neq \{0\}$. Since $\mathcal{S}(\Sigma_k(\alpha)) = \{0\}$ we may choose a set $I \subseteq \{1, 2, \ldots, n\}$ with $|I| = k$ so that $\sum_{i \in I} a_i + H \not\subseteq \Sigma_k(\alpha)$. Let $\{R_1, R_2, \ldots, R_\ell\} = \{a_i + H : i \in I\}$ and for $1 \leq j \leq \ell$ let $d_j = \rho_{R_j}(\alpha|_I)$. Next we define the following sumset

$$D = \Sigma_{d_1}(\alpha|^{R_1}) + \Sigma_{d_2}(\alpha|^{R_2}) + \ldots + \Sigma_{d_\ell}(\alpha|^{R_\ell})$$

and we set $C' = C \cup D$ and $H' = \mathcal{S}(D)$. By construction D is a proper subset of an H-coset and D is disjoint from C. Thus $H' = \mathcal{S}(C')$. If $\psi : G \to G/H'$ is the canonical homomorphism, then $\psi(D)$ has trivial stabilizer, so the image under ψ of each $\Sigma_{d_j}(\alpha|^{R_j})$ has trivial stabilizer. Thus, applying our argument inductively and then applying Kneser's theorem (all in the quotient group) we find that

$$\begin{aligned} |D| &\geq \sum_{j=1}^{\ell} (\Xi_{H'}^{R_j}(\alpha, d_j) - |H'|(d_j - 1)) - |H'|(\ell - 1) \\ &\geq \sum_{j=1}^{\ell} \Xi_{H'}^{R_j}(\alpha', d_j) - |H'|(k - 1) \end{aligned} \tag{1}$$

The next inequality follows immediately from our definitions and the observation that each H-coset other than $R_1, R_2, \ldots, R_\ell$ contributes at least as much to $\Xi_H(\alpha', k-1)$ as it does to $\Xi_{H'}(\alpha', k-1)$.

$$\Xi_H(\alpha', k-1) - \Xi_{H'}(\alpha', k-1) \geq \sum_{j=1}^{\ell} \Big(|H| \rho_{R_j}(\alpha', k-1) - \Xi_{H'}^{R_j}(\alpha', k-1) \Big) \tag{2}$$

If $\ell > 1$, then for every $1 \leq j \leq \ell$ we have $d_j \leq k - 1$ and we find

$$|H| \rho_{R_j}(\alpha', k-1) - |H| d_j \geq \Xi_{H'}^{R_j}(\alpha', k-1) - \Xi_{H'}^{R_j}(\alpha', d_j).$$

So in this case the right hand side of Equation (2) is at least $\sum_{j=1}^{\ell} (|H| d_j - \Xi_{H'}^{R_j}(\alpha', d_j)) = |H|k - \sum_{j=1}^{\ell} \Xi_{H'}^{R_j}(\alpha', d_j) \geq |H|(k-1) - \sum_{j=1}^{\ell} \Xi_{H'}^{R_j}(\alpha', d_j)$. On the other hand, if $\ell = 1$, then $d_1 = k$ and we find that the right hand side of Equation (2) is equal to $|H|(k-1) - \Xi_{H'}^{R_1}(\alpha', k-1) \geq |H|(k-1) - \Xi_{H'}^{R_1}(\alpha', d_1)$. So, in either case we obtain the following.

$$\Xi_H(\alpha', k-1) - \Xi_{H'}(\alpha', k-1) \geq |H|(k-1) - \sum_{j=1}^{\ell} \Xi_{H'}^{R_j}(\alpha', d_j) \tag{3}$$

Now combining our results from equations (1) and (3) we have

$$\begin{aligned}
|C'| &= |C| + |D| \\
&\geq (|X| + \Xi_H(\alpha', k-1) - |H|(k-1)) + \left(\sum_{j=1}^{\ell} \Xi_{H'}^{R_j}(\alpha', d_j) - |H'|(k-1) \right) \\
&\geq |X| + \Xi_{H'}(\alpha', k-1) - |H'|(k-1).
\end{aligned}$$

Thus, we deduce that C' is a portion, and this contradicts our choice of C. □