14 Transpositions

Definition. A *cycle* is a permutation A with the property that the cycle representation of A has exactly one cycle. For instance $A = (a_1 a_2 \dots a_k)$. We call k the *length* of the cycle.

Note: It may seem that there is ambiguity about an expression such as (164)(29)(8735). Is this one permutation with three cycles, or a product of the three cycles (164), (29), and (8735)? Fortunately, the permutation (164)(29)(8735) is equal to the product of the three cycles (164), (29), and (8735), so there is no trouble.

Definition. A *transposition* is a cycle of length 2. So, in cycle notation, a transposition has the form (ab). Note that every transposition is its own inverse: $(ab)(ab) = I$.

Lemma 14.1. *Every permutation can be expressed as a product of transpositions.*

Proof. A quick check reveals that a cycle $(a_1, a_2, \dots, a_k)$ can be represented as follows:

$$(a_1 a_2 a_3 \dots a_k) = (a_1 a_k) \dots (a_1 a_4)(a_1 a_3)(a_1 a_2)$$

Since every permutation is a product of cycles, every permutation may be represented as a product of transpositions. □

Example: Represent the permutation $(13584)(2967) \in S_9$ as a product of transpositions.

$$(13584)(2967) = (14)(18)(15)(13)(27)(26)(29)$$

Note: Every permutation can be expressed as a product of transpositions in many (actually infinitely many) ways. For instance, the permutation (13584)(2967) from the above example can also be expressed in all of the following ways

$$\begin{aligned}(13584)(2967) &= (72)(38)(17)(28)(47) \\ &= (69)(64)(68)(12)(17)(15)(13)(56)(24) \\ &= (91)(95)(98)(94)(93)(92)(97)(96)(45)(83)(12)\end{aligned}$$

We have just represented a particular permutation as a product of 5, 9, and 11 transpositions—all odd numbers. This is not a coincidence! In fact for every permutation either all such expressions will all have an even number of terms, or all such expressions will have an odd number of terms. We next prove that this property holds for the identity.

Lemma 14.2. *If $T_1, \ldots, T_m$ are transpositions and $T_1 T_2 \cdots T_m = I$, then m is even.*

Proof. We return to our original way of thinking about composition using a figure where each bijection in is represented by arrows from $1, 2, \ldots, n$ to $1, 2, \ldots, n$ as follows:

Now, if we start at the rightmost 1 and follow the arrows we go along a path, finally ending up at the leftmost 1 (since the product $T_1 T_2 \cdots T_m$ is the identity). Let's imagine this path as a string which we call *strand* 1. Similarly, for every $2 \leq i \leq n$ we have a strand starting and ending at i. The figure may be complicated since the strands may cross one another many times, but nevertheless, we can reason about these crossings. Make the following definitions:

1. Let $c_{i,j}$ be the number of times strand i and strand j cross.

2. Let c be the total number of times one strand crosses another.

3. Let t_k be the number of times one strand crosses another in the transposition T_k.

For any two distinct strands, say i and j, it must be that $c_{i,j}$ is even, since these strands must cross an even number of times in order to end in the same positions in which they begin. It follows from this that the total number of crossings c must also be even.
Next let us think about the number of crossings contributed by a single transposition T_k. Suppose (without loss) that $T_k = (ab)$ where $a < b$. If $b = a + 1$ then the only strands crossing in T_k are strand a and $a+1$ so $t_k = 1$. If $b = a+2$ then strands a and $a+2$ cross in T_k but strand $a+1$ also gets crossed by strand a and $a+2$ for a total of 3 crossings. More generally, if $b = a + p$ then strands a and $a+p$ will cross one another, and both of these strands will cross all of the strands numbered $a+1, a+2, \ldots, a+p-1$. This gives a total of $1 + 2(p-1)$ crossings, so $t_k = 1 + 2(p-1)$ is odd. The total number of crossings can also be counted by summing the contributions from each transposition, giving us the equation

$$c = t_1 + t_2 + \ldots + t_m$$

Now c is even, but each t_k is odd, and it follows that m must be even, as desired. $\square$

Now we are ready to prove that this even/odd property holds for every permutation.

Theorem 14.3. *For every permutation $A \in S_n$, either every representation of A as a product of transpositions has an odd number of transpositions, or every such representation has an even number of transpositions.*

Proof. Let $T_1, \ldots, T_j$ and $U_1, \ldots, U_k$ be transpositions satisfying

$$\begin{aligned} A &= T_1 T_2 \cdots T_j \\ &= U_1 U_2 \cdots U_k \end{aligned}$$

To prove the theorem it suffices to show that either j and k are both even, or j and k are both odd. For any product of permutations $B = B_1 B_2 \cdots B_{m-1} B_m$ the inverse is always given by $B^{-1} = B_m^{-1} B_{m-1}^{-1} \cdots B_2^{-1} B_1^{-1}$. Since every transposition is its own inverse, we can therefore express A^{-1} as

$$A^{-1} = U_k U_{k-1} \cdots U_2 U_1$$

Now we have

$$I = AA^{-1} = T_1 \cdots T_j U_k \cdots U_1$$

By the previous lemma we deduce that $j + k$ is even, so either j and k are both even, or j and k are both odd, as desired. □

Definition. We call a permutation $A \in S_n$ *even* if it can be represented as a product of an even number of transpositions, and *odd* if it can be represented as a product of an odd number of transpositions. By Theorem 14.3 every permutation is either even and not odd, or is odd and not even. We call this characteristic (even or odd) the *parity* of A.

Note: If A is a cycle of length k, say $A = (a_1 a_2 \ldots a_k)$ then we can express A as $A = (a_1 a_k) \ldots (a_1 a_3)(a_1 a_2)$. Therefore a cycle of even length is an odd permutation, and a cycle of odd length is an even permutation!

Observation 14.4. *If $A, B \in S_n$ then the product AB satisfies*

$$AB \text{ is } \left\{ \begin{array}{c} \text{even} \\ \text{odd} \end{array} \right\} \text{ if } \left\{ \begin{array}{c} A, B \text{ are either both even or both odd} \\ \text{one of } A, B \text{ is even and the other is odd} \end{array} \right\}.$$

Proof. Express A and B as products of transpositions

$$A = T_1 T_2 \cdots T_j$$
$$B = U_1 U_2 \cdots U_k$$

Now $AB = T_1 T_2 \cdots T_j U_1 U_2 \cdots U_k$ so AB is even if $j + k$ is even and odd if $j + k$ is odd. $\square$

The previous observation shows that parity of permutations acts just like the parity of integers: Adding two integers that are both even or both odd gives an even integer; adding two integers with one odd and the other even gives an odd integer.

Proposition 14.5. *A permutation is* $\left\{ \begin{array}{c} \textit{even} \\ \textit{odd} \end{array} \right\}$ *if, in cycle notation, there are an* $\left\{ \begin{array}{c} \textit{even} \\ \textit{odd} \end{array} \right\}$ *number of cycles of even length.*

Proof. This follows from the previous observation and the fact that a cycle of odd length is an even permutation while a cycle of even length is an odd one. $\square$

Example: The parity of the permutation (13)(94)(657)(28) is odd since this permutation has three cycles of even length.