17 Isomorphism

Cayley tables

Definition. If a group $\mathcal{G}$ has elements $G_1, G_2, \ldots, G_m$ then we can make a multiplication table for it. The multiplication table will be $m \times m$ and will have G_iG_j in position (i,j). We call this multiplication table a *Cayley table*. (Note that if we order the elements differently, we get a different Cayley table)

Example: For the group $C_5 = \{R_a \mid a \in \mathbb{Z}_5\} = \{I, R_1, R_2, R_3, R_4\}$, the Cayley table is easy to compute, since it is given by the rule $R_aR_b = R_{a+b}$. So, the Cayley table for $\mathbb{Z}_5$ is

$\cdot$	I	R_1	R_2	R_3	R_4
I	I	R_1	R_2	R_3	R_4
R_1	R_1	R_2	R_3	R_4	I
R_2	R_2	R_3	R_4	I	R_1
R_3	R_3	R_4	I	R_1	R_2
R_4	R_4	I	R_1	R_2	R_3

Example: For the group $D_3 = \{I, R_1, R_2, M_0, M_1, M_2\}$ the Cayley table is

$\cdot$	I	R_1	R_2	M_0	M_1	M_2
I	I	R_1	R_2	M_0	M_1	M_2
R_1	R_1	R_2	I	M_1	M_2	M_0
R_2	R_2	I	R_1	M_2	M_0	M_1
M_0	M_0	M_2	M_1	I	R_2	R_1
M_1	M_1	M_0	M_2	R_1	I	R_2
M_2	M_2	M_1	M_0	R_2	R_1	I

As a reminder of how these entries are computed, to determine M_1R_2 we compute

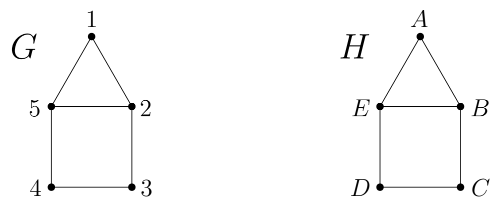
$$M_1R_2(x) = M_1(R_2(x)) = M_1(x+2) = -(x+2)+1 = -x-1 = -x+2 = M_2(x)$$

It follows that $M_1R_2 = M_2$.

Note: Cayley tables are useful since they give us a total picture of how the product operation works. However in practice they can be cumbersome to determine since they have m^2 entries when the group has size m.

Isomorphism

It is common in mathematics to find two objects with the same structure but different names. For instance, consider the two graphs G and H shown in the figure below.



The graphs G and H are not equal since they have different vertex sets. However, we could change from G to H by simply changing names. More precisely, if the vertices of G are ordered $(1, 2, 3, 4, 5)$ and those of H are ordered (A, B, C, D, E), then we could change from G to H by changing the name of the i^{th} vertex of G to the i^{th} vertex of H. In this case we say that the graphs G and H are isomorphic. More generally in mathematics, two objects are called isomorphic if they have the same structure (but possibly different names).

Definition. If $\mathcal{F}$ and $\mathcal{G}$ are groups, we say that $\mathcal{F}$ and $\mathcal{G}$ are *isomorphic* if the elements of $\mathcal{F}$ can be ordered $F_1, \ldots, F_m$ and the elements of $\mathcal{G}$ can be ordered $G_1, \ldots, G_m$ so that starting from a Cayley table for $\mathcal{F}$ we can move to a Cayley table for $\mathcal{G}$ by replacing each F_i by G_i.

Example: The group S_3 has Cayley table

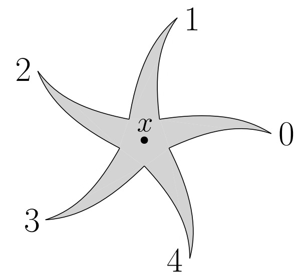
$\cdot$	I	(123)	(132)	(12)	(13)	(23)
I	I	(123)	(132)	(12)	(13)	(23)
(123)	(123)	(132)	I	(13)	(23)	(12)
(132)	(132)	I	(123)	(23)	(12)	(13)
(12)	(12)	(23)	(13)	I	(132)	(123)
(13)	(13)	(12)	(23)	(123)	I	(132)
(23)	(23)	(13)	(12)	(132)	(123)	I

We can change this Cayley table for S_3 into the Cayley table for D_3 shown before by changing each element of S_3 in the list $(I, (123), (132), (12), (13), (23))$ into the corresponding element of D_3 from the list $(I, R_1, R_2, M_0, M_1, M_2)$. Thus S_3 is isomorphic to D_3.

Symmetry

Definition. If $\mathbf{x} \in \mathbb{R}^2$ and $t \in \mathbb{R}$ we define $R_{\mathbf{x},t}$ to be the transformation of $\mathbb{R}^2$ given by rotating by an angle of t around the point $\mathbf{x}$. We call any such transformation a *rotation*.

Example: Consider the starfish shape in the figure below



The symmetries of this shape are precisely the identity I and the rotations $R_{\mathbf{x},\frac{2\pi}{5}}$, $R_{\mathbf{x},\frac{4\pi}{5}}$, $R_{\mathbf{x},\frac{6\pi}{5}}$, and $R_{\mathbf{x},\frac{8\pi}{5}}$ (we will discuss why later). This set of symmetries is a subgroup of Trans($\mathbb{R}^2$) with Cayley table

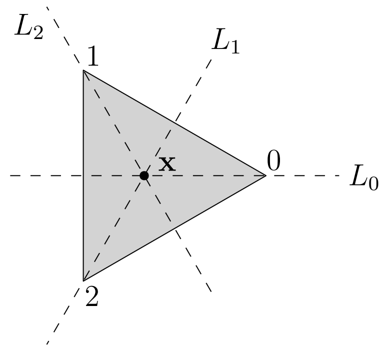
$\cdot$	I	$R_{\mathbf{x},\frac{2\pi}{5}}$	$R_{\mathbf{x},\frac{4\pi}{5}}$	$R_{\mathbf{x},\frac{6\pi}{5}}$	$R_{\mathbf{x},\frac{8\pi}{5}}$
I	I	$R_{\mathbf{x},\frac{2\pi}{5}}$	$R_{\mathbf{x},\frac{4\pi}{5}}$	$R_{\mathbf{x},\frac{6\pi}{5}}$	$R_{\mathbf{x},\frac{8\pi}{5}}$
$R_{\mathbf{x},\frac{2\pi}{5}}$	$R_{\mathbf{x},\frac{2\pi}{5}}$	$R_{\mathbf{x},\frac{4\pi}{5}}$	$R_{\mathbf{x},\frac{6\pi}{5}}$	$R_{\mathbf{x},\frac{8\pi}{5}}$	I
$R_{\mathbf{x},\frac{4\pi}{5}}$	$R_{\mathbf{x},\frac{4\pi}{5}}$	$R_{\mathbf{x},\frac{6\pi}{5}}$	$R_{\mathbf{x},\frac{8\pi}{5}}$	I	$R_{\mathbf{x},\frac{2\pi}{5}}$
$R_{\mathbf{x},\frac{6\pi}{5}}$	$R_{\mathbf{x},\frac{6\pi}{5}}$	$R_{\mathbf{x},\frac{8\pi}{5}}$	I	$R_{\mathbf{x},\frac{2\pi}{5}}$	$R_{\mathbf{x},\frac{4\pi}{5}}$
$R_{\mathbf{x},\frac{8\pi}{5}}$	$R_{\mathbf{x},\frac{8\pi}{5}}$	I	$R_{\mathbf{x},\frac{2\pi}{5}}$	$R_{\mathbf{x},\frac{4\pi}{5}}$	$R_{\mathbf{x},\frac{6\pi}{5}}$

The group of symmetries of this shape is isomorphic to C_5. To see why, note that replacing $R_{\mathbf{x},\frac{2\pi j}{5}}$ with R_j brings us from this Cayley table to that of C_5 from earlier in this section.

Note: Here we see the value of isomorphism. If we were to move our starfish figure to be centered at a different point $\mathbf{x}'$ in the plane we would have another shape with a different set of symmetries, but the group of symmetries would still be isomorphic to C_5. More generally, all of the following shapes have symmetry group isomorphic to C_5.

Definition. If L is a line in the plane, then we define M_L to be the transformation of $\mathbb{R}^2$ given my performing a mirror reflection about the line L.

Example: Consider the triangle centred at $\mathbf{x}$ in the figure below.



The symmetries of this triangle are precisely the identity I, the rotations $R_{\mathbf{x},\frac{2\pi}{3}}$, $R_{\mathbf{x},\frac{4\pi}{3}}$, and the mirror symmetries M_{L_0}, M_{L_1}, and M_{L_2}. This set of symmetries is a subgroup of $\text{Trans}(\mathbb{R}^2)$ with Cayley table

$\cdot$	I	$R_{\mathbf{x},\frac{2\pi}{3}}$	$R_{\mathbf{x},\frac{4\pi}{3}}$	M_{L_0}	M_{L_1}	M_{L_2}
I	I	$R_{\mathbf{x},\frac{2\pi}{3}}$	$R_{\mathbf{x},\frac{4\pi}{3}}$	M_{L_0}	M_{L_1}	M_{L_2}
$R_{\mathbf{x},\frac{2\pi}{3}}$	$R_{\mathbf{x},\frac{2\pi}{3}}$	$R_{\mathbf{x},\frac{4\pi}{3}}$	I	M_{L_1}	M_{L_2}	M_{L_0}
$R_{\mathbf{x},\frac{4\pi}{3}}$	$R_{\mathbf{x},\frac{4\pi}{3}}$	I	$R_{\mathbf{x},\frac{2\pi}{3}}$	M_{L_2}	M_{L_0}	M_{L_1}
M_{L_0}	M_{L_0}	M_{L_2}	M_{L_1}	I	$R_{\mathbf{x},\frac{4\pi}{3}}$	$R_{\mathbf{x},\frac{2\pi}{3}}$
M_{L_1}	M_{L_1}	M_{L_0}	M_{L_2}	$R_{\mathbf{x},\frac{2\pi}{3}}$	I	$R_{\mathbf{x},\frac{4\pi}{3}}$
M_{L_2}	M_{L_2}	M_{L_1}	M_{L_0}	$R_{\mathbf{x},\frac{4\pi}{3}}$	$R_{\mathbf{x},\frac{2\pi}{3}}$	I

The group of symmetries of this shape is isomorphic to D_3. To see why, note that replacing $R_{\mathbf{x},\frac{2\pi j}{3}}$ with R_j and replacing each M_{L_i} with M_i brings us from the above Cayley table to the earlier one for D_3.

Note: Again here we see the value of isomorphism. We could place a regular pentagon in a variety of different positions in the plane and these shapes would all have symmetry groups that are different, but isomorphic. More generally, all of the following shapes, no matter how they are positioned in the plane will have symmetry group isomorphic to D_5.