8 Colouring Planar Graphs

The Four Colour Theorem

Lemma 8.1 *If G is a simple planar graph, then*

(i) $12 \leq \sum_{v \in V(G)}(6 - deg(v))$ *with equality for triangulations.*

(ii) *G has a vertex of degree ≤ 5.*

Proof: For (i), note that by Lemma 7.5 we have $12 \leq 6|V(G)| - 2|E(G)| = \sum_{v \in V(G)}(6 - deg(v))$ with equality for triangulations. Part (ii) follows immediately from this. □

Theorem 8.2 (Heawood) *Every loopless planar graph is 5-colourable.*

Proof: We proceed by induction on $|V(G)|$. As a base, note that the result is trivial when $|V(G)| = 0$. For the inductive step, let G be a planar graph with $|V(G)| > 0$. By removing parallel edges, we may also assume that G is simple. Now by (ii) of the previous lemma, we may choose $v \in V(G)$ with $deg(v) \leq 5$. By the inductive hypothesis, we may choose a 5-colouring of $G - v$. If there is a colour which does not appear on a neighbor of v, then we may extend this colouring to a 5-colouring of G. Thus, we may assume that v has exactly 5 neighbors, v_1, v_2, v_3, v_4, v_5 appearing in this order clockwise around v (in our embedding), and we may assume that v_i has colour i for $i = 1 \ldots 5$.

For every $1 \leq i < j \leq 5$ let G_{ij} be the subgraph of $G - v$ induced by the vertices of colour i and j. Note that the colouring obtained by switching colours i and j on any component of G_{ij} is still a colouring of $G - v$. Now, consider the component of G_{13} which contains the vertex v_1. If this component does not contain v_3, then by switching colours 1 and 3 on it, we obtain a 5-colouring of $G - v$ where no neighbor of v has colour 1, and this may be extended to a 5-colouring of G. Thus, we may assume that the component of G_{13} containing v_1 also contains v_3. So, in particular, there is a path in $G - v$ containing only vertices of colour 1 and 3 joining v_1 and v_3. This path may be completed to a cycle by adding v, and this cycle separates v_2 and v_4. It follows that the component of G_{24} containing v_2 does not contain v_4. By switching colours on this component, we obtain a 5-colouring of $G - v$ where no neighbor of v has colour 2. This may then be extended to a 5-colouring of G as required. □

Theorem 8.3 (The Four Colour Theorem - Appel, Haken) *Every loopless planar graph is 4-colourable.*

Proof: The proof involves a finite set $\mathcal{X}$ of planar graphs, and splits into two parts. First, it is proved that every (sufficiently well connected) planar graph contains at least one of the graphs in $\mathcal{X}$ as a subgraph. Second, it is proved that every graph in $\mathcal{X}$ is reducible in the sense that whenever G contains a graph in $\mathcal{X}$ as a subgraph, this subgraph may be deleted or replaced by something smaller in such a way that every 4-colouring of this new graph can be extended to a 4-colouring of the original graph. We give two easy examples of this in the next two lemmas. See "http://www.math.gatech.edu/˜thomas/FC/fourcolor.html" for a more detailed description (you can also access this page by typing "4 color theorem" into Google and clicking "I'm Feeling Lucky") $\square$

Lemma 8.4 (Birkhoff) *Let G be a plane graph which contains the subgraph in Figure 1 embedded as shown. Let G' be the (plane) graph obtained from G by deleting vertices w, x, y, z, identifying a and c, and then adding an edge between d and f (as shown in Figure 2). Then G is 4-colourable if G' is 4-colourable.*

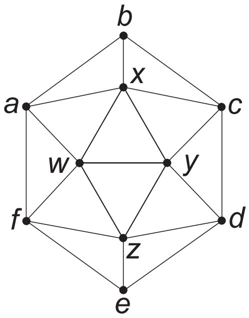

Figure 1

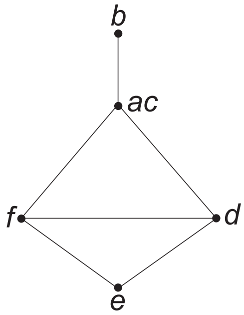

Figure 2

Proof: Consider a 4-colouring of G'. We may assume that vertex ac has colour 1, vertex d has colour 2, and vertex f has colour 3. By removing the edge df and then splitting ac back to a and c, we obtain a 4-colouring of $G - \{w, x, y, z\}$ where vertices a and c have colour 1, d has colour 2, and f has colour 3. If b is not colour 4, then we may assign x colour 4, w colour 2, y colour 3, and z either colour 1 or 4 (depending on the colour of e) to achieve a 4-colouring of G. Thus, we may assume that b has colour 4. If e has colour 4, then giving

w, x, y, z the colours $3, 2, 4, 1$ respectively yields a colouring, so we may also assume that e has colour 1.

For every 4-colouring of $G - \{w, x, y, z\}$ using the colours $\{1, 2, 3, 4\}$ and every $1 \leq i < j \leq 4$, we let G_{ij} be the subgraph of $G - \{w, x, y, z\}$ induced by the vertices of colours i and j. If the component of G_{14} containing e does not contain b, then switching colours on this component changes e to colour 4 and does not effect any of a, b, c, d, f bringing us back to a previously handled case. Thus, we may assume that there is a path with vertices coloured 1 and 4 joining b and e. It follows that the component of G_{23} containing d does not contain f. By switching colours on this component, we get a colouring of $G - \{w, x, y, z\}$ where a, b, c, d, e, f have colours $1, 4, 1, 3, 1, 3$ respectively. Now consider the component of G_{12} containing e. If this component does not contain a or c, then we may switch colours on it, and extend to a colouring of G by assigning w, x, y, z the colours $4, 3, 2, 1$ respectively. Thus, by symmetry, we may assume that there is a path of vertices with colours 1 and 2 joining e and a. It follows from this that the component of G_{34} containing f does not contain b or d. By switching colours on this component, and then assigning w, x, y, z the colours $3, 2, 4, 2$ we obtain a 4-colouring of G. This completes the proof. $\square$

Lemma 8.5 *Let G be a triangulation of the plane. Then must contain one of the following configurations.*

(i) *A vertex with degree ≤ 4.*

(ii) *Two adjacent vertices with degree 5.*

(iii) *A triangle with vertices of degree 5, 6, 6.*

Proof: We shall assume that every vertex of G has degree ≥ 5 and show that one of the other two outcomes must occur. For every vertex v, put a *charge* of $3(6 - deg(v))$ on v. By (i) of Lemma 8.1 we have that $36 = \sum_{v \in V(G)} 3(6 - deg(v))$, so the sum of the charges is 36. Next, move one unit of charge from each vertex v of degree 5 to each neighbor of v with degree ≥ 7. Now, consider a vertex u with charge > 0 (one must exist since they sum to 36). First suppose that u has degree 5. Then it began with a charge of 3, so it must have lost ≤ 2, so it has ≤ 2 neighbors of degree ≥ 7. But then, either G has a neighbor of degree 5 (config. (ii)) or two adjacent neighbors of degree 6 (config (iii)) so we are done. The degree of u cannot be 6, since such vertices have 0 charge. If u has degree 7, then it began with a charge of -3, so it must have ≥ 4 neighbors of degree 5, two of which must be adjacent. Similarly, if u has degree 8, then it began with a charge of -6, so it must have ≥ 7 neighbors of degree 5, two of which must be adjacent. Finally, u cannot have degree $d \geq 9$ since in this case its initial charge would be $3(6-d) = 18 - 3d \leq -d$ and in this case it is impossible for u to end up with positive charge. $\square$

Tait's Theorem

Theorem 8.6 (Tait) *A triangulation G is 4-colourable if and only if G^* is 3-edge-colourable.*

Proof: For the "only if" direction, let $\phi : V(G) \to \mathbb{Z}_2 \times \mathbb{Z}_2$ be a 4-colouring of G. Now, define a labeling $\psi^* : E(G^*) \to \mathbb{Z}_2 \times \mathbb{Z}_2$ by the rule that if $e^* \in E(G^*)$ and e has ends x and y, then $\psi^*(e^*) = \phi(x) + \phi(y)$. Since ϕ is a colouring, $\psi^*(e^*) \neq (0,0)$ for every $e^* \in E(G^*)$. Let $a^* \in V(G^*)$ and assume that a^* is incident with the faces $x^*, y^*, z^* \in F(G^*)$. Then $\sum_{e^* \in \delta(a^*)} \psi(e^*)$ adds each of $\phi(x)$, $\phi(y)$, and $\phi(z)$ twice, so this sum is zero. The only possibility for a triple of nonzero elements in $\mathbb{Z}_2 \times \mathbb{Z}_2$ to have zero sum is if these elements are distinct. Thus ψ^* is a 3-edge-colouring of G^*.

For the "if" direction, let $\psi^* : E(G^*) \to (\mathbb{Z}_2 \times \mathbb{Z}_2) \setminus \{(0,0)\}$ be a 3-edge-colouring of G^* and let ψ be the dual map given by the rule $\psi(e) = \psi^*(e^*)$. Next we prove a key fact.

Claim: If $v_1, e_1, \ldots, v_n$ is a closed walk in G, then $\sum_{i=1}^{n-1} \psi(e_i) = (0,0)$.

Proof of Claim: It suffices to prove the claim for closed walks without repeated vertices, so we may assume $v_1, \ldots, v_{n-1}$ are distinct. The claim holds trivially for walks of length 2 which traverse the same edge twice. Otherwise, we may assume that $\{e_1, \ldots, e_{n-1}\}$ is the edge set of a cycle C in G. Now, let A be the set of faces of G which are inside C. By construction, every $a \in A$, is a triangle and ψ assigns each edge of this triangle a distinct nonzero element from $\mathbb{Z}_2 \times \mathbb{Z}_2$. It follows that the sum of ψ on the edges of this triangle is zero. Now, form a sum by adding for every $a \in A$ the sum of ψ over the edges of the triangle bounding a. As observed, this sum must be zero. However, since every edge not in C is counted twice, and every edge in C is counted once, this is also the sum of ϕ on the edges of C.

Now, choose a vertex $u \in V(G)$ and define the map $\phi : V(G) \to \mathbb{Z}_2 \times \mathbb{Z}_2$ by the rule that $\phi(v)$ is the sum of ψ on the edges of a walk from u to v. It follows from the claim that this sum is independent of the choice of walk. Further, if $v, w \in V(G)$ are joined by the edge e, then we may construct a walk from u to w by first walking to v and then traversing the edge e. It follows that $\phi(w) = \phi(v) + \psi(e) \neq \phi(v)$, so ϕ is a 4-colouring of G. □

Corollary 8.7 *The following statements are equivalent.*

(i) *Every loopless planar graph is 4-colourable (The Four Colour Theorem).*

(ii) *Every loopless triangulation of the plane is 4-colourable.*

(iii) *Every 2-edge-connected 3-regular plane graph is 3-edge-colourable.*

Proof: To see that (ii) $\Leftrightarrow$ (iii), note that if G, G^* are connected dual planar graphs, then G is a loopless triangulation if and only if G^* is 2-edge-connected and 3-regular (loops are dual to cut-edges by (i) of Proposition 7.4). It follows from this and Tait's Theorem that (ii) $\Leftrightarrow$ (iii). It is immediate that (i) $\Rightarrow$ (ii). To see that (ii) $\Rightarrow$ (i), assume (ii) holds and let G be a loopless plane graph. By adding edges to G we may form a loopless triangulation. By (ii) this new graph has a 4-colouring, and this is also a 4-colouring of G. □

Choosability

Theorem 8.8 (Voigt) *There exists a loopless planar graph which is not 4-choosable.*

Proof: Homework.

Theorem 8.9 (Thomassen) *Every loopless planar graph is 5-choosable.*

Proof: Since adding edges cannot reduce the list chromatic number (and the result is trivial for graphs with < 2 vertices), it suffices to prove the following stronger statement.

Claim: Let G be a connected plane graph with all finite faces of length 3, let v_1, v_2 be distinct adjacent vertices which lie on the infinite face, and let $L : V(G) \to \mathbb{N}$ be a list assignment. If the following conditions are satisfied, then G is L-choosable:

- $|L(v)| \geq 5$ if v does not lie on the infinite face.
- $|L(v)| \geq 3$ if $v \neq v_1, v_2$ and v lies on the infinite face.
- $|L(v_1)| = |L(v_2)| = 1$ and $L(v_1) \neq L(v_2)$.

We prove the claim by induction on $|V(G)|$. As a base case, observe that the result holds trivially when $|V(G)| = 2$. For the inductive step, let G and L satisfy the above conditions, and assume that the claim holds for any graph with fewer vertices.

Suppose the infinite face is not bounded by a cycle.

In this case, there exists a proper 1-separation (H_1, H_2) of G with $V(H_1) \cap V(H_2) = \{u\}$ so that u lies on the infinite face. Since v_1 and v_2 are adjacent, we must have either $v_1, v_2 \in V(H_1)$ or $v_1, v_2 \in V(H_2)$ and we may assume the former case without loss. By induction, we may choose a colouring ϕ of H_1 so that every vertex receives a colour from its list. Now, modify the list of u by setting $L(u) = \{\phi(u)\}$. Choose a neighbor u' of u in H_2 which lies on the infinite face, choose a colour $q \in L(u')$ so that $q \neq \phi(u)$ and set $L(u') = \{q\}$. By applying the claim inductively to H_2 where u and u' play the roles of v_1 and v_2, we obtain a colouring of H_2 so that every vertex receives a colour from its list. Merging these two colourings gives us a colouring of G. Thus, we may assume that the infinite face is bounded by a cycle.

Suppose the cycle bounding the infinite face is not induced.

In this case, there exists a proper 2-separation (H_1, H_2) of G with $V(H_1) \cap V(H_2) = \{u, w\}$ where u, w lie on the cycle C bounding the infinite face, and u, v are adjacent in G but not in C. Since v_1 and v_2 are adjacent, we must have either $v_1, v_2 \in V(H_1)$ or $v_1, v_2 \in V(H_2)$ and we may assume the former case without loss. By induction, we may choose a colouring ϕ of H_1 so that every vertex receives a colour from its list. Modify the lists of u and w by setting $L(u) = \{\phi(u)\}$ and $L(w) = \{\phi(w)\}$. Now by applying the claim inductively to H_2 where u and w play the roles of v_1 and v_2, we obtain a colouring of H_2 so that every vertex receives a colour from its list. Merging these two colourings gives us a colouring of G. Thus, we may assume that the cycle bounding the infinite face is induced.

Let $v_1, v_2, v_3, \ldots, v_k$ be an ordering of the vertices of C so that $v_i v_{i+1} \in E(G)$ for every $1 \leq i \leq k-1$. Let $u_1, u_2, \ldots, u_\ell$ be the neighbors of v_3 which do not lie on the infinite face. Now, $|L(v_3)| \geq 3$ so we may choose a set $S \subseteq L(v_3)$ of size 2 which is disjoint from $L(v_2)$. Delete the vertex v_3 and then modify the lists of the vertices $u_1, \ldots, u_\ell$ by removing from them any colour which appears in S. By induction, we may choose a colouring of $G - v$ where every vertex receives a colour from its list. Now, none of the vertices $v_2, u_1, \ldots, u_\ell$ has a colour in S, and v_3 has only one neighbor not appearing in this list, so we may extend our colouring to a list colouring of G by giving v_3 one of the colours in S. This completes the proof. $\square$