6 Graph Colouring

In this section, we shall assume (except where noted) that graphs are loopless.

Upper and Lower Bounds

Colouring: A k-*colouring* of a graph G is a map $\phi : V(G) \to S$ where $|S| = k$ with the property that $\phi(u) \neq \phi(v)$ whenever there is an edge with ends u, v. The elements of S are called *colours*, and the vertices of one colour form a *colour class*. The *chromatic number* of G, denoted $\chi(G)$, is the smallest integer k such that G is k-colourable. If G has a loop, then it does not have a colouring, and we set $\chi(G) = \infty$.

Independent Set: A set of vertices is *independent* if they are pairwise nonadjacent. We let $\alpha(G)$ denote the size of the largest independent set in G. Note that in a colouring, every colour class is an independent set.

Clique: A set of vertices is a *clique* if they are pairwise adjacent. We let $\omega(G)$ denote the size of the largest clique in G.

Observation 6.1

$$\begin{aligned} \chi(G) &\geq \omega(G) \\ \chi(G) &\geq \frac{|V(G)|}{\alpha(G)} \end{aligned}$$

Proof: The first part follows from the observation that any two vertices in a clique must receive different colours. The second follows from the observation that each colour class in a colouring has size $\leq \alpha(G)$. □

Greedy Algorithm: Order the vertices $v_1, v_2, \ldots, v_n$ and then colour them (using positive integers) in order by assigning to v_i the smallest possible integer which is not already used on a neighbor of v_i.

Maximum and Minimum Degree: We let $\Delta(G)$ denote the maximum degree of a vertex in G and we let $\delta(G)$ denote the minimum degree of a vertex in G.

Degeneracy: A graph G is k-degenerate if every subgraph of G has a vertex of degree at most k.

Observation 6.2

$$\begin{aligned} \chi(G) &\leq \Delta(G) + 1 \\ \chi(G) &\leq k + 1 \quad \textit{if } G \textit{ is } k\textit{-degenerate} \end{aligned}$$

Proof: The first part follows by applying the greedy algorithm to any ordering of $V(G)$. For the second part, let $|V(G)| = n$, and order the vertices starting from the back and working forward by the rule that v_i is chosen to be a vertex of degree $\leq k$ in the graph $G - \{v_{i+1}, v_{i+2}, \ldots, v_n\}$. When the greedy algorithm is applied to this ordering, each vertex has $\leq k$ neighbors preceding it, so we obtain a colouring with $\leq k + 1$ colours as desired. □

Theorem 6.3 (Brooks) *If G is a connected graph which is not complete and not an odd cycle, then $\chi(G) \leq \Delta(G)$.*

Proof: Let $\Delta = \Delta(G)$. If G is $(\Delta - 1)$-degenerate, then we are done by the previous observation. Thus, we may assume that there is a subgraph $H \subseteq G$ so that H has minimum degree Δ. But then H must be Δ-regular, and no vertex in H can have a neighbor outside $V(H)$, so we find that $H = G$. It follows from this that G is Δ-regular and every proper subgraph of H is $(\Delta - 1)$-degenerate. The theorem is trivial if $\Delta = 2$, so we may further assume that $\Delta \geq 3$.

If G has a proper 1-separation (H_1, H_2), then H_1 and H_2 are $(\Delta - 1)$-degenerate, so each of these graphs is Δ-colourable. By permuting colours in the colouring of H_2, we may arrange that these two Δ-colourings assign the same colour to the vertex in $V(H_1) \cap V(H_2)$, and then combining these colourings gives a Δ-colouring of G. Thus, we may assume that G is 2-connected.

If G is 3-connected, choose $v_n \in V(G)$. If every pair of neighbors of v_n are adjacent, then G is a complete graph and we are finished. Otherwise, let v_1, v_2 be neighbors of v_n, and note that $G - \{v_1, v_2\}$ is connected.

If G is not 3-connected, choose $v_n \in V(G)$ so that $G - v_n$ is not 2-connected. Consider the block-cutpoint graph of $G - v_n$, and for $i = 1, 2$, let H_i be a leaf block of $G - v_n$ which is adjacent in the block-cutpoint graph to the cut-vertex x_i. Since G is 2-connected, for $i = 1, 2$ there exists a vertex $v_i \in V(H_i) \setminus \{x_i\}$ which is adjacent to v_n. Note that because

H_i is 2-connected, $H_i - v_i$ is connected for $i = 1, 2$ and it then follows that $G - \{v_1, v_2\}$ is connected.

So, in both cases, we have found a vertex v_n and two nonadjacent neighbors v_1, v_2 of v_n so that $G - \{v_1, v_2\}$ is connected. Next, choose an ordering $v_3, v_4, \ldots, v_n$ of the vertices of $G - \{v_1, v_2\}$ so that $i < j$ whenever $dist(v_i, v_n) > dist(v_j, v_n)$ (this can be achieved by taking a breadth first search tree rooted at v_n). We claim that the greedy algorithm will use at most Δ colours when following this order. Since v_1 and v_2 are nonadjacent, they both get colour 1. Since $v_3, \ldots, v_{n-1}$ have at least one neighbor following them, they have at most $\Delta - 1$ neighbors preceding them, so they will also receive colours which are $\leq \Delta$. Finally, since v_1 and v_2 got the same colour, there are at most $\Delta - 1$ distinct colours used on the neighbors of v_n, so v_n will also get a colour which is $\leq \Delta$. □

Colouring Structure

Theorem 6.4 (Gallai-Roy-Vitaver) *If D is an orientation of G and the longest directed path in D has length t, then $\chi(G) \leq t + 1$. Furthermore, equality holds for some orientation of G.*

Proof: We may assume without loss that G is connected. Now, let D' be a maximal acyclic subgraph of D, and note that $V(D') = V(G)$. Define a function $\phi : V(G) \to \{0, 1, \ldots, t\}$ by the rule that $\phi(v)$ is the length of the longest directed path in D' ending at v. We claim that ϕ is a colouring of G. To see this, let $(u, v) \in E(D)$. If $(u, v) \in E(D')$ and $P \subseteq D'$ is the longest directed path in D' ending at u, then appending the edge (u, v) to P yields a longer directed path in D' ending at v (it cannot form a directed cycle since D' is acyclic). It follows that $\phi(v) > \phi(u)$. If $(u, v) \notin D'$, then it follows from the maximality of D' that there must exist a directed path $Q \subseteq D'$ from v to u. Now, if P is the longest directed path in D' ending at v, we find that $P \cup Q$ is a directed path ending at u. Thus $\phi(u) > \phi(v)$. It follows that f is a $(t + 1)$-colouring of G, as required.

To see that there exists an orientation of G for which equality holds, let $k = \chi(G)$ and let $\phi : V(G) \to \{1, 2, \ldots, k\}$ be a k-colouring of G. Now, orient the edges of G to form an acyclic digraph D by the rule that every edge uv with $\phi(u) < \phi(v)$ is oriented from u to v. Now the colours increase along every directed path in D, so every such path must have length at most $k - 1$. □

Mycielski's Construction: Let G be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. We construct a new graph G' from G by the following procedure: For every $1 \leq i \leq n$, add a new vertex u_i and add an edge from u_i to every neighbor of v_i in $\{v_1, v_2, \ldots, v_n\}$. Finally, add one new vertex w and add an edge from w to every u_i.

Theorem 6.5 (Mycielski) *Let G' be a graph obtained from G by applying Mycielski's construction. Then $\chi(G') = \chi(G) + 1$. Further, if G is triangle free, then so is G'.*

Proof: We shall use the notation in the description of Mycielski's Construction and we shall assume that $\chi(G) = k$. If $\phi : V(G) \to \{1, 2, \ldots, k\}$ is a k-colouring of G, we may extend ϕ to a $(k+1)$-colouring of G' by assigning $\phi(u_i) = \phi(v_i)$ for $1 \leq i \leq n$ and then setting $\phi(w) = k + 1$. It follows from this that $\chi(G') \leq \chi(G) + 1$. Also, note that it follows from the definitions that G' will be triangle free if G is triangle free.

It remains to show that $\chi(G') \geq \chi(G)+1$. Suppose (for a contradiction) that $\phi : V(G') \to \{1, \ldots, k\}$ is a colouring of G' and assume (without loss) that $\phi(w) = k$. Note that no vertex in $\{u_1, \ldots, u_n\}$ can get colour k and let $S = \{v_i : 1 \leq i \leq n \text{ and } \phi(v_i) = k\}$. Now we shall modify ϕ by the rule that for every $v_i \in S$ we change $\phi(v_i)$ to $\phi(u_i)$. We claim that the restriction of ϕ to G is still a colouring of G. Since S is an independent set, we need only check that ϕ does not have a conflict on edges $v_i v_j$ where $v_i \in S$ and $v_j \notin S$. However, in this case the colour of v_i was changed to $\phi(u_i)$, but $u_i v_j \in E(G')$. It follows that $\phi(v_i) \neq \phi(v_j)$ after our change. Now we have now found a $(k-1)$-colouring of G, which contradicts our assumption. $\square$

Critical Graphs: If G is a graph with $\chi(G) = k$ and $\chi(H) < k$ for every proper subgraph $H \subset G$, then we say that G is *k-colour critical* or *k-critical*.

Observation 6.6 *If G is k-critical, then $\delta(G) \geq k - 1$.*

Proof: Suppose (for a contradiction) that G is k-critical and that $v \in V(G)$ satisfies $deg(v) < k - 1$. Then $G - v$ has a $(k-1)$-colouring, and this colouring extends to a $(k-1)$-colouring of G, a contradiction. $\square$

Theorem 6.7 *If G is $(k+1)$-critical, then G is k-edge-connected.*

Proof: Suppose (for a contradiction) that G is not k-edge-connected, and choose a partition $\{X, Y\}$ of $V(G)$ so that the number of edges between X and Y is at most $k-1$. Now, by our $(k+1)$-critical assumption, we may choose k colourings of both $G-Y$ and $G-X$ using the colours $\{1, \ldots, k\}$. For $1 \leq i \leq k$ let $X_i \subseteq X$ and $Y_i \subseteq Y$ be the sets of vertices which receive colour i in these colourings.

Now, we shall form a bipartite graph H with bipartition $(\{X_1, \ldots, X_k\}, \{Y_1, \ldots, Y_k\})$ by the rule that we add an edge from X_i to Y_j if there is *no* edge in G from a vertex in X_i to a vertex in Y_j. It follows from our assumptions that $E(H) \geq k^2 - (k-1) > k(k-1)$. Now, every set of m vertices in H can cover at most mk edges. It follows from this that the smallest vertex cover of H must have size at least k. But then, the König-Egervary Theorem (3.6) implies that H has a perfect matching M.

Now we shall use M to modify our k-colouring of $G-X$ by the rule that if $X_iY_j \in M$, we change all vertices in Y_j to colour i. This only permutes colour classes, so it results in a proper k-colouring of $G-X$. However, by this construction, we have that for every colour $1 \leq i \leq k$, there is no edge between a vertex in X of colour i and a vertex in Y of colour i. Thus, we have obtained a k-colouring of G. This contradicts our assumption, thus completing the proof. $\square$

Subdivision: Let $e = uv$ be an edge of the graph G and modify G to form a new graph G' by removing the edge e and then adding a new vertex w which is adjacent to u and v. We say that G' is obtained from G by *subdividing* the edge e. Any graph obtained from G by a sequence of such operations is called a *subdivision* of G.

Theorem 6.8 *Every simple graph with minimum degree* ≥ 3 *contains a subdivision of* K_4.

Proof: For inductive purposes, we shall prove the following stronger statement.

Claim: Let G be a graph with a special vertex. If G satisfies the following conditions, then it contains a subdivision of K_4.

- $|V(G)| \geq 2$.
- Every non-special vertex has degree ≥ 3.
- There are ≤ 2 edges in parallel, and any such edge is incident with the special vertex.

We prove the claim by induction on $|V(G)| + |E(G)|$. Note that G must have a vertex of degree ≥ 3 and has at most two parallel edges, so $|V(G)| \geq 3$. Let $u \in V(G)$ be the special vertex. If u has at most one neighbor, then the result follows by applying induction to $G - u$ (if u has a neighbor, use it as the special vertex). If u has exactly two neighbors, say v_1, v_2, then the result follows by applying induction to the graph G' obtained from $G - u$ by adding a new edge between v_1 and v_2 (in G' the only possible parallel edges are between v_1 and v_2 and at most one of v_1, v_2 can have degree < 3 so this may be taken as the special vertex). If $deg(u) \geq 4$, then let e be an edge in parallel if G contains one, and otherwise let e be any edge incident with u. Now $G - e$ has no parallel edges and has at most one vertex of degree < 3, so the result follows by applying induction to this graph. The only remaining case is when u has ≥ 3 neighbors, and has degree ≤ 3, so G is simple and $deg(u) = 3$. Let $\{v_1, v_2, v_3\}$ be the neighbors of u. If v_1, v_2, v_3 are pairwise adjacent, then G contains a K_4 subgraph and we are done. Otherwise, assume without loss that v_1 and v_2 are not adjacent. Now, form a graph G' from $G - u$ by adding the edge $v_1 v_2$. By induction on G' with the special vertex v_3, we find that G' contains a subdivision of K_4. However, this implies that G contains a subdivision of K_4 as well. □

Corollary 6.9 (Dirac) *Every graph of chromatic number* ≥ 4 *contains a subdivision of* K_4.

Proof: If G has $\chi(G) \geq 4$, then G contains a 4-critical subgraph G'. Now G' is a simple graph of minimum degree ≥ 3, so by the above theorem, G' (and thus G) contains a subdivision of K_4. □

Counting Colourings

For the purposes of this subsection, we shall permit graphs to have loops.

Counting Colourings For any graph G and any positive integer t, we let $\chi(G; t)$ denote the number of proper t-colourings $\phi : V(G) \to \{1, 2, \ldots, t\}$ of G. Note that ϕ need not be onto (so not all t colours must be used).

Observation 6.10

(i) $\chi(G, t) = 0$ *if* G *has a loop.*

(ii) $\chi(\bar{K}_n; t) = t^n$

(iii) $\chi(K_n; t) = t(t-1)(t-2)\ldots(t-n+1)$

(iv) $\chi(G; t) = t(t-1)^{n-1}$ *if* G *is a tree on* n *vertices.*

Proof: Parts (i) and (ii) follow immediately from the definition. For part (iii), order the vertices $v_1, v_2, \ldots, v_n$, and colour them in this order. Since there are $(t - i + 1)$ choices for the colour of v_i (and every colouring arises in this manner), we conclude that $\chi(K_n; t) = t(t-1)\ldots(t-n+1)$. For part (iv), proceed by induction on $|V(G)|$. As a base case, observe that the formula holds whenever $|V(G)| = 1$. For the inductive step, let G be a tree with $|V(G)| \geq 2$, and assume that formula holds for every tree with fewer vertices. Now, choose a leaf vertex v. Since every t-colouring of $G - v$ extends to a t-colouring of G in exactly $(t - 1)$ ways, we have $\chi(G; t) = \chi(G - v; t)(t - 1) = t(t - 1)^{n-1}$. $\square$

Contraction: Let $e \in E(G)$ be a non-loop edge with ends u, v. Modify G by deleting the edge e and then identifying the vertices u and v. We say that this new graph is obtained from G by *contracting* e and we denote it by $G \cdot e$.

Proposition 6.11 (Chromatic Recurrence)
$\chi(G; t) = \chi(G - e; t) - \chi(G \cdot e; t)$ *whenever* e *is a non-loop edge of* G.

Proof: Let $e = uv$. Then we have

$$\begin{aligned} \chi(G - e; t) &= |\{\phi : V(G) \to \{1, \ldots, t\} : \phi \text{ is a colouring and } \phi(u) \neq \phi(v)\}| \\ &\quad +|\{\phi : V(G) \to \{1, \ldots, t\} : \phi \text{ is a colouring and } \phi(u) = \phi(v)\}| \\ &= \chi(G; t) + \chi(G \cdot e; t). \end{aligned}$$

Proposition 6.12 (Chromatic Polynomial) $\chi(G; t)$ *is a polynomial for every graph* G.

Proof: We proceed by induction on $|E(G)|$. If G has no non-loop edge, then it follows from Observation 6.10 that either $E(G) = \emptyset$ and $\chi(G; t) = |V(G)|^t$ or $E(G) \neq \emptyset$ and $\chi(G; t) = 0$. Thus, we may assume that G has a non-loop edge e. By the chromatic recurrence we have $\chi(G; t) = \chi(G - e; t) - \chi(G \cdot e; t)$. Now, it follows from our inductive hypothesis that both $\chi(G - e; t)$ and $\chi(G \cdot e; t)$ are polynomials, so we conclude that $\chi(G; t)$ is a polynomial as well.

Theorem 6.13 (Whitney) *If* $G = (V, E)$ *is a graph, then*

$$\chi(G; t) = \sum_{S \subseteq E} (-1)^{|S|} t^{comp(V,S)}$$

Proof: For every set $S \subseteq E$, let q_S denote the number of labellings $\phi : V \rightarrow \{1, \ldots, t\}$ for which every edge $e \in S$ has the same colour on both endpoints. By inclusion-exclusion, we find

$$\chi(G;t) = \sum_{S \subseteq E} (-1)^{|S|} q_S.$$

Now, for a set $S \subseteq E$, a labelling $\phi : V \rightarrow \{1, \ldots, t\}$ will have the same colour on both ends on all edges in S if and only if for every component H of (V, S), this labelling assigns the same value to all vertices in H. The number of such labellings, q_S is precisely $t^{comp(V,S)}$, and substituting this in the above equation gives the desired result. $\square$

Edge Colouring

Edge Colouring A *k-edge colouring* of a graph G is a map $\phi : E(G) \rightarrow S$ where $|S| = k$ with the property that $\phi(e) \neq \phi(f)$ whenever e and f share an endpoint. As before, the elements of S are called *colours*, and the edges of one colour form a *colour class*. The *chromatic index* of G, denoted $\chi'(G)$, is the smallest k so that G is k-edge colourable.

Line Graph For any graph G, the *line graph* of G, denoted $L(G)$, is the simple graph with vertex set $E(G)$, and adjacency determined by the rule that $e, f \in E(G)$ are adjacent vertices in $L(G)$ if they share an endpoint in G. Note that $\chi'(G) = \chi(L(G))$.

Observation 6.14 $\Delta(G) \leq \chi'(G) \leq 2\Delta(G) - 1$ *for every graph* G

Proof: If v is a vertex of degree $\Delta(G)$, then the edges of G incident with v form a clique in $L(G)$. Thus $\chi'(G) = \chi(L(G)) \geq \omega(L(G)) \geq \Delta(G)$. Every edge in G is adjacent to at most $2(\Delta(G) - 1)$ other edges, so we have $\chi'(G) = \chi(L(G)) \leq \Delta(L(G)) + 1 \leq 2\Delta(G) - 1$. $\square$

Lemma 6.15 *If* $d \geq \Delta(G)$*, then* G *is a subgraph of a* d*-regular graph* H*. Furthermore, if* G *is bipartite, then* H *may be chosen to be bipartite.*

Proof: Let $G = (V, E)$ and let $G' = (V', E')$ be a copy of G so that every vertex $v \in V$ has a corresponding copy $v' \in V'$. Now, construct a new graph H by taking the disjoint union of G and G' and then adding $d - deg(v)$ new edges between the vertices v and v' for every $v \in V(G)$. It follows that H is d-regular and $G \subseteq H$. Furthermore, if (A, B) is a bipartition of G, and (A', B') is the corresponding bipartition of G', then H has bipartition $(A \cup B', A' \cup B)$. $\square$

Theorem 6.16 (König) $\chi'(G) = \Delta(G)$ *For every bipartite graph* G.

Proof: By Observation 6.14 we have $\chi'(G) \geq \Delta(G)$. We shall prove $\chi'(G) \leq \Delta(G)$ by induction on $\Delta = \Delta(G)$. As a base case, observe that the theorem holds trivially when $\Delta = 0$. For the inductive step, we may assume that $\Delta > 0$ and that the result holds for all graphs with smaller maximum degree. Now, by Lemma 6.15, we may choose a Δ-regular bipartite graph G' which contains G as a subgraph. It follows from Corollary 3.3 of Hall's Matching Theorem that G' contains a perfect matching M. Now $G' - M$ has maximum degree $\Delta - 1$, so by induction, $G' - M$ has a proper $\Delta - 1$ edge colouring $\phi : E(G' - M) \to \{1, 2, \ldots, \Delta - 1\}$. Now giving every edge in M colour Δ extends this to a proper Δ-edge colouring of G' (and thus G). $\square$

Factors: A *k-factor* in a graph $G = (V, E)$ is a set $S \subseteq E$ so that (V, S) is k-regular.

Proposition 6.17 *If* G *is a* $2k$*-regular graph, then* $E(G)$ *may be partitioned into* k *2-factors.*

Proof: For each component of G, choose an Eulerian tour, and orient the edges of G according to these walks to obtain a directed graph D. By construction every vertex in D has indegree and outdegree equal to k. Now, let V' be a copy of V so that every $v \in V$ corresponds to a vertex $v' \in V'$ and construct a new bipartite graph H with vertex set $V \cup V'$ and bipartition (V, V'), by the rule that $u \in V$ and $v \in V'$ are adjacent if (u, v) is a directed edge of D. By construction, H is a k-regular bipartite graph, so by König's Theorem we may partition the edges of H into k perfect matchings. Each perfect matching in H corresponds to a 2-factor in G, so this yields the desired decomposition. $\square$

Theorem 6.18 (Shannon) $\chi'(G) \leq 3\lceil \frac{\Delta(G)}{2} \rceil$ *for every graph* G.

Proof: Let $k = \lceil \frac{\Delta(G)}{2} \rceil$. By Lemma 6.15 we may choose a $2k$-regular graph H so that $G \subseteq H$. By the above proposition, we may choose a partition of $E(H)$ into k 2-factors $\{F_1, F_2, \ldots, F_k\}$. The edges in each 2-factor may be coloured using ≤ 3 colours, so by using a new set of 3 colours for each 2-factor, we obtain a proper $3k = 3\lceil \frac{\Delta(G)}{2} \rceil$ edge colouring of G.

Kempe Chain: Let $\phi : E(G) \to S$ be an edge-colouring of the graph G and let $s, t \in S$. Let G_{st} be the subgraph of G consisting of all vertices, and all edges with colour in $\{s, t\}$.

We define an (s,t)-*Kempe Chain* to be any component of G_{st}. If we modify ϕ by switching colours s and t on a Kempe Chain K, we obtain a new colouring which we say is obtained from the original by *switching on* K.

Theorem 6.19 (Vizing) *$\chi'(G) \leq \Delta(G) + 1$ for every simple graph G.*

Proof: Let $\Delta = \Delta(G)$ and proceed by induction on $|E(G)|$. Choose an edge $f \in E(G)$ and apply the induction hypothesis to find a $(\Delta + 1)$-edge-colouring of $G - f$. We say that a colour is *missing* at a vertex v if no edge incident with v has this colour, and is *present* otherwise. Call a path $P \subseteq G$ *acceptable* If P has vertex-edge sequence $v_1, e_1, v_2, e_2, \ldots, v_k$ where $e_1 = f$ and every e_i with $i > 1$ has a colour which is missing at an earlier vertex in the path (i.e. missing at some v_j with $j < i$.).

Consider a maximal acceptable path P with vertex-edge sequence $v_1, e_2, \ldots, v_k$, and suppose (for a contradiction) that no colour is missing at > 1 vertex of this path. If S is the set of colours missing at the vertices $v_1, \ldots, v_{k-1}$, then $|S| \geq k$ (since v_1, v_2 are missing ≥ 2 colours, and every other vertex ≥ 1). By assumption, no colour in S is missing at v_k, but then there is an edge incident with v_k with colour in S with its other endpoint not in $\{v_1, \ldots, v_{k-1}\}$, thus contradicting the maximality of P.

Now, over all acceptable paths in all possible edge-colourings of $G - f$, choose an acceptable path P with vertex edge sequence $v_1, e_1, \ldots, v_k$ so that some colour s is missing at both v_j for some $1 \leq j \leq k-1$ and missing at v_k and so that:

1. k is as small as possible.

2. j is as large as possible (subject to 1.)

If $j = 1$ and $k = 2$, then some colour is missing at both v_1 and v_2 and this colour may be used on the edge f to give a proper $(\Delta + 1)$-edge-colouring of G. We now assume (for a contradiction) that this does not hold. If $j = k - 1$, let t be the colour of e_{k-1} (note that t must be missing at some vertex in $v_1, \ldots, v_{k-2}$), and modify the colouring by changing e_{k-1} to the colour s. Now, $P - v_k$ is an acceptable path which is missing the colour t at both v_{k-1} and at some earlier vertex, thus contradicting our choice. Thus, we may now assume that $j < k - 1$. Let r be a colour which is missing at the vertex v_{j+1}. Note that by our choices r must be present at $v_1, \ldots, v_j$ and s must be present at $v_1, \ldots, v_{j-1}$ and v_{j+1}. Now, let K be the (s,r)-Kempe Chain containing v_{j+1}, note that K is a path, and then modify the colouring by switching on K. If v_j is not the other endpoint of K, then after this recolouring, the path with vertex and edge sequence $v_1, e_1, \ldots, v_j, e_j, v_{j+1}$ is an acceptable path missing the colour s at both v_j and v_{j+1} which contradicts our choice of P. It follows that K has ends v_j and v_{j+1}. But then, after switching on K, the path P is still acceptable, and is now missing the colour t on both v_{j+1} and v_k, giving us a final contradiction. □

Choosability

Choosability: Let G be a graph, and for every $v \in V(G)$ let $L(v)$ be a set of colours. We say that G is *L-choosable* if there exists a colouring ϕ so that $\phi(v) \in L(v)$ for every $v \in V(G)$. We say that G is *k-choosable* if G is L-choosable whenever every list has size $\geq k$ and we define $\chi_\ell(G)$ to be the minimum k so that G is k-choosable. We define choosability for edge-colouring similarly, and we let $\chi'_\ell(G)$ denote the smallest integer k so that G is k-edge-choosable. Note, that by using the same list for every vertex, we have $\chi(G) \leq \chi_\ell(G)$ and $\chi'(G) \leq \chi'_\ell(G)$.

Observation 6.20 *If $m = \binom{2k-1}{k}$, then $\chi_\ell(K_{m,m}) > k$.*

Proof: Let (A_1, A_2) be the bipartition of our $K_{m,m}$ and for $i = 1, 2$ assign every element of A_i a distinct k element subset of $\{1, 2, \ldots, 2k-1\}$. Now, for $i = 1, 2$, however we choose one element from each list of a vertex in A_i, there must be at least k different colours appearing on the vertices in A_i. However, then some colour is used on both A_1 and A_2, and this causes a conflict. □

Kernel: A *kernel* of a digraph D is an independent set $X \subseteq V(D)$ so that $X \cup N^+(X) = V(D)$. A digraph is *kernel-perfect* if every induced subdigraph has a kernel.

Lemma 6.21 *If D is a kernel-perfect digraph and $L : V(D) \to \mathbb{N}$ is a list assignment with the property that $|L(v)| > deg^-(v)$ for every $v \in V(D)$, then D is L-choosable.*

Proof: We proceed by induction on $|V(D)|$. As a base, note that the lemma is trivial when $|V(D)| = 0$. Otherwise, choose s in the range of L and let D' be the subgraph of D induced by those vertices whose list contains s. By assumption, we may choose a kernel X of D'. Now, X is an independent set of vertices whose list contain s, and we shall use the colour s on precisely those vertices in X. To complete our colouring, we must now find a list colouring of the digraph $D'' = D - X$ after s has been removed from all of the lists. However, since X was a kernel of D', every vertex in D'' whose list originally contained s loses at least one in indegree when passing from D to D''. Thus, the lemma may be applied inductively to obtain the desired colouring of D''. □

Preference Oriented Line Graphs: Let G be a graph with a system of preferences $\{>_v\}_{v \in V(G)}$. The *preference oriented line graph* of G is the directed graph obtained by orienting the edges of the line graph $L(G)$ by the rule that if $e, f \in E(G)$ are incident with v and $e >_v f$, then we orient the edge between e and f from e to f.

Lemma 6.22 *If D is the preference oriented line graph of a bipartite graph, then D is kernel-perfect.*

Proof: Let G be the bipartite graph with preference system $\{<_v\}_{v \in V(G)}$ for which D is the preference oriented line graph. Now, by the Gale-Shapley Theorem, G has a stable matching M. We claim that M is a kernel in D. To see this, note that since M is a matching in G, it is an independent set in D. Further, for every $e \notin M$, there must be an edge $f \in M$ sharing an endpoint, say v, with e so that v prefers f to e. However, this means that in D there will be an edge directed from f to e. It follows that M is a kernel, as desired. □

Theorem 6.23 *Every bipartite graph G satisfies $\chi'_\ell(G) = \Delta(G)$.*

Proof: Let (A, B) be a bipartition of G, let $\Delta = \Delta(G)$, and choose a Δ-edge-colouring $\phi : E(G) \to \{1, 2, \ldots, \Delta\}$. Now, we define a system of preferences on G by the rule that every vertex in A prefers edges in order of their colour, and every vertex in B prefers edges in reverse order of their colour. Now, let D be the preference oriented line graph of G with this system of preferences. By construction, every vertex in D has indegree $\Delta - 1$. Now, by Lemma 6.22, D is kernel perfect, and by Lemma 6.21 we see that D is Δ-choosable. □