Advice for solving graph theory problems

Proving theorems from scratch is a difficult - but rewarding - art. It requires focus, patience, and inspiration. With a hard problem, it is impossible to simply read out the question and then start writing the solution. There are two distinct phases to solving such problems. First off, you need to experiment, and imagine the situation.. try to understand the properties involved, and search for the key idea(s) which will get you to the solution. This document contains a number of tools which I hope will be helpful in this search. Second, after you have found the key idea, you need to formalize your argument and write out a proper proof. This is also a skill which takes time and patience to master.

1. Consider special cases

Many times a problem may look daunting since it requires you to consider a wide variety of graphs. However, frequently if you focus on a single graph or a family of graphs, you find that you can solve the problem for these special cases. With luck, the idea(s) you use to solve these cases can generalize to solve the full problem. This process of specializing, solving the problem there, and then trying to lift the solution back to the general case is one of the best things to try if you're stuck.

It should also be noted that there are many ways to specialize. For instance, many problems have natural parameters (i.e. n, k), and it is natural to consider special cases by setting one or more parameters to some fixed value (i.e. put $n = 3$). Other times, there may be no obvious such parameters to fix, but you may gain insight by restricting to a class of graphs which might be easy to work with - for instance, try to solve the problem for graphs where every vertex has degree 3.

2. Draw pictures

Sometimes when you cannot see what to do, one of the best things to try is simply drawing a picture. Many times the simple act of drawing the situation you are considering can help trigger an idea.

One instance when this is particularly useful is when you are given a problem which asks you to prove that a certain class of graphs satisfy some property. If the class is complicated, it can be quite tricky to imagine what is going on before you draw a few of them on the page.

3. Try to find a counterexample

On occasion, the best way to figure out how to prove something is to just set out and try to prove the opposite. If you are to prove that all graphs with property A have less than k edges, try to construct a graph with property A but more than k edges. Frequently, while trying to build a counterexample, you find a fundamental obstruction which turns out to be the key to the proof. This is especially helpful in conjunction with previous bit of advice. Simply trying to draw a graph which would violate the theorem is a terrific way to see why the theorem is true.

4. Induction

Induction is an incredibly powerful tool for solving graph theory problems. Most graph theorists are very quick to try induction on $|V(G)|$ or $|E(G)|$ or something else.. after all, as soon as you say you are proceeding by induction on k, you have armed yourself with the knowledge that the problem is true for all smaller values of k (assuming you have settled the base case).

There are no black and white rules for when this method will succeed, but frequently problems which succumb to an inductive proof are those where you make a fairly subtle change to the graph (i.e. delete a single vertex or edge) since in this case the solution for the smaller graph tends to convey considerable information about the original.

5. Argue by contradiction

Contradiction is another powerful tool for solving problems in discrete mathematics. Part of the reason contradiction is so useful for us is that it permits us to immediately apply a new assumption. For instance, if you wish to prove that every graph satisfying property A has no cycle of length 3, why not suppose (for a contradiction) that G is a graph with property A and with a cycle of length 3. Perhaps this cycle will lead you to a contradiction.

6. Local modification of a (supposedly) optimal solution

Sometimes when you are asked to show the existence of a certain structure (i.e. a specially structured subgraph with many edges), the best thing to do is simply choose the best one, and then investigate it (i.e. choose such a subgraph with as many edges as possible). Consider making a small modification to your supposedly optimal solution (i.e. moving one vertex). Since the modified example cannot be better than the original, this may give you some useful information about the structure.

7. Choose an extreme example

Frequently problems can get complicated simply because there are so many possibilities. Quite often, you can cut down on some of these cases by simply considering an extreme example. For instance, if you know that G has a cycle with property A, consider choosing a shortest (or longest) cycle with property A. The maximality/minimality of the case you are considering may simplify the analysis considerably.

8. Consider any related theorems/proofs

As soon as you have a toolkit of theorems, you have the added power of using them whenever you have a suitable graph. However, especially in discrete mathematics, it is useful to remember not just the theorems, but also the proofs. Time and again a key idea you see in a proof somewhere can be the tool you need to solve a different problem.