Hoffman's Circulation Theorem

Edge Cuts: For a digraph $G = (V, E)$ and $X \subseteq V$ we let $\delta^+(X) = \{(x, y) \in E : x \in X \text{ and } y \notin X\}$ and set $\delta^-(X) = \delta^+(V \setminus X)$. For $v \in V$ we write $\delta^\pm(v) = \delta^\pm(\{v\})$.

Circulation: A function $\phi : E \to \mathbb{R}$ is a *circulation* if

$$\sum_{e \in \delta^+(v)} \phi(e) = \sum_{e \in \delta^-(v)} \phi(e) \quad \text{for every } v \in V.$$

Note: if ϕ is a circulation and $X \subseteq V$ then

$$\sum_{e \in \delta^+(X)} \phi(e) - \sum_{e \in \delta^-(X)} \phi(e) = \sum_{x \in X} \left(\sum_{e \in \delta^+(x)} \phi(e) - \sum_{e \in \delta^-(x)} \phi(e) \right) = 0$$

Theorem 1 (Hoffman's Circulation Theorem) *Let $G = (V, E)$ be a digraph and let $\ell, u : E \to \mathbb{R}^+$ satisfy $\ell(e) \leq u(e)$ for every $e \in E$. Then either there exists a circulation $\phi : E \to \mathbb{R}$ with $\ell(e) \leq \phi(e) \leq u(e)$ for every $e \in E$ or there exists $X \subseteq V$ so that*

$$\sum_{e \in \delta^+(X)} u(e) < \sum_{e \in \delta^-(X)} \ell(e)$$

Proof: Define the *slack* of a set X to be

$$s(X) = \sum_{e \in \delta^+(X)} u(e) - \sum_{e \in \delta^-(X)} \ell(e).$$

It suffices to prove the existence of a flow ϕ under the assumption that every set has slack ≥ 0. If every edge e satisfies $\ell(e) = u(e)$, then we define $\phi = u = \ell$. Now for every $v \in V$ we have $\sum_{e \in \delta^+(v)} \phi(e) - \sum_{e \in \delta^-(v)} \phi(e) = s(\{v\}) \geq 0$ and $\sum_{e \in \delta^-(v)} \phi(e) - \sum_{e \in \delta^+(v)} \phi(e) = s(V \setminus \{v\}) \geq 0$ so ϕ is a flow. We shall now modify ℓ, u one edge at a time (maintaining nonnegative slack everywhere) until we achieve $\ell = u$. To do this, choose an edge f with $\ell(f) \neq u(f)$. Choose a set X with minimum slack so that $f \in \delta^+(X)$ and choose a set Y with minimum slack so that $f \in \delta^-(Y)$. Set S be the set of edges with one end in $X \setminus Y$ and one end in $Y \setminus X$ and note that $e \in S$. Now we have

$$s(X) + s(Y) = s(X \cap Y) + s(X \cup Y) + \sum_{e \in S} (u(e) - \ell(e)) \geq u(f) - \ell(f).$$

So, we may choose $x, y \geq 0$ with $x \leq s(X)$ and $y \leq s(Y)$ and $x + y = u(f) - \ell(f)$. Now increase $\ell(f)$ by x and decrease $u(f)$ by y. Then $u(f) = \ell(f)$ and it follows from our choice of X, Y that the resulting functions ℓ, u still have nonnegative slack for every set. $\square$