Integral Cayley graphs over Abelian groups

Here we give a character-theoretic proof of a theorem of Bridges and Mena which characterizes which Cayley graphs over abelian groups are integral (i.e. have all eigenvalues integers). Throughout we shall assume that G is a multiplicative abelian group. The collection of subsets of G naturally forms a boolean algebra (using $\cap$, $\cup$, and complement) and we let $\mathcal{B}(G)$ denote the boolean subalgebra generated by the subgroups of G.

Theorem 1 (Bridges, Mena)**.** *Cayley*(G, S) *is integral if and only if* $S \in \mathcal{B}(G)$.

"only if"

Let G^* denote the dual group of G, consisting of all complex characters (i.e. homomorphisms from G to (the multiplicative group of) $\mathbb{C}$). It is well known that G^* is a group under pointwise multiplication, and that $G^* \cong G$. We define F to be the matrix indexed by $G^* \times G$ and given by the rule that for $\alpha \in G^*$ and $x \in G$ we have $F_{\alpha,x} = \alpha(x)$. Note that each row of F is a character. Furthermore, it follows from the fact that the characters form an orthonormal basis that $FF^\top = I$, so $F^\top = F^{-1}$. Finally, observe that if n is the exponent of G, then every element of F is an n^{th} root of unity.

We define a relation $\sim$ on G by the rule that if $x, y \in G$, then $x \sim y$ if x and y generate the same subgroup of G. It follows immediately that $\sim$ is an equivalence relation. In the remainder, for any vector $v \in \mathbb{C}^I$ and any $n \in \mathbb{R}$, we let v^n denote the vector in $\mathbb{C}^I$ given by $(v^n)_i = (v_i)^n$.

Observation 2. *Let* $x, y \in G$ *and let* F_x, F_y *denote the column vectors of* F *indexed by* x *and* y *respectively. If* $x \sim y$ *then there exist integers* $j, k \in \mathbb{Z}$ *so that* $(F_x)^j = F_y$ *and* $(F_y)^k = F_x$.

Proof. Since $x \sim y$, we may choose $j, k \in \mathbb{Z}$ so that $x^j = y$ and $y^k = x$. Now, for any character $\alpha \in G^*$ we have $\alpha(y) = \alpha(x^j) = (\alpha(x))^j$ and it follows that $F_y = (F_x)^j$. A similar argument shows that $F_x = (F_y)^k$. □

Lemma 3. *Let* $v \in \mathbb{Q}^G$ *and* $u \in \mathbb{Q}^{G^*}$ *and assume that* $u = Fv$. *If* $x, y \in G$ *satisfy* $x \sim y$ *then* $v_x = v_y$.

Proof. Let F_x and F_y denote the column vectors of F indexed by x and y and let ℓ (m) be the smallest integer so that every term of F_x (F_y) is a ℓ^{th} (m^{th}) root of unity. It follows from the previous observation that $\ell = m$. Now, fix a primitive ℓ^{th} root of unity ω and express each entry of F_x and F_y in the form ω^k for some $k \in \{0, 1, \ldots, \ell - 1\}$. Using this interpretation, and observing that u is rational, we obtain an expression for the dot product of F_x and u as

$$F_x \cdot u = \sum_{i=0}^{\ell-1} a_i \, \omega^i$$

where each $a_i \in \mathbb{Q}$. Now, let $P(z) \in \mathbb{C}[z]$ denote the polynomial $P(z) = \sum_{i=0}^{\ell-1} a_i z^i - v_x$. Next, choose $j \in \{0, 1, \ldots, \ell - 1\}$ so that $F_y = (F_x)^j$ and note that by our assumptions, j and ℓ are relatively prime. The polynomial P has rational coefficients and has ω as a root. It follows from this and the fact that the polynomial

$$\Phi_\ell(z) = \prod_{i \in \{1..\ell\} : gcd(i,\ell)=1} (z - \omega^i)$$

is irreducible over $\mathbb{Q}$, that ω^j is also a root of P. But then we have

$$0 = P(\omega^j) = \sum_{i=0}^{\ell-1} a_i \omega^{ij} - v_x = F_y \cdot u - v_x$$

which implies that $v_y = F_y \cdot u = v_x$ as desired. □

It is well known that for every $S \subseteq G$ the Cayley graph $Cayley(G, S)$ has each character $\alpha \in G^*$ as an eigenvector with eigenvalue $\alpha(S) = \sum_{g \in S} \alpha(g)$. Alternately, if we view α as a vector and let $\mathbf{1}_S \in \mathbb{C}^G$ denote the characteristic vector of S, this eigenvalue may be written as $\alpha \cdot \mathbf{1}_S$. Before proving the "only if" direction, we need one easy observation concerning $\mathcal{B}(G)$ which we leave without proof.

Observation 4.

- *The atoms of* $\mathcal{B}(G)$ *are precisely the equivalence classes of* $\sim$.
- $S \in \mathcal{B}(G)$ *if and only if* S *is a union of equivalence classes of* $\sim$.

Proof of Theorem 1 "only if". Since $Cayley(G, S)$ is integral we must have that $\alpha \cdot \mathbf{1}_S \in \mathbb{Q}$ for every $\alpha \in G^*$. Equivalently, $F\mathbf{1}_S$ is rational valued. But then, it follows from the previous lemma that whenever $x, y \in G$ satisfy $x \sim y$ we have $(\mathbf{1}_S)_x = (\mathbf{1}_S)_y$. This implies that $S \in \mathcal{B}(G)$, as desired. □

"if"

If X is a graph on the vertex set V and B is a basis of $\mathbb{C}^V$ then we say that X is B-*integral* if B is a set of eigenvectors for X and all eigenvalues of X are integral. For any family of graphs $\mathcal{G}$ on a common vertex set, we let $U(\mathcal{G})$ be the closure of $\mathcal{G}$ under the operation $\cup$. We begin with an easy lemma.

Lemma 5.

- *If G is B-integral and $\mathbf{1} \in B$, then $\overline{G}$ is B-integral.*
- *If G and $G \cap H$ are B-integral, then $G \cap \overline{H}$ is B-integral.*
- *If $\mathcal{G}$ is an intersection-closed family of B-integral graphs, then $U(\mathcal{G})$ is B-integral.*

For any set of graphs $\mathcal{G}$ on a common vertex set V, we let $\mathcal{B}(\mathcal{G})$ denote the set of all graphs on V which may be expressed using members of $\mathcal{G}$ and the operations $\cap$, $\cup$, and complement.

Lemma 6. *Let $\mathcal{G}$ be an intersection closed family of B-integral graphs and assume that $\mathbf{1} \in B$. Then $\mathcal{B}(\mathcal{G})$ is B-integral.*

Proof. Set $\mathcal{G}_0 = \mathcal{G}$ and for every $k \in \mathbb{N}$ we recursively define

$$\mathcal{G}_{k+1} = \{X_1 \cap X_2 \cap \ldots \cap X_n : \text{either } X_i \in \mathcal{G}_k \text{ or } \overline{X_i} \in \mathcal{G}_k \text{ for every } 1 \leq i \leq n\}$$

It is immediate that each $\mathcal{G}_k$ is intersection closed and it follows from De Morgan's law that $\mathcal{B}(\mathcal{G}) = \cup_{k=0}^{\infty} \mathcal{G}_k$. To complete the proof, we shall show that every graph in $\mathcal{G}_k$ is B-integral by induction on k. As a base, we observe that this holds for $k = 0$ by assumption. For the inductive step, let X be a graph in G_{k+1} and suppose that $X = X_1 \cap X_2 \ldots \cap X_\ell \cap \overline{Y_1} \cap \overline{Y_2} \ldots \cap \overline{Y_m}$ where $X_1 \ldots X_\ell, Y_1 \ldots Y_m \in \mathcal{G}_k$. Then we have

$$X = \left(\cap_{i=1}^{\ell} X_i\right) \cap \overline{\left(\cup_{j=1}^{m} Y_j\right)}$$

and it now follows from the lemma that X is B-integral, as desired. □

Proof of Theorem 1 "if". Let $\mathcal{G} = \{Cayley(G, H) : H \leq G\}$. For every $X \in \mathcal{G}$ we have that X has G^* as a basis of eigenvectors, and X is a disjoint union of cliques, so X is G^*-integral. It now follows from the theorem that $\mathcal{B}(\mathcal{G}) = \{Cayley(G, S) : S \in \mathcal{B}(G)\}$ is G^*-integral, and this completes the proof. □