9 Lovasz's Theta Function

Alternate Formulations

Theta: Recall that an orthonormal representation of a graph $G = (V, E)$ is a family of unit vectors $\{u_i\}_{i \in V}$ so that $u_i \cdot u_j = 0$ whenever i, j are distinct and nonadjacent. The Lovasz theta function is:

$$\theta(G) = \min \left\{ \theta \ : \begin{array}{c} \text{there exists an orthonormal representation } \{u_i\}_{i\in V} \text{ of } G \\ \text{and a unit vector } c \text{ so that } (c \cdot u_i)^2 \geq \frac{1}{\theta} \text{ for every } i \in V \end{array} \right\} \tag{1}$$

A Variant of Theta: Let $\{u_i\}_{i \in V}$ be an orthogonal representation of $G = (V, E)$ with handle c and value θ. Then define a family of vectors $\{v_i\}_{i \in V}$ (orthogonal to c) by

$$v_i = \frac{u_i}{c \cdot u_i} - c$$

Now we have:

$$v_i \cdot v_j = \left(\frac{u_i}{c \cdot u_i} - c \right) \cdot \left(\frac{u_i}{c \cdot u_i} - c \right) = \frac{u_i \cdot u_j}{(c \cdot u_i)(c \cdot u_j)} - 1$$

Which implies

$$v_i \cdot v_i = \tfrac{1}{(c \cdot u_i)^2} - 1 \leq \theta - 1 \tag{$\star$}$$

$$v_i \cdot v_j = -1 \quad \text{if } i \neq j \text{ and } i \not\sim j \tag{$\star\star$}$$

Next we show that given a family $\{v_i\}_{i \in V}$ satisfying ($\star$) and ($\star\star$), we can modify the vectors to form a new family $\{v_i'\}_{i \in V}$ so that $\{v_i'\}_{i \in V}$ satisfy ($\star$) with equality and still satisfy ($\star\star$). For each $i \in V$, choose a new vector w orthogonal to all vectors in the family, and then replace v_i by $v_i' = v_i + tw$. This has no effect on the dot products between the vectors and by choosing a suitable t we may arrange that $v_i' \cdot v_i' = \theta - 1$.

Now, suppose we have a family $\{v_i\}_{i \in V}$ satisfying ($\star$) with equality and ($\star\star$) and construct a new family $\{u_i\}_{i \in V}$ and a handle c as follows. Choose c to be a unit vector orthogonal to every vector in the family and then define

$$u_i = \tfrac{1}{\sqrt{\theta}}(v_i + c)$$

Now $u_i \cdot u_j = \frac{1}{\theta}(v_i + c) \cdot (v_j + c) = \frac{1}{\theta}(v_i \cdot v_j + 1)$. It follows that $u_i \cdot u_i = 1$ and $u_i \cdot u_j = 0$ if $i \neq j$ and $i \not\sim j$. So $\{u_i\}_{i \in V}$ is an orthogonal representation. Furthermore $u_i \cdot c = \frac{1}{\sqrt{\theta}}(v_i + c) \cdot c = \frac{1}{\sqrt{\theta}}$ so this is a representation of value θ.

It follows from this that every graph G has vector representation $\{u_i\}_{i \in V}$ with handle c and value $\theta = \theta(G)$ so that every $u_i \in V$ satisfies $u_i \cdot c = \frac{1}{\sqrt{\theta}}$. Furthermore, it gives us the following alternative definition of theta.

$$\theta(G) = \min \left\{ \theta \ : \begin{array}{l} \text{there exists a family } \{v_i\}_{i \in V} \text{ of vectors with } ||v_i|| = \theta - 1 \\ \text{and } v_i \cdot v_j = -1 \text{ whenever } i \neq j \text{ and } i \not\sim j \end{array} \right\} \tag{2}$$

Theta Bar: We define $\bar{\theta}(G) = \theta(\bar{G})$. By scaling the above representation we have

$$\begin{aligned} \bar{\theta}(G) &= \min\{\theta \geq 2 : \text{there exist unit vectors } \{u_i\}_{i \in V} \text{ s.t. } u_i \cdot u_j = \tfrac{-1}{\theta - 1} \text{ whenever } ij \in E\ \} \\ &\geq \min\{\theta \geq 2 : \text{there exist unit vectors } \{u_i\}_{i \in V} \text{ s.t. } u_i \cdot u_j \leq \tfrac{-1}{\theta - 1} \text{ whenever } ij \in E\ \} \\ &= \chi_v(G) \end{aligned}$$

Our earlier proof that $\chi_v(G) \leq \chi_f(G)$ in fact shows that $\bar{\theta}(G) \leq \chi_f(G)$ giving the following chain of inequalities:

$$\omega(G) \leq \chi_v(G) \leq \bar{\theta}(G) \leq \chi_f(G) \leq \chi_c(G) \leq \chi(G)$$

A Semidefinite Programming Formulation: Using the the equivalence between Gram matrices and positive semidefinite matrices, we may formulate the θ function as the following SDP over matrices $X \in \mathbb{R}^{V \times V}$

$$\theta(G) \ = \left\{ \begin{array}{l} \min \theta \\ X \succeq 0 \\ X_{ii} = \theta - 1 \text{ for every } i \in V \\ X_{ij} = -1 \text{ if } i \not\sim j \end{array} \right\} \tag{3}$$

A Spectral Formulation: If a square matrix A is modified by adding tI, this shifts the spectrum by t. It follows from this easy observation that $\theta(G)$ may be reformulated as the following minimization over all symmetric matrices $A \in \mathbb{R}^{V \times V}$.

$$\theta(G) \ = \left\{ \begin{array}{l} \min \theta \\ A \text{ has smallest eigenvalue } -\theta \\ A_{ii} = -1 \text{ for every } i \in V \\ A_{ij} = -1 \text{ if } i \not\sim j \end{array} \right\} \tag{4}$$

Theorem 9.1 (Duality) *For every graph $G = (V, E)$, we have the following (here we optimize over all symmetric matrices $Y \in \mathbb{R}^{V \times V}$).*

$$\theta(G) = \left\{ \begin{array}{l} \max Y \cdot J \\ Y \succeq 0 \\ Y_{ij} = 0 \textit{ if } ij \in E \\ tr(Y) = 1 \end{array} \right\} \tag{5}$$

Proof: Let $\theta = \theta(G)$ be the optimum of the program in equation (3) and let λ be the right hand side of the equation in the statement of the theorem (so then our goal is to prove $\theta = \lambda$). First note that if X is feasible for the program in equation (3) and Y is feasible for the above program, then

$$0 \leq X \cdot Y = (\theta I - J) \cdot Y = \theta - \lambda$$

so $\theta \geq \lambda$.

For the other direction, let $V = \{1, 2, \ldots, n\}$ and let $i_1 j_1, i_2 j_2, \ldots i_m j_m$ be the edges of G with $i_k < j_k$ for every $1 \leq k \leq m$. Now define

$$\begin{aligned} &\mathcal{Y} = \{Y \in \mathbb{R}^{V \times V} : Y \succeq 0 \text{ and } \textstyle\sum_{i=1}^{n} Y_{ii} \leq 1\} \\ &\hat{Y} = (Y_{i_1 j_1}, Y_{i_2 j_2}, \ldots Y_{i_m j_m}, Y \cdot J) \quad \text{for every } Y \in \mathcal{Y} \\ &\hat{\mathcal{Y}} = \{\hat{Y} : Y \in \mathcal{Y}\} \end{aligned}$$

It follows from the convexity of the cone of positive semidefinite matrices that $\mathcal{Y}$ is convex. But then, the $(n^2 + 1)$ dimensional space $\{(Y, Y \cdot J) : Y \in \mathcal{Y}\}$ is convex, so $\hat{\mathcal{Y}}$ (which is a projection of this) must also be convex. If the vector $z = (0, 0, \ldots 0, \theta) \in \hat{\mathcal{Y}}$ then there exists $Y \in \mathcal{Y}$ which is feasible for the program in the statement of the theorem which has value θ so $\lambda \geq \theta$ and we are done. Thus, we may now assume (for a contradiction) that $z \notin \hat{\mathcal{Y}}$. Since $\hat{\mathcal{Y}}$ is closed and convex, it follows that there exists a vector $a = (a_1, a_2, \ldots, a_m, w)$ and a real number α so that $a^\top z > \alpha$ but $a^\top \hat{Y} \leq \alpha$ for every $\hat{Y} \in \hat{\mathcal{Y}}$. The matrix $Y \in \mathbb{R}^{V \times V}$ with $Y_{1,1} = 1$ and all other entries 0 is in $\mathcal{Y}$ and the corresponding element of $\hat{\mathcal{Y}}$ is $(0, 0, \ldots, 0, 1)$ so $w \leq \alpha$ and $\theta > 1$. On the other hand, we have $a^\top z = w\theta > \alpha$. It follows from this that $w, \alpha > 0$. By scaling, we may then assume $w = 1$ so $\alpha < \theta$. Now define a matrix $A \in \mathbb{R}^{V \times V}$ by the rule

$$A_{ij} = \begin{cases} \frac{1}{2} a_k + 1 & \text{if } i, j = \{i_k, j_k\} \\ 1 & \text{otherwise} \end{cases}$$

If $Y \in \mathcal{Y}$ then $a^\top \hat{Y} \leq \alpha$ so $A \cdot Y = J \cdot Y + \sum_{k=1}^m a_k Y_{i_k,j_k} \leq \alpha$. In particular, for any unit vector $u \in \mathbb{R}^V$ we have $uu^\top \in \mathcal{Y}$ so $u^\top A u = A \cdot uu^\top \leq \alpha$. It then follows that the largest eigenvalue of A is at most α. Now applying equation (4) to the matrix $-A$ shows that $\theta \leq \alpha$ which is a contradiction. □

Lemma 9.2 *Let $\{u_i\}_{i \in V}$ be an orthonormal representation of G and let $\{v_i\}_{i \in V}$ be an orthonormal representation of $\bar{G}$ and let c, d be any vectors. Then*

$$\sum_{i \in V} (u_i \cdot c)^2 (v_i \cdot d)^2 \leq (c \cdot c)(d \cdot d).$$

Proof: It is immediate that $\{u_i \otimes v_i\}_{i \in V}$ is an orthonormal representation of the complete graph on V, so these are pairwise orthogonal unit vectors and we have

$$(c \cdot c)(d \cdot d) = (c \otimes d) \cdot (c \otimes d) \geq \sum_{i \in V} ((u_i \otimes v_i) \cdot (c \otimes d))^2 = \sum_{i \in V} (u_i \cdot c)^2 (v_i \cdot d)^2$$

Corollary 9.3 $\theta(G)\theta(\bar{G}) \geq n$

Theorem 9.4 *For every graph $G = (V, E)$*

$$\theta(G) = \max \left\{ \sum_{i \in V} (d \cdot v_i)^2 \ : \ \begin{array}{l} \{v_i\}_{i \in V} \text{ is an orthonormal rep. of } \bar{G} \\ d \text{ is a unit vector} \end{array} \right\} \tag{6}$$

Proof: It is an immediate consequence of the previous lemma that $\theta(G) \geq \sum_{i \in V} (d \cdot v_i)$. For the other direction, let Y be a matrix which achieves the optimum in equation (5). Since Y is positive semidefinite, we may choose vectors $\{w_i\}_{i \in V}$ so that $B_{ij} = w_i \cdot w_j$. Note that

$$w_i \cdot w_j = 0 \text{ if } i \neq j \text{ and } i \not\sim j \qquad \sum_{i \in V} w_i \cdot w_i = 1 \qquad \left(\sum_{i \in V} w_i \right)^2 = \theta(G)$$

Now set

$$v_i = \frac{w_i}{||w_i||} \text{ for every } i \in V \qquad d = \frac{\sum_{i \in V} w_i}{|| \sum_{i \in V} w_i ||}$$

Then $\{v_i\}_{i\in V}$ is an orthonormal representation of $\bar{G}$ and by Cauchy-Schwartz we get

$$\begin{aligned}\sum_{i\in V}(d\cdot v_i)^2 &= \left(\sum_{i\in V}(w_i\cdot w_i)\right)\left(\sum_{i\in V}(d\cdot v_i)\right)\\ &\geq \left(\sum_{i\in V}||w_i||d\cdot v_i\right)^2\\ &= \left(d\cdot\sum_{i\in V}w_i\right)^2\\ &= \left(\sum_{i\in V}w_i\right)^2 = \theta(G)\end{aligned}$$

which completes the proof. □

Further Properties

Proposition 9.5 $\theta(G\boxtimes H)=\theta(G)\theta(H)$

Proposition 9.6 *If G is vertex transitive and $|V(G)|=n$ then $\theta(G)\theta(\bar{G})=n$*

Theorem 9.7 *If G is vertex transitive self-complementary graph on n vertices, then the Shannon Capacity of G is $\sqrt{n}$.*