Flows on Bidirected Graphs

Matt DeVos*

Applied Math Department

Princeton University

Princeton, NJ 08544

matdevos@math.princeton.edu

Abstract

The study of nowhere-zero flows began with a key observation of Tutte that in planar graphs, nowhere-zero k-flows are dual to k-colorings (in the form of k-tensions). Tutte conjectured that every graph without a cut-edge has a nowhere-zero 5-flow. Seymour proved that every such graph has a nowhere-zero 6-flow.

For a graph drawn on an orientable surface of higher genus, flows are not dual to colorings, but to local-tensions. By Seymour's theorem, every graph on an orientable surface without the obvious obstruction has a nowhere-zero 6-local-tension. Bouchet conjectured that the same holds true on non-orientable surfaces. Equivalently, Bouchet conjectured that every bidirected graph without the obvious obstruction should have a nowhere-zero 6-flow.

Improving on some earlier results, we show that Bouchet's conjecture is true with 6 replaced by 12. For 4-edge-connected bidirected graphs, we resolve Bouchet's conjecture (and extend Jaeger's 4-flow theorem), by showing that every such graph (without the obvious obstruction) has a nowhere-zero 4-flow. We also exhibit a graph to show that this result is best possible.

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1 Introduction

Throughout the paper, we consider only finite graphs, which may have loops and parallel edges. If G is a graph and $X \subseteq V(G)$, we let $\delta(X)$ denote the set of edges with exactly one endpoint in X. If $X = \{x\}$ we will abbreviate this notation to $\delta(x)$. We let $G[X]$ denote the subgraph induced by X. If $S \subseteq E(G)$, then we let G/S denote the graph obtained from G by contracting the edges in S. If $S = \{e\}$, then we will abbreviate this notation to G/e.

If G is a graph, a *signature* of G is a map $\sigma : E(G) \to \{\pm 1\}$. We say that an edge $e \in E(G)$ is *balanced* if $\sigma(e) = 1$ and *unbalanced* if $\sigma(e) = -1$. For any $S \subseteq E(G)$, we let $\sigma(S) = \prod_{e \in S} \sigma(e)$ and for any $H \subseteq G$, we let $\sigma(H) = \sigma(E(H))$. A circuit $C \subseteq G$ is said to be *balanced* (*unbalanced*) if $\sigma(C) = 1$ (-1). We say that G is *completely balanced* if every circuit of G is balanced.

Let $v \in V(G)$, and modify σ to make a new signature σ' by changing $\sigma'(e) = -\sigma(e)$ for every $e \in \delta(v)$. We will say that σ' is obtained from σ by making a *flip* at the vertex v. We will say that two signatures of G are *equivalent* if one can be obtained from the other by a sequence of flips. In general, σ and σ' are equivalent if and only if there is an edge-cut S so that σ and σ' differ precisely on S. Also, σ and σ' are equivalent if and only if every circuit C of G is either balanced with respect to both signatures, or unblanced with respect to both signatures. A *signed graph* is a pair (G, σ) so that G is a graph and σ is a signature of G. For convenience, we will sometimes not explicitly state the signature, and say that G is a signed graph. In this case, it is understood that σ_G is the signature of G.

We will use $H(G)$ to denote the set of half-edges of G as in [1]. Each half-edge h is contained in exactly one edge e and is incident with exactly one vertex, which must be an endpoint of e. A non-loop edge contains exactly one half-edge incident with each endpoint. A loop edge contains two half-edges each incident with the unique endpoint. For every vertex $v \in V(G)$, we will let $H_G(v)$ denote the set of half-edges incident with v (we sometimes abbreviate this by $H(v)$). For every half-edge h, we will let e_h denote the edge containing h. For an edge e, we will let h_e^1, h_e^2 denote the two half-edges contained in e.

If (G, σ) is a signed graph, an *orientation* of (G, σ) is a map $\tau : H(G) \to \{\pm 1\}$ with the property that $\tau(h_e^1)\tau(h_e^2) = \sigma(e)$ for every edge e. If h is a half-edge incident with the vertex

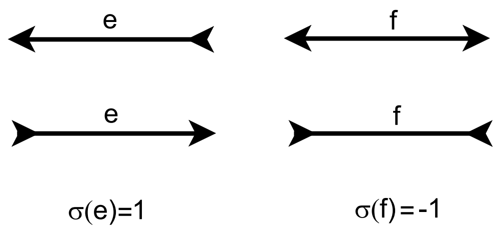

Figure 1: Orientations of signed edges

v and $\tau(h) = 1$, then h is directed toward the vertex v. If $\tau(h) = -1$ then h is directed away from v (see Figure 1). A bidirected graph, is a triple (G, σ, τ) so that G is a graph, σ is a signature of G, and τ is an orientation of (G, σ). Again, for convenience, we will sometimes refer to a graph G as a bidirected graph. In this case, it is understood that σ_G is the signature of G and τ_G is the orientation of (G, σ_G).

Let (G, σ, τ) be a bidirected graph, let Γ be an abelian group, and let $f : E(G) \to \Gamma$ be a map. We define the *boundary* of f be the map $\partial f : V(G) \to \Gamma$ given by the rule

$$\partial f(v) = \sum_{h \in H(v)} \tau(h) f(e_h)$$

We define f to be a *flow* if $\partial f = 0$. If $0 \notin f(E(G))$, we will say that f is *nowhere-zero* (which we will abbreviate as *NZ*). If f is a flow, $\Gamma = \mathbf{Z}$, and $|f(e)| < k$ for every $e \in E(G)$, then we will call f a *k-flow*.

Let f be a flow of G and let $e \in E(G)$. Now, modify τ to form a new orientation τ' by changing $\tau'(h_e^i) = -\tau(h_e^i)$ for $i = 1, 2$, and modify f to form a new map f' by changing $f'(e) = -f(e)$. After these adjustments, f' is a flow of (G, σ, τ'). Further, f' is NZ if and only if f is NZ and f' is a k-flow if and only if f is a k-flow. Thus, as in the case of ordinary graphs, for any signed graph (G, σ) and any orientations τ, τ' of (G, σ), we have that (G, σ) will have a NZ k-flow (Γ-flow) with respect to τ if and only if G has a NZ k-flow (Γ-flow) with respect to τ'. We will say that a signed graph (G, σ) *has a nowhere zero k-flow* (Γ*-flow*) if there exists an orientation τ of (G, σ) so that (G, σ, τ) has a NZ k-flow (Γ-flow).

Let v be a vertex of G and let f be a flow of (G, σ, τ) as before. Modify σ to make a new signature σ' by making a flip at the vertex v, and modify τ to make a new orientation

τ' by changing $\tau'(h) = -\tau(h)$ for every $h \in H(v)$. Now f is a flow of (G, σ', τ'). Again, this adjustment preserves the properties of NZ and k-flow. Thus, for any graph G and any two equivalent signatures σ, σ' of G, we have that (G, σ) has a NZ k-flow (Γ-flow) if and only if (G, σ') has a NZ k-flow (Γ-flow).

Bouchet made the following conjecture, which has been the motivating force for the present work

Conjecture 1.1 (Bouchet's 6-Flow Conjecture [1]). *Every bidirected graph with a nowhere zero* $\mathbf{Z}$*-flow has a nowhere zero* 6*-flow.*

Bouchet [1] proved that the above conjecture holds with 6 replaced by 216, and gave an example (also appearing later in this introduction) to show that 6 if true would be best possible. Zyka and independently Fouquet proved that the above conjecture is true with 6 replaced by 30. For 4-connected graphs Khelladi [7] proved that the conjecture holds with 6 replaced by 18.

In this paper, we will prove the following two theorems. Theorem 1.2 can be seen as a generalization of Jaeger's 4-Flow Theorem. This is an interesting fact, since for bidirected graphs Theorem 1.2 is best possible, whereas for ordinary directed graphs it has been conjectured by Tutte that every 4-edge-connected graph has a nowhere-zero 3-flow.

Theorem 1.2. *Every 4-edge-connected bidirected graph with a nowhere zero* $\mathbf{Z}$*-flow has a nowhere zero 4-flow.*

Theorem 1.3. *Every bidirected graph with a nowhere zero* $\mathbf{Z}$*-flow has a nowhere zero* 12*-flow.*

We begin here by stating some basic properties of signed graphs which we will require later in the paper. We will also state some properties of local-tensions which were the original motivation for the study of flows on bidirected graphs. The remainder of this section consists entirely of known results most of which are contained in Bouchet's original paper [1].

Let G be a signed graph. If $C_1, C_2 \subseteq G$ are two vertex disjoint unbalanced circuits and $P \subseteq G$ is a path which has one end in $V(C_1)$, one end in $V(C_2)$, and no interior vertices in $V(C_1) \cup V(C_2)$, then we will call $C_1 \cup C_2 \cup P$ a *barbell*. Also, if C_1, C_2 are two edge disjoint unbalanced circuits and $|V(C_1) \cap V(C_2)| = 1$, then we will call $C_1 \cup C_2$ a *barbell.* We will call a subgraph $H \subseteq G$ a *signed-circuit* (abbreviated *s-circuit*) if H is either a balanced circuit or H is a barbell. We will say that G is *sign-bridgeless* (abbreviated *s-bridgeless*) if every edge of G is contained in an s-circuit. The following proposition follows easily from the fact (see [18]) that the s-circuits of G are precisely the minimal subgraphs $H \subseteq G$ so that G has a $\mathbf{Z}$-flow ϕ with $supp(\phi) = E(H)$.

Proposition 1.4. *A signed graph G is s-bridgeless if and only if G has a NZ $\mathbf{Z}$-flow.*

The following characterization of s-bridgeless graphs also appears in [1].

Proposition 1.5. *If (G, σ) is a connected signed graph, then (G, σ) is s-bridgeless, and thus has a NZ $\mathbf{Z}$-flow, if and only if (G, σ) does not have one of the following properties.*

(1) *σ is equivalent to a signature σ' with the property that $\sigma'(e) = -1$ for exactly one edge $e \in E(G)$.*

(2) *$G[X]$ is completely balanced and $|\delta(X)| = 1$ for some $X \subseteq V(G)$.*

As the notation suggests, s-circuits are indeed the circuits of a matriod. We have chosen not to explicitly introduce this matriod since we will not require any special properties of it.

A theorem of Edmonds [3] allows us to work with cellular embeddings of graphs in a purely combinatorial way. Following [10], we define a *rotation scheme* of a graph G to be a family $\{\pi_v\}_{v \in V(G)}$ where π_v is a cyclic permutation of $\delta(v)$ for every $v \in V(G)$ and we define an *embedded* graph to be a triple triple $\langle G, \pi, \sigma \rangle$ so that π is a rotation scheme of G and σ is a signature of G. A *facial walk* is a closed walk in G which corresponds to a face boundary in $\langle G, \pi, \sigma \rangle$. A cycle $C \subseteq G$ is *one-sided* (*two-sided*) if it is unbalanced (balanced) with respect to σ. Thus, $\langle G, \pi, \sigma \rangle$ is an embedding in an orientable surface if and only if σ is equivalent to the trivial signature.

Let G be an ordinary directed graph, let Γ be an abelian group, and let $\phi : E(G) \to \Gamma$ be a map. Let $W = v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_{k+1}$ be a walk in the undirected graph G, and define

$$\epsilon_i = \begin{cases} 1 & \text{if } e_i \text{ is directed from } v_i \text{ to } v_{i+1} \\ -1 & \text{if } e_i \text{ is directed from } v_{i+1} \text{ to } v_i \end{cases}$$

Now, we define a map h_ϕ called the *height* function as follows

$$h_\phi(W) = \sum_{i=1}^{k} \epsilon_i \phi(e_i)$$

If $h_\phi(W) = 0$ for every closed walk W (or equivalently for every closed path), we will call ϕ a *tension.* If $\langle G, \pi, \sigma\rangle$ is an embedding of G, then we will call ϕ a *local-tension* if $h_\phi(W) = 0$ for every closed walk W which is a contractible curve in the embedding (or equivalently for every facial walk). As with flows, we will say that a tension (or local-tension) ϕ is *nowhere-zero* (abbreviated NZ) if $\phi(e) \neq 0$ for every $e \in E(G)$ and we will call ϕ a k-tension (or k-local-tension) if $\Gamma = \mathbf{Z}$ and $|\phi(e)| < k$ for every $e \in E(G)$. As with flows, if G is an undirected graph and D_1, D_2 are directed graphs obtained by orienting G, then D_1 has a NZ k-tension (k-local-tension) if and only if D_2 has a NZ k-tension (k-local-tension). In this case, we will say that G *has a nowhere zero* k*-tension* (k*-local tension*). The following proposition was first discovered by Tutte.

Proposition 1.6 (Tutte). *A graph* G *has a nowhere-zero* k*-tension if and only if* G *is* k*-colorable.*

Let $\langle G, \pi, \sigma\rangle$ be a directed graph embedded on the surface Σ and let $\phi : E(G) \to \Gamma$ be a local-tension. Now, for any two path homotopic walks W_1, W_2, we have that $h_\phi(W_1) = h_\phi(W_2)$. Fix a vertex $u \in V(G)$. We will think of u as a point in Σ as well as a vertex of G. Next, define a map $\Omega_\phi : \pi_1(\Sigma, u) \to \Gamma$ by the rule $\Omega_\phi([p]) = h_\phi(W)$ for a closed walk W with initial and terminal vertex equal to u and with W path homotopic to p. It follows immediately from the definitions that Ω_ϕ is well defined and that Ω_ϕ is a homomorphism. Further, Ω_ϕ is trivial if and only if ϕ is a tension. The following proposition shows a key property of the projective plane.

Proposition 1.7. *If* $\langle G, \pi, \sigma\rangle$ *is a directed graph embedded on the projective plane, then every* $\mathbf{Z}$*-local-tension of* $\langle G, \pi, \sigma$ *is also a* $\mathbf{Z}$*-tension.*

Proof: If ϕ is a local-tension of $\langle G, \pi, \sigma\rangle$, then the map Ω_ϕ is a homomorphism from $\mathbf{Z}_2$ to $\mathbf{Z}$. Thus, Ω_ϕ is trivial, and ϕ is a tension. □

Next we will explore the duality between local tensions on surfaces and flows in bidirected graphs. First we show how a directed embedded graph dualizes to an embedded bidirected graph. This is pictured in Figure 2.

Let $\langle G, \pi, \sigma \rangle$ be an embedded directed graph and let $\langle G^*, \pi^*, \sigma^* \rangle$ be the (undirected) dual graph. It will be convenient to think of G and G^* drawn in the surface in the usual manner. We will assume for simplicity that G is loopless, that each face of G in the embedding is bounded by a simple circuit, and that every vertex $v^* \in V(G^*)$ has degree ≥ 3. The general case may be handled in a similar manner.

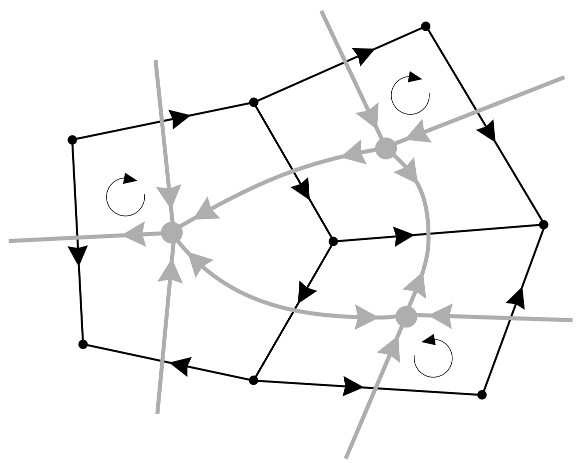

Figure 2: directed graph/bidirected graph duality

Now, we will produce an orientation τ^* of the signed graph (G^*, σ^*) by the following process. Let $v^* \in V(G^*)$ be a vertex. We will show how to define τ^* on $H(v^*)$. By repeating this procedure for every $u^* \in V(G^*)$ we obtain a map $\tau^* : H(G^*) \to \{-1, 1\}$ as desired. If $\pi^*_{v^*} = (e^*_1, e^*_2, \ldots, e^*_k)$, then we may choose a facial walk $W = v_1, e_1, \ldots, e_k, v_{k+1} = v_1$ of $\langle G, \pi, \sigma \rangle$ which bounds the face corresponding to v^*. For $1 \leq i \leq k$, let h^*_i be the half edge contained in e^*_i which is incident with v^*. Now we define τ on $H(v^*)$ by the following rule:

$$\tau^*(h^*_i) = \begin{cases} 1 & \text{if } e_i \text{ is directed from } v_i \text{ to } v_{i+1} \\ -1 & \text{otherwise} \end{cases}$$

It follows from this definition that τ^* is an orientation of (G^*, σ^*).

For any map $\phi : E(G) \to \Gamma$, we define the map $\phi^* : E(G^*) \to \Gamma$ by the rule $\phi^*(e^*) = \phi(e)$. With this definition, $\partial\phi^*(v^*) = h_\phi(W)$ for every map $\phi : E(G) \to \Gamma$ and every facial walk $W = v_1 e_1, \ldots, e_k, v_{k+1}$ of v^* with $\pi^*_{v^*} = (e^*_1, \ldots, e^*_k)$. The following proposition follows from this observation.

Proposition 1.8. *The map ϕ is a local-tension of $\langle G, \pi, \sigma \rangle$ if and only if ϕ^* is a flow of (G^*, σ^*, τ^*).*

Based on Proposition 1.6, Proposition 1.7, and Proposition 1.8, we have the following examples depicted in Figures 3 and 4. The bidirected graphs G^* and H^* from Figure 4 are obtained by orienting the embedded graphs G^* and H^* in Figure 3. Since $G \cong K_4$, the bidirected graph G^* does not have a nowhere zero 3-flow. Since $H \cong K_6$, the bidirected graph H^* does not have a nowhere zero 5-flow. These two graphs show that Theorem 1.2 does not hold under the weaker assumption of 3-edge-connectivity and that Theorem 1.2 is also false when 4-flow is replaced by 3-flow.

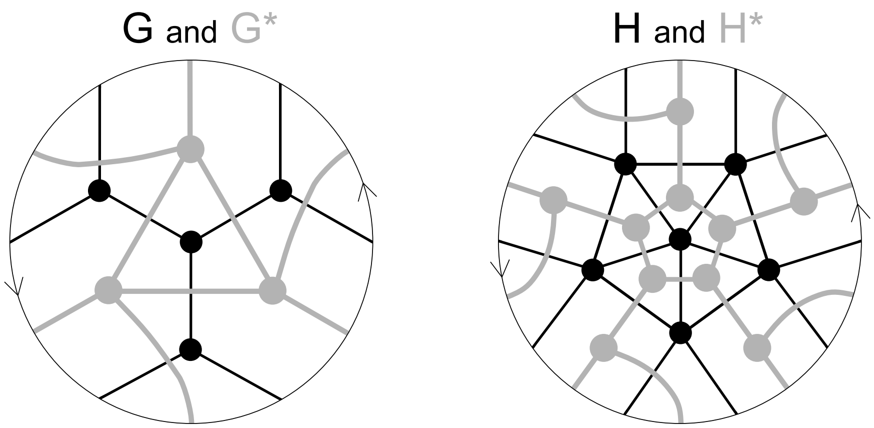

Figure 3: Embeddings in the projective plane

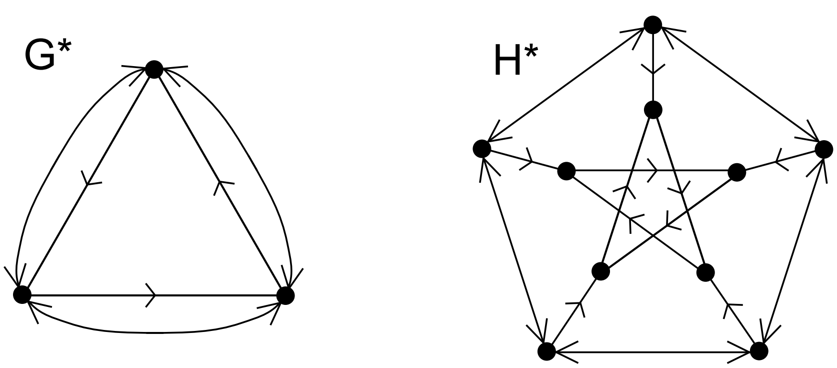

Figure 4: Orientations of G^* and H^*

Next we state Seymour's 6-Flow theorem in dual form.

Theorem 1.9 (Seymour [12]). *If $\langle G, \pi, \sigma \rangle$ is an orientable embedding of a directed graph and $\langle G, \pi, \sigma \rangle$ has a nowhere trivial $\mathbf{Z}$-local tension, then $\langle G, \pi, \sigma \rangle$ has a nowhere trivial 6-local tension.*

If Bouchet's 6-flow conjecture is true, then the above statement is still true without the assumption that the surface is orientable.

2 Nowhere-Zero 4-Flows

In this section, we will prove a simple lemma which characterizes the bidirected graphs which have nowhere-zero 2-flows. With the aid of this lemma and a helpful matroid, we will extend Jaeger's proof of the 4-flow theorem to prove Theorem 1.2 (restated here for convenience).

Theorem 1.2: *Every s-bridgeless 4-edge-connected bidirected graph has a NZ 4-flow*

The following simple observation will be of frequent use to us. Let G be a bidirected graph, let Γ be an abelian group, and let $\phi : E(G) \to \Gamma$ be a map. Define $S = \{e \in E(G) \mid \sigma_G(e) = -1\}$. Then we have that:

$$\sum_{v \in V(G)} \partial\phi(v) = \sum_{e \in S} 2\tau_G(h_e^1)\phi(e) \tag{1}$$

Lemma 2.1. *Let G be a connected bidirected graph. Then G has a NZ 2-flow if and only if G is eulerian and $\sigma_G(G) = 1$.*

Proof: First we will prove the only if portion of the lemma. If G has a NZ 2-flow, then it is clear that every vertex of G must have even degree. Since every negatively signed edge of G contributes ± 2 to the sum in equation (1), we also must have that $\sigma_G(G) = 1$.

To prove the if direction, let G be a connected eulerian graph with $\sigma_G(G) = 1$. Let $v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_{k+1} = v_1$ be an eulerian walk of G (so $v_{k+1} = v_1$), and assume that $h_{e_i}^1$ is incident with v_i and $h_{e_i}^2$ is incident with v_{i+1}. We construct a mapping $\phi : E(G) \to \pm 1$ as follows. Start by chosing $\phi(e_1) \in \pm 1$ arbitrarily. If we have already chosen the values of $\phi(e_1), \ldots, \phi(e_i)$, then choose $\phi(e_{i+1}) = \pm 1$ so that

$$\tau_G(h_{e_i}^2)\phi(e_i) + \tau_G(h_{e_{i+1}}^1)\phi(e_{i+1}) = 0$$

Now, by construction, we must have that $\partial\phi(v_i) = 0$ for every $1 < i \leq k$. By equation (1), and the fact that G has an even number of negatively signed edges, we see that $\partial\phi(v_1)$ must be a multiple of 4. Since $\phi(e_1), \phi(e_k) \in \{\pm 1\}$, it follows that $\partial\phi(v_1) = 0$, and we conclude that ϕ is a NZ 2-flow of G. □

For the next part of this section, we will need a matriod associated with a signed graph. This matriod is sometimes called the even-cycle matriod. Let (G, σ) be a signed graph and let A be a 0-1 incidence matrix for G with rows corresponding to vertices and columns corresponding to edges. Then A is a representation of the (binary) cycle matriod of G, which we will denote by $M(G)$. Now, construct a new matrix B by adding a new row to A with a 1 in the column corresponding to edge e if and only if $\sigma(e) = -1$. This new matrix B gives us a binary matriod on $E(G)$ which we will denote as $N(G, \sigma)$. If σ and σ' are equivalent signatures of G, then $N(G, \sigma) = N(G, \sigma')$. We will say that a set of edges $S \subseteq E(G)$ is *even* if every vertex in $(V(G), S)$ has even degree. Throughout the rest of this section, we will associate subgraphs $H \subseteq G$ with their edge sets $E(H)$.

For convenience, we will assume that (G, σ) is a connected signed graph and state several properties of $N(G, \sigma)$.

Proposition 2.2. *If (G, σ) does not contain an unbalanced circuit, then B is a base of $N(G, \sigma)$ if and only if it is a spanning tree. Otherwise, B is a base of $N(G, \sigma)$ if and only if B contains a spanning tree, B contains unique circuit C of G, and C is unbalanced circuit.*

Proposition 2.3. *C is a circuit of $N(G, \sigma)$ if and only if C is either a balanced circuit of G, or C is the union of two edge disjoint circuits C_1, C_2 of G with the property that C_1 and C_2 are both unbalanced and $|V(C_1) \cap V(C_2)| \leq 1$.*

Throughout the rest of this section, we will let ρ be the rank function of $M(G)$ and we will let ρ' be the rank function of $N(G, \sigma)$.

Proposition 2.4. *For every $X \subseteq E(G)$*

$$\rho'(X) = \begin{cases} \rho(X) + 1 & \text{if } X \text{ contains an unbalanced circuit} \\ \rho(X) & \text{otherwise} \end{cases}$$

Lemma 2.5. *Let k be a positive integer, let (G, σ) be a $2k$-edge-connected signed graph, and assume that $|Y| \geq k$ for every $Y \subseteq E(G)$ with the property that $G \setminus Y$ is completely balanced. Then G contains k disjoint bases of $N(G, \sigma)$*

Proof: By the matriod union theorem, we need only to verify that for every $X \subseteq E(G)$ we have:

$$k\rho'(X) + |E(G) \setminus X| \geq k\rho'(E(G))$$

If X does not contain a spanning tree, then since G is 2k-edge-connected,

$$k\rho'(X) + |E(G) \setminus X| \geq k\rho(X) + |E(G) \setminus X| \geq k\rho(E(G)) + k \geq k\rho'(E(G))$$

If G contains a spanning tree, then $\rho'(X) = \rho'(E(G))$ and we are done, unless X does not contain an unbalanced circuit. In this case $\rho'(X) = \rho'(E(G)) - 1$, and by the assumption we have that $|E(G) \setminus X| \geq k$. Thus the above equation is still satisfied. $\square$

Proposition 2.6. *If (G, σ) is a 4-edge-connected s-bridgeless signed graph, then $N(G, \sigma)$ contains two disjoint bases.*

Proof: If (G, σ) does not contain an unbalanced circuit, then a base of $N(G, \sigma)$ is a spanning tree of G, so the proposition follows from a well known theorem of Tutte [15] and Nash-Williams [11]. Otherwise, by Proposition 1.5, $G \setminus e$ contains an unbalanced circuit for every $e \in E(G)$, so the proposition follows by applying the above lemma with $k = 2$. $\square$

Proof of Theorem 1.2: Let G be a 4-edge-connected s-bridgeless bidirected graph and let B_1, B_2 be disjoint bases of $N(G, \sigma_G)$. For every edge $e \notin B_i$, let $C_i(e)$ denote the fundamental circuit of e with respect to B_i in $N(G, \sigma_G)$. Note that $C_i(e)$ is even and that $\sigma_G(C_i(e)) = 1$. For $i = 1, 2$, let $S_i = \triangle_{e \in E(G) \setminus B_i} C_i(e)$. Now S_i is an even subgraph of G and $\sigma_G(S_i) = 1$ for $i = 1, 2$. Furthermore, since $B_2 \subseteq S_1$ and $B_1 \subseteq S_2$, we have that S_1, S_2 are connected. Thus, by Lemma 2.5 we may choose 2-flows ϕ_1, ϕ_2 of G such that $supp(\phi_i) = S_i$ for $i = 1, 2$. Now $\phi_1 + 2\phi_2$ is a nowhere-zero 4-flow of G. $\square$

3 Restricted Flows in Digraphs

If Γ is an abelian group, $S \subseteq T$, and $\phi : S \to \Gamma$, we will frequently think of ϕ as defined on T with the understanding that $\phi(x) = 0$ for every $x \in T \setminus S$. If G is a directed graph and $X \subseteq V(G)$, we let $\delta^+(X)$ denote the set of edges with initial vertex in X and terminal vertex in $V(G) \setminus X$. We let $\delta^-(X) = \delta^+(V(G) \setminus X)$. It will be sometimes be convenient to think of G as a bidirected graph with signature σ_G the constant 1 map. In particular, we use this association to define the boundary ∂f of a map $f : E(G) \to \Gamma$. The goal of this section is to prove the following lemma, which will be used to build up the connectivity required in the proof of our 12-flow theorem.

Lemma 3.1. *Let G be a directed graph, let Γ be an abelian group, and assume that G has a nowhere zero Γ-flow. If $u \in V(G)$ is a vertex with $deg(u) \leq 3$ and $\gamma : \delta(u) \to \Gamma \setminus \{0\}$ satisfies $\partial\gamma(u) = 0$, then there is a nowhere zero Γ-flow ϕ of G so that $\phi|_{\delta(u)} = \gamma$.*

After a few definitions, we will prove a lemma of Seymour, from which the above lemma will easily follow.

Let G be a directed graph, let $T \subseteq E(G)$, and let Γ be an abelian group. For any map $\gamma : T \to \Gamma$, we will let $\mathcal{F}_\gamma(G)$ denote the number of NZ Γ-flows ϕ of G with $\phi(e) = \gamma(e)$ for every $e \in T$. For every $X \subseteq V(G)$, let $\alpha_X : E(G) \to \{-1, 0, 1\}$ be given by the rule

$$\alpha_X(e) = \begin{cases} +1 & \text{if } e \in \delta^+(X) \\ -1 & \text{if } e \in \delta^-(X) \\ 0 & \text{otherwise} \end{cases}$$

If $\gamma_1, \gamma_2 : T \to \Gamma$, we will call γ_1, γ_2 *similar* if for every $X \subseteq V(G)$, it holds that

$$\sum_{e \in T} \alpha_X(e)\gamma_1(e) = 0 \quad \text{if and only if} \quad \sum_{e \in T} \alpha_X(e)\gamma_2(e) = 0 \tag{2}$$

Lemma 3.2 (Seymour - personal communication). *Let G be a directed graph and let $T \subseteq E(G)$. If $\gamma_1, \gamma_2 : T \to \Gamma$ are similar, then $\mathcal{F}_{\gamma_1}(G) = \mathcal{F}_{\gamma_2}(G)$.*

Proof: We proceed by induction on the number of edges in $E(G) \setminus T$. If this set is empty, then $\mathcal{F}_{\gamma_i}(G) \leq 1$ and $\mathcal{F}_{\gamma_i}(G) = 1$ if and only if γ_i is a flow of G for $i = 1, 2$. Thus, the result follows by the assumption. Otherwise, choose an edge $e \in E(G) \setminus T$. If e is a cut-edge then $\mathcal{F}_{\gamma_i}(G) = 0$ for $i = 1, 2$. If e is a loop, then we have inductively that

$$\mathcal{F}_{\gamma_1}(G) = (|K| - 1)\mathcal{F}_{\gamma_1}(G \setminus e) = (|K| - 1)\mathcal{F}_{\gamma_2}(G \setminus e) = \mathcal{F}_{\gamma_1}(G)$$

Otherwise, applying induction to $G \setminus e$ and G/e we have

$$\mathcal{F}_{\gamma_1}(G) = \mathcal{F}_{\gamma_1}(G/e) - \mathcal{F}_{\gamma_1}(G \setminus e) = \mathcal{F}_{\gamma_2}(G/e) - \mathcal{F}_{\gamma_2}(G \setminus e) = \mathcal{F}_{\gamma_2}(G)$$

□

Proof of Lemma 3.1 If ϕ is a NZ Γ-flow of G, then $\phi|_{\delta(v)}$ is similar to γ. Thus by Lemma 3.2, we have that $\mathcal{F}_\gamma(G) = \mathcal{F}_{\phi|_{\delta(v)}}(G) \neq 0$. □

4 Modular Flows on Bidirected Graphs

If $\phi : S \to X_1 \times X_2 \times \ldots \times X_n$ we will let ϕ_i denote the projection of ϕ onto X_i. If G is a bidirected graph and $\phi : E(G) \to \mathbf{Z}_2 \times \mathbf{Z}_3$ is a flow, we will say that ϕ is *balanced* if $\sigma_G(supp(\phi_1)) = 1$. For brevity, we will abbreviate "balanced nowhere zero" by BNZ. The purpose of this section is to prove the following lemma.

Lemma 4.1. *If G is a connected bidirected graph with a BNZ $\mathbf{Z}_2 \times \mathbf{Z}_3$-flow, then G has a NZ 12-flow.*

The proof of this Lemma 4.1 is based on the following theorem. Actually, this result is just an application of a key theorem of Bouchet [1] which applies to general chain groups.

Theorem 4.2 (Bouchet [1]). *Let G be a bidirected graph, let ϕ be a $\mathbf{Z}$-flow of G, and let $k > 0$. Then there exists a 2k-flow ϕ' of G so that $\phi'(e) \cong \phi(e)$ (modulo k) for every $e \in E(G)$.*

We start by establishing two lemmas.

Lemma 4.3. *Let G be a connected bidirected graph with an unbalanced circuit, and let $p : V(G) \to \mathbf{Z}$ be a map with $\sum_{v \in V(G)} p(v)$ even. Then there exists a map $\eta : E(G) \to \mathbf{Z}$ such that $\partial\eta = p$.*

Proof: Let C be an unbalanced circuit of G, and let $u \in V(C)$ and $e \in E(C)$ be incident. Choose a spanning tree $T \subseteq G$ so that $C \setminus e \subseteq T$. Since T is a tree, we may choose a map $\eta' : E(T) \to \mathbf{Z}$ so that $\partial(\eta'(v)) = p(v)$ for every $v \in V(T) \setminus u$. Now, $\sum_{v \in V(G)} p(v)$ and $\sum_{v \in V(G)} \partial\eta'(v)$ are both even, so $t = p(u) - \partial\eta'(u)$ is even. Since C is unbalanced, we may choose a map $\zeta : E(C) \to \{-1, 0, 1\}$ so that

$$\partial\zeta(v) = \begin{cases} 2 & \text{if } v = u \\ 0 & \text{otherwise} \end{cases}$$

Now $\eta = \eta' + t/2\zeta$ is a map with $\partial\eta = p$ as required. $\square$

Lemma 4.4. *Let G be a connected bidirected graph, let p be a prime, let ψ be a $\mathbf{Z}_p$-flow of G, and assume that either p is odd or that $\sigma_G(supp(\psi)) = 1$. Then there is a $\mathbf{Z}$-flow ϕ of G so that $\phi(e) \cong \psi(e)$ (modulo p) for every $e \in E(G)$.*

Proof Choose $\phi' : E(G) \to \mathbf{Z}$ so that $\phi'(e) \cong \psi(e)$ (modulo p) for every $e \in E(G)$. Since ψ is a $\mathbf{Z}_p$-flow, we will have $\partial\phi'(v)$ is a multiple of p for every $v \in V(G)$. By equation 1 in Section 2, we have that $\sum_{v\in V(G)} \partial\phi'(v)$ is even. If $p = 2$, then by assumption, $\sigma_G(supp(\psi)) = 1$, so in this case $\sum_{v\in V(G)} \partial\phi'(v)$ is a multiple of 4. In either case, by the above lemma, we may choose $\eta : E(G) \to \mathbf{Z}$ so that $\partial\eta = (1/p)\partial\phi'$. Now $\phi = \phi' - p\eta$ is a flow and $\phi(e) \cong \psi(e)$ (modulo p) for every $e \in E(G)$ as required. □

We are now ready to prove Lemma 4.1

Proof of Lemma 4.1 Let ψ be a BNZ $\mathbf{Z}_2 \times \mathbf{Z}_3$-flow of G. By Lemma 4.4 we may choose integer flows ϕ, ϕ' so that $\phi(e) \cong \psi(e)$ (modulo 2) and $\phi'(e) \cong \psi_2(e)$ (modulo 3) for every $e \in E(G)$. Now $\omega = 3\phi + 2\phi'$ is an integer flow with the property that $\omega(e) \not\cong 0$ (modulo 6) for every $e \in E(G)$. By theorem 4.2 we may now choose an integer flow ω' so that $\omega'(e) \cong \omega(e)$ (modulo 6) and $|\omega'(e)| < 12$ for every $e \in E(G)$. Now ω' is a NZ 12-flow of G. □

5 12-Flow Reductions

If G is a bidirected graph, we will call G a *shrubbery* if it has the following properties:

(i) G is cubic

(ii) If A is a component of G, then $A\backslash e$ contains an unbalanced circuit for every $e \in E(G)$.

(iii) $|\delta(X)| \geq 4$ for every $X \subseteq V(G)$ so that $|X| > 1$ and $G[X]$ is completely balanced.

(iv) G has no balanced circuits of length four.

In this section, we will combine the results of the previous two sections to prove the following lemma.

Lemma 5.1. *To prove that every s-bridgeless bidirected graph has a NZ 12-flow, it is sufficient to prove that every shrubbery has a BNZ* $\mathbf{Z}_2 \times \mathbf{Z}_3$*-flow.*

Throughout the remainder of this paper, we will frequently modify a signed or bidirected graph G to obtain a new graph G'. If no new edges were created in this process, we will consider G' to be a signed or bidirected graph with signature $\sigma_{G'} = \sigma_G|_{E(G')}$ and (if G is bidirected) $\tau_{G'} = \tau_G|_{E(G')}$. If $E(G') \not\subseteq E(G)$, and we wish to consider G' as a signed or bidirected graph, we will explicitly give a signature and orientation (if necessary) of any newly created edge.

Let G be a graph, let $v \in V(G)$, and let $\{A_1, A_2\}$ be a partition of $H(v)$. Let G' be the graph obtained from G by the following process. First, add two new vertices v_1, v_2 and for every half edge $h \in A_i$, change h so that it is incident with the vertex v_i and change the edge e_h accordingly. Finally, add a single new edge f between v_1, v_2, and delete the vertex v. We will say that G' is obtained from G by *uncontracting* an edge at v in accordance with $\{A_1, A_2\}$. If σ is a signature of G, then let σ' be the signature of G' given by the following rule.

$$\sigma'(e) = \begin{cases} 1 & \text{if } e = f \\ \sigma(e) & \text{otherwise} \end{cases}$$

In this case, we will say that the signed graph (G', σ') is obtained from (G, σ) by *uncontracting a balanced edge* at v. The proof of Lemma 5.1 will require an observation and a proposition.

Observation 5.2. *If G is an s-bridgeless bidirected graph, and $S \subseteq E(G)$ is a set of balanced edges, then G/S is s-bridgeless.*

Proof: If ϕ is a NZ $\mathbf{Z}$-flow of G, then $\phi|_{E(G)\setminus S}$ is a NZ $\mathbf{Z}$-flow of G/S. $\square$

Proposition 5.3. *Let G be a s-bridgeless signed graph and let $v \in V(G)$ be a vertex with $deg(v) \geq 4$. Then we may form a new signed graph G' by uncontracting a balanced edge at v so that the new vertices v_1, v_2 formed by this uncontraction have $deg_{G'}(v_i) \geq 3$ and so that G' is s-bridgeless*

Proof: Let $B \subseteq G$ be an s-circuit with $v \in V(B)$. Then, choose a partition $\{B_1, B_2\}$ of $H_B(v)$ so that the graph B' obtained from B by uncontracting a balanced edge at v in accordance with $\{B_1, B_2\}$ is an s-circuit. Now, extend $\{B_1, B_2\}$ to a partition $\{A_1, A_2\}$ of $H_G(v)$ so that $|A_1|, |A_2| \geq 2$, and let G' be the graph obtained from G by uncontracting a balanced edge e at v in accordance with $\{A_1, A_2\}$. By construction, e is contained in an s-circuit of G'. Furthermore, For every s-circuit $D \subseteq G$ and every $f \in E(D)$, we find that $E(D) \cup \{e\}$ contains an s-circuit $D' \subseteq G'$ with $f \in E(D')$. It follows from this that G' is s-bridgeless. $\square$

Proof of Lemma 5.1 By Lemma 4.1, it will suffice to prove that every s-bridgeless bidirected graph G has a BNZ $\mathbf{Z}_2 \times \mathbf{Z}_3$-flow under the assumption that every shrubbery has a BNZ $\mathbf{Z}_2 \times \mathbf{Z}_3$-flow. We will proceed by induction on $\sum_{v \in V(G)} |deg(v) - 5/2|$.

Inductively, we may assume that G is connected. Since G is s-bridgeless, G has no vertices of degree one. Suppose that G has a vertex v of degree two. If v is incident with a loop, then the proposition is trivial. Otherwise, let $\delta(v) = \{e, f\}$. Now, by possibly replacing σ_G with an equivalent signature (and adjusting τ_G accordingly), we may assume that $\sigma_G(e) = 1$. By Observation 5.2 and induction, we may choose a BNZ $\mathbf{Z}_2 \times \mathbf{Z}_3$-flow ϕ of G/e. Now, we may extend the domain of ϕ to $E(G)$ by setting $\phi(e) = \pm\phi(f)$ so that ϕ is a BNZ $\mathbf{Z}_2 \times \mathbf{Z}_3$-flow of G.

If G contains a vertex v with $deg(v) \geq 4$, then by Proposition 5.3, we may uncontract a balanced edge at v so that the resulting graph G' is s-bridgeless. Inductively, we may choose a BNZ $\mathbf{Z}_2 \times \mathbf{Z}_3$-flow ϕ of G'. Now $\phi|_{E(G)}$ is a BNZ $\mathbf{Z}_2 \times \mathbf{Z}_3$-flow of G. Thus, we may assume that G is cubic.

If there is a subset $X \subseteq V(G)$ with $|X| > 1$ so that $G[X]$ is completely balanced and so that $|\delta(X)| \leq 3$, then we may assume that $\sigma_G(e) = 1$ for every $e \in E(G)$ and that every half edge contained in an edge of $\delta(X)$ and incident with a vertex $x \in X$ is directed toward x. Let G_1 be the graph obtained from G by identifying X to a single new vertex x, and let G_2 be the graph obtained from G by identifying $V(G) \setminus X$ to a single new vertex y and by modifying σ_{G_2} and τ_{G_2} so that every edge in $\delta(y)$ has signature 1 and is directed away from y. By Observation 5.2 and induction, we may choose a BNZ $\mathbf{Z}_2 \times \mathbf{Z}_3$-flow ϕ of G_1. Now, G_2 is completely balanced, so by Lemma 3.1 we may choose a NZ $\mathbf{Z}_2 \times \mathbf{Z}_3$-flow ψ of G_2 so that $\psi(e) = \phi(e)$ for every edge $e \in \delta(y) = \delta(x)$. By construction, the map $\omega : E(G) \to \mathbf{Z}_2 \times \mathbf{Z}_3$ given by the rule

$$\omega(e) = \begin{cases} \phi(e) & \text{if } e \in E(G_1) \\ \psi(e) & \text{otherwise} \end{cases}$$

is a BNZ $\mathbf{Z}_2 \times \mathbf{Z}_3$-flow of G as required.

If there is a balanced 4-circuit $C \subseteq G$, then we may assume that $\sigma_G(e) = 1$ for every $e \in E(G)$. Let G' be the graph obtained from G by deleting $E(C)$ and then identifying $V(C)$ to a single new vertex v. By Observation 5.2, G' has a NZ $\mathbf{Z}$-flow, so by induction we may choose a BNZ $\mathbf{Z}_2 \times \mathbf{Z}_3$-flow ϕ of G'. It is now straightforward to verify that ϕ can be extended to a BNZ $\mathbf{Z}_2 \times \mathbf{Z}_3$-flow of G.

If G is completely balanced, then by Seymour's 6-flow theorem, G has a BNZ $Z_2 \times Z_3$-flow as required. Otherwise, since G is connected, G must be a shrubbery. Thus, G has a BNZ $\mathbf{Z}_2 \times \mathbf{Z}_3$-flow by assumption. This completes the proof. $\square$

6 Circuits in Signed Graphs

In this section, we will establish two Lemmas concerning the existence of circuits in signed 3-connected cubic graphs with certain special properties. These two Lemmas will be of key importance in the proof of our 12-flow theorem.

If G is a graph and S is an edge-cut of G, we will say that S *seperates circuits* if both components of $G \setminus S$ contain a circuit. We will say that G is *cyclically k-edge-connected* if every edge-cut of G which seperates circuits has size $\geq k$.

A subgraph $H \subseteq G$ is *peripheral* if $G \setminus V(H)$ is connected and no edge in $E(G) \setminus E(H)$ has both ends in $V(H)$. Note that if G is cubic and H is a circuit, the above condition is equivalent to $G \setminus E(H)$ is connected. The following proposition contains two properties of peripheral circuits which we will require.

Proposition 6.1 (Tutte [16]). *If G is a 3-connected graph, then*

(i) the peripheral circuits of G generate the cycle-space of G over $\mathbf{Z}_2$.

(ii) for every $xy \in E(G)$, there exist peripheral circuits $C_1, C_2 \subseteq G$ such that $xy \in E(C_1) \cap E(C_2)$ and $V(C_1) \cap V(C_2) = \{x, y\}$.

A component H of G is *trivial* if H consists of a single isolated vertex. If P is a path, we will let $Ends(P)$ denote the set of ends of P and we will let $Int(P) = V(P) \setminus Ends(P)$.

Proposition 6.2. *Let G be a cyclically 4-edge-connected cubic graph and let $C \subseteq G$ be a peripheral circuit of G. For every subset $S \subseteq E(C)$ with $|S| \geq 2$, there is a subpath $P \subseteq G \setminus E(C)$ so that the ends of P are in distinct components of $C \setminus S$ and so that $C \cup P$ is peripheral.*

Proof: Choose a path $P \subseteq G \setminus E(C)$ so that the ends of P are in distinct components of $C \setminus S$. Subject to this, choose P so as to lexicographically maximize the sizes of the components of $G' = G \setminus E(C \cup P)$. By this we mean that P is chosen to maximize the size of the largest component of G', subject to this P is chosen to maximize the size of the second largest component of G' and so forth. If G' contains a single nontrivial component, then P

satisfies the Proposition and we are finished. Otherwise, let H be a non-trivial component of G' of minimal size. We will prove that P can be rerouted using H so as to increase the size of another non-trivial component of G' thus contradicting the choice of P. Note that since C is peripheral, every non-trivial component of G' must include a vertex of $Int(P)$, so in particular $Int(P) \not\subseteq V(H)$. Let A_1, A_2 be the components of $C \setminus S$ which contain a vertex in $Ends(P)$.

If $V(H) \cap V(B) \neq \emptyset$ for some component B of $C \setminus S$ distinct from A_1, A_2, then choose a path $Q \subseteq H$ so that one end of Q is in $V(B)$ and the other end is in $Int(P)$. Now some subpath of $P \cup Q$ contradicts the choice of P. Thus, we may assume that $V(H) \cap V(C) \subseteq V(A_1) \cup V(A_2)$.

Let $R \subseteq P \cup A_1 \cup A_2$ be a path with $Ends(R) \subseteq V(H)$ and assume that one end of R is in $Int(P)$ and that some vertex $v \in Int(R)$ is contained in a non-trivial component of G' distinct from H. In this case, we may choose a path $Q \subseteq H$ with $Ends(Q) = Ends(R)$. Again, $P \cup Q$ contains a path which contradicts the choice of P.

It follows from the above argument that H is disjoint from either A_1 or A_2. We will assume that $V(H) \cap V(A_2) = \emptyset$. Let $X = V(H) \cap V(C \cup P)$ and let $V(P) \cap V(A_1) = \{x\}$. It follows from the above arguments that either X is an interval of P (in which case $|\delta(V(H))| = 2$) or $X \cup \{x\}$ induces a connected subgraph of $P \cup A_1$ (in which case $|\delta(V(H) \cup \{x\})| = 3$). Either possibility contradicts the cyclic 4-edge-connectivity of G. □

If G is a signed cubic graph, we will say that a balanced circuit $C \subseteq G$ is a *halo* if $G \setminus E(C)$ contains a pair (P_1, P_2) of vertex disjoint paths, called a *cross* of C, with the following properties:

(i) $Ends(P_i) \subseteq V(C)$ for $i = 1, 2$

(ii) $C \cup P_1 \cup P_2$ is isomorphic to a subdivision of K_4.

(iii) $P_i \cup C$ contains an unbalanced circuit for $i = 1, 2$.

(iv) Every component of $G \setminus E(C)$ contains either P_1 or P_2

Let $K = Ends(P_1) \cup Ends(P_2)$. If $Q \subseteq C$ is a path with $Ends(Q) \subseteq K$ and with $Int(Q) \cap K = \emptyset$, then we will call Q a *side* of C with respect to P_1, P_2. If Q, R are sides which are vertex disjoint, we call them *opposite* sides.

Lemma 6.3. *Let G be a signed s-bridgeless 3-connected cubic graph. Assume that G contains an unbalanced circuit, but that G does not contain two vertex-disjoint unbalanced circuits. Then G contains a halo.*

Proof: We proceed by induction on $|V(G)|$. First, we consider the case that G contains a 3-edge-cut S which seperates circuits. Let A_1, A_2 be the components of $G \setminus S$. Since G does not contain two disjoint unbalanced circuits, we may assume that A_1 is completely balanced. By possibly replacing σ with an equivalent signature, we may assume that $\sigma(e) = 1$ for every $e \in E(A_1)$. Now, for $i = 1, 2$, let G_i be the graph obtained from G by deleting all edges in A_i and identifying every vertex in $V(A_i)$ to a single new vertex x_i. By induction, we may choose a halo C of G_1. If $x_1 \notin V(C)$, then C is also a halo of G. If $x_1 \in V(C)$, then let $e, f \in E(C)$ be the edges of C incident with x_1. By (ii) of Proposition 6.1, we may choose a peripheral circuit $D \subseteq G_2$ with $e, f \in E(D)$. Now $(C \setminus x_1) \cup (D \setminus x_2) \cup \{e, f\}$ is a halo of G. Thus, we may assume that G is cyclically 4-edge-connected.

By (i) of Proposition 6.1, we may choose a peripheral circuit $D \subseteq G$ so that D is unbalanced. Since $G \setminus E(D)$ is completely balanced, by possibly replacing σ with an equivalent signature, we may assume that $\sigma(e) = 1$ for every $e \in E(G) \setminus E(D)$. Let $S = \{e \in E(D) \mid \sigma(e) = -1\}$. Since G is s-bridgeless, $|S| > 1$, so we may apply Proposition 6.2 to choose a path $Q \subseteq G \setminus E(D)$ such that $Q \cup D$ is peripheral and so that the ends of Q are in distinct components of $G \setminus S$. Let $R_1, R_2 \subseteq D$ be the subpaths of D with $Ends(R_1) = Ends(Q) = Ends(R_2)$ and assume that $Q \cup R_1$ is a balanced circuit. Now, since the ends of Q are in distinct components of $D \setminus S$, we have that $|E(R_1) \cap S|$ is an even number greater than zero. Thus, we may choose two edge disjoint subpaths $W_1, W_2 \subseteq R_1$ with $W_1 \cup W_2 = R_1$ so that $\sigma(W_1) = -1 = \sigma(W_2)$. Let $Ends(W_1) \cap Ends(W_2) = \{x\}$ and choose a vertex $y \in Int(Q)$. Since $D \cup Q$ is peripheral, we may choose a path $P \subseteq G \setminus E(D \cup Q)$ with $Ends(P) = \{x, y\}$. By construction, $C = Q \cup R_1$ is a halo of G, and (P, R_2) is a cross of C. □

Lemma 6.4. *Let G be a signed 3-connected cubic graph and assume that G does not contain two disjoint unbalanced circuits. Let C be a halo of G, let (P_1, P_2) be a cross of C, and let Q_1, Q_2 be opposite sides of C with respect to (P_1, P_2). Then there exists a cross (P'_1, P'_2) of C and opposite sides Q'_1, Q'_2 of C with respect to (P'_1, P'_2) so that $Q'_i \subseteq Q_i$ and so that $|E(Q'_i)| = 1$ for $i = 1, 2$.*

Proof: Choose a cross (P'_1, P'_2) and opposite sides Q'_1, Q'_2 of C with respect to (P'_1, P'_2) so that $Q'_1 \subseteq Q_1$ and $Q'_2 \subseteq Q_2$. Subject to this, choose (P'_1, P'_2) so as to minimize the size of $|E(Q'_1)| + |E(Q'_2)|$. If this quantity is equal to two then we are finished. Otherwise, we may assume that $|E(Q_1')| \geq 2$, and we may choose a vertex $v \in Int(Q_1')$. By property (iv) of halos, we may choose a path $R \subseteq G \setminus E(C \cup P_1' \cup P_2')$ with $Ends(R) = \{v, u\}$ for some vertex $u \in Int(P_1') \cup Int(P_2')$. We will assume that $u \in Int(P_1')$. Let $Ends(P_1') \cap Ends(Q_1') = \{w\}$, let $W \subseteq Q_1'$ be the path with $Ends(W) = \{w, v\}$, and let $Y \subseteq P_1'$ be the path with $Ends(Y) = \{w, u\}$. Now, $R \cup W \cup Y$ must be a balanced circuit, since it is vertex disjoint from the unbalanced circuit $D \cup P_2'$. It follows from this that $R \cup P_1'$ contains a path P_1'' so that (P_1'', P_2') is a cross of C which contradicts the choice of (P_1', P_2'). This completes the proof. $\square$

7 Nowhere-Zero 12-Flows

In this section, we will prove our 12-flow theorem (restated for convenience).

Theorem 1.3 *Every s-bridgeless bidirected graph has a NZ 12-flow.*

We will start by extending the definition of shrubberies to include graphs which are not cubic. After establishing two lemmas concerning shrubberies, we will prove a lengthy lemma based on Seymour's 6-flow theorem. The 12-flow theorem will follow easily from this lemma.

We define a *shrubbery* to be a bidirected graph G with the following properties:

(i) $\Delta(G) \leq 3$

(ii) If $A \subseteq G$ is a component of G and every vertex in A has degree three, then $A \setminus e$ contains an unbalanced circuit for every $e \in E(A)$.

(iii) For every $X \subseteq V(G)$ with $|X| \geq 2$, if $G[X]$ is completely balanced then

$$|\delta(X)| + \sum_{x \in X} (3 - deg(x)) > 3$$

(iv) G has no balanced circuits of length 4

If G is a shrubbery, then a *watering* of G is a map $\phi : E(G) \to \mathbf{Z}_2 \times \mathbf{Z}_3$ so that

$$\partial\phi(v) = \begin{cases} (0,0) & \text{if } deg(v) = 3 \\ (0, \pm 1) & \text{if } deg(v) = 1, 2 \end{cases}$$

If $\phi(e) \neq 0$ for every $e \in E(G)$ we will call ϕ a nowhere-zero watering (again abbreviated NZ). If $\sigma(supp(\phi_1)) = 1$ then we will call ϕ a balanced watering. Note that as in the case of flows, if G' is a shrubbery obtained from G by replacing σ_G with an equivalent signature σ'_G, and replacing the orientation τ_G with an orientation τ'_G of (G', σ'_G), then G will have a NZ watering ϕ with $\sigma_G(supp(\phi_1)) = \epsilon$ if and only if G' has a NZ watering ϕ' with $\sigma_{G'}(supp(\phi'_1)) = \epsilon$ Also note that if G is cubic then a watering of G is a $\mathbf{Z}_2 \times \mathbf{Z}_3$-flow. The following observation follows immediately from the definitions:

Observation 7.1. *If G is a shrubbery and H is an induced subgraph of G, then H is a shrubbery*

We will call an edge e a *chord* of the circuit C if both ends of e are in $V(C)$, but $e \notin E(C)$. We denote the set of chords of C by $\mathcal{C}(C)$. If $e \in \mathcal{C}(C)$ and there is an unbalanced circuit $C' \subseteq C \cup e$ with $e \in E(C')$, then we will say that e is an unbalanced chord with respect to C. We denote the set of unbalanced chords of C by $\mathcal{U}(C)$. For any graph G, we will let $\mathcal{D}(G) = \{v \in V(G) | deg(v) = 2\}$. Let G be a shrubbery and let $C \subseteq G$ be a circuit of G. We will call C a *lucky* circuit if it has one of the following properties.

(i) C is unbalanced

(ii) $|\mathcal{D}(G) \cap V(C)| + |\mathcal{U}(C)| \geq 2$

The following Lemma will be a key tool in our proof.

Lemma 7.2. *Let G be a shrubbery and let $C \subseteq G$ be a lucky circuit. Then, for any NZ watering ϕ' of $G' = G \setminus V(C)$, there exists a NZ watering ϕ of G so that $\phi(e) = \phi'(e)$ for every $e \in E(G')$ and so that $supp(\phi_1) = E(C) \cup supp(\phi'_1)$.*

Proof: We may assume by possibly flipping on vertices in $V(C)$ that if C is a balanced circuit, then $\sigma_G(e) = 1$ for every $e \in E(C)$. Since every vertex $v \in V(G) \setminus V(C)$ adjacent to a vertex in $V(C)$ has degree < 3 in the graph G', we may extend ϕ' to $\delta(V(C))$ so that $\phi'(e) = (0, \pm 1)$ for every $e \in \delta(V(C))$ and so that

$$\partial(\phi')(v) = \begin{cases} 0 & \text{if } deg(v) = 3 \\ (0, \pm 1) & \text{if } deg(v) = 1, 2 \end{cases}$$

holds for every $v \in V(G) \setminus V(C)$. Now, for every edge $e \in \mathcal{U}(C)$ let α_e be a variable in $\mathbf{Z}_3$ and for every vertex $v \in V(C) \cap \mathcal{D}(G)$, let β_v be a variable in $\mathbf{Z}_3$. Extend ϕ' to $E(C) \cup \mathcal{C}(C)$ by the following rule

$$\phi'(e) = \begin{cases} (1, 0) & \text{if } e \in E(C) \\ (0, 1) & \text{if } e \in \mathcal{C}(C) \setminus \mathcal{U}(C) \\ (0, \alpha_e) & \text{if } e \in \mathcal{U}(C) \end{cases}$$

let $q : V(C) \to \mathbf{Z}_3$ be given by the rule

$$q(v) = \begin{cases} \beta_v & \text{if } v \in \mathcal{D}(G) \\ 0 & \text{otherwise} \end{cases}$$

and let $p : V(C) \to \mathbf{Z}_3$ be given by $p = q - (\partial\phi'_2)|_{V(C)}$.

Claim: We may choose an assignment of ± 1 to the variables α_e and β_v and we may choose a map $\mu : E(C) \to \mathbf{Z}_3$ so that $\partial\mu = p$.

Case 1: C is unbalanced

Choose arbitrary ± 1 assignments to the variables α_e and β_v. Since C is unbalanced, for every vertex $u \in V(C)$, we may choose a map $\eta^u : E(C) \to \mathbf{Z}_3$ so that $\partial\eta^u(v) = 0$ for every $v \in V(C) \setminus \{u\}$ and so that $\partial\eta^u(u) = 1$. Now $\mu = \sum_{v \in V(C)} p(v)\eta^v$ has $\partial\mu = p$.

Case 2: C is balanced

An edge $e \in \mathcal{C}(C)$ will have $\sigma_G(e) = -1$ if and only if $e \in \mathcal{U}(C)$. Thus, an edge $e \in \mathcal{U}(C)$ with $\alpha_e = x$ will contribute $-x$ to the sum $\sum_{v \in V(C)} \partial\phi'_2(v)$. A vertex $u \in \mathcal{D}(G)$ with $\beta_u = y$ will contribute y to the sum $\sum_{v \in V(C)} q(v)$. Since $|\mathcal{D}(G) \cap V(C)| + |\mathcal{U}(C)| \geq 2$, we may assign values ± 1 to the variables α_e and β_v so that $\sum_{v \in V(C)} p(v) = 0$. Now, since every edge of C has signature 1, we may choose a map $\mu : E(C) \to \mathbf{Z}_3$ with $\partial\mu = p$.

Let $\mu' : E(C) \to \mathbf{Z}_2 \times \mathbf{Z}_3$ be given by the rule $\mu' = (0, \mu)$. Now, $\phi = \phi' + \mu'$ is a NZ watering of G and by construction, $supp(\phi_1) = supp(\phi'_1) \cup E(C)$. □

Lemma 7.3. *Let G be a 2-connected balanced shrubbery and let $x_1, x_2 \in \mathcal{D}(G)$. Then there is a path $P \subseteq G$ so that $Ends(P) = \{x_1, x_2\}$ and $|Int(P) \cap \mathcal{D}(G)| \geq 2$*

The proof of this lemma will require the following theorem which is a special case of a result of Messner and Watkins.

Theorem 7.4 (Messner and Watkins [9]). *Let G be a 2-connected graph with maximum degree three and let $v_1, v_2, v_3 \in V(G)$. Then there is a circuit $C \subseteq G$ with $v_1, v_2, v_3 \in V(C)$ unless there is a partition of $V(G)$ into $\{A_1, A_2, B_1, B_2, B_3\}$ with the following properties:*

(i) $v_i \in B_i$ for $1 \leq i \leq 3$

(ii) there are no edges between A_1 and A_2 or B_i and B_j for $1 \leq i < j \leq 3$.

(iii) there is exactly one edge between A_i and B_j for every $i = 1, 2$ and $j = 1, 2, 3$.

Proof of Lemma 7.3 We proceed by induction on $|V(G)|$. If there exists $Y \subseteq V(G) \setminus \{x_1, x_2\}$ so that $\delta(Y)$ seperates cycles, and so that $|\delta(Y)| = 2$, then choose a minimal set Y with these properties. By construction, $G[Y]$ is 2-connected. Let $y_1, y_2 \in Y$ be the two vertices incident with an edge of $\delta(Y)$. Inductively, we may choose a path $Q \subseteq G[Y]$ with $Ends(Q) = \{y_1, y_2\}$ and with $|Int(Q) \cap \mathcal{D}(G)| \geq 2$. Since G is 2-connected, we may choose two vertex disjoint paths R_1, R_2 with initial vertex in $\{x_1, x_2\}$ and terminal vertex in $\{y_1, y_2\}$. Now $P = Q \cup R_1 \cup R_2$ is a path satisfying the theorem.

Let G' be the graph obtained from G by adding a new vertex q and joining q to the vertices x_1 and x_2. By the above arguments, we may assume that G' is cyclically 3-edge-connected. Choose $u, v \in \mathcal{D}(G) \setminus \{x_1, x_2\}$. If there is a circuit $C \subseteq G'$ with $u, v, q \in V(C)$, then $C \setminus q$ is a path of G which satisfies the proposition. Thus, we may assume that no such circuit exists. By Theorem 7.4 we may choose a partition $\{A_1, A_2, B_1, B_2, B_3\}$ of $V(G')$ with the properties above. Note that since G is cyclically 3-edge-connected, $G[A_i]$ is a path for $1 \leq i \leq 3$. We will assume that $u \in B_1$, $v \in B_2$, and $q \in B_3$. If there is a vertex $w \in \mathcal{D}(G) \cap (A_1 \cup A_2 \cup B_1 \cup B_2)$ distinct from u, v, then let C be a circuit of G with $w, q \in V(C)$. Since $V(C)$ must also contain one of u, v, we have that $C \setminus q$ satisfies the Lemma. Thus, we may assume that no such vertex exists. In this case, since G is a shrubbery, we must have that $|A_1| = |A_2| = |B_1| = |B_2| = 1$. But this contradicts the assumption that G has no balanced 4-circuit. □

We will say that a graph G is an *theta* if G is a subdivision of K_2^3. If G is a theta and G contains an unbalanced circuit, then we will call G an *unbalanced* theta.

Observation 7.5. *If G is a cubic shrubbery, then G contains a loop or an unbalanced theta.*

Proof: Let H be a connected component of G. If H is not 2-connected, then let H' be a leaf-block of H. Otherwise, let $H' = H$. Now, H' contains an unbalanced circuit, so either H' is a loop or H' contains an unbalanced theta. □

Let G be a graph and let x be a vertex of G of degree two which is adjacent to y, z. Let G' be the graph obtained from G by deleting x and by adding the edge yz. In this case, we say that G' is obtained from G by *suppressing* the vertex x. Finally, we are ready to prove the workhorse lemma of this section. This lemma will easily imply Thoerem 1.3.

Lemma 7.6. *Let G be a shrubbery, and let $\epsilon = \pm 1$. Then G has a nowhere-zero watering ϕ. Furthermore, if G contains an unbalanced theta or a loop, then we may choose ϕ so that $\sigma_G(supp(\phi_1)) = \epsilon$.*

Proof: We proceed by induction on $|E(G)|$. The theorem is trivial if G has at most one edge. For simplicity of presentation, we will assume that G does not satisfy the theorem, and proceed to find a contradiction. Inductively, we may assume that G is connected. Let $\mathcal{D} = \mathcal{D}(G)$.

(1) G is 2-connected

Assume that f is a cut-edge of G and that A, B are the components of $G \setminus f$. Now, we may apply induction to A, B to find NZ waterings ϕ, ψ. If G contains an unbalanced theta or a loop, then so does A or B, so in this case, we may also choose ϕ, ψ so that $\sigma(supp(\phi_1))\sigma(supp(\psi_1)) = \epsilon$. Next, choose $\alpha, \beta = \pm 1$ so that the mapping $\omega : E(G) \to \mathbf{Z}_2 \times \mathbf{Z}_3$ given by

$$\omega(e) = \begin{cases} \alpha\phi(e) & \text{if } e \in E(A) \\ \beta\psi(e) & \text{if } e \in E(B) \\ (0,1) & \text{if } e = f \end{cases}$$

Is a NZ watering of G. By construction, if G contained an unbalanced theta or a loop, then $\sigma(supp(\omega_1)) = \epsilon$.

(2) G contains an unbalanced theta

If G does not contain an unbalanced theta, then by Lemma 7.2 and induction, it will suffice to prove that G contains a lucky circuit. If G contains an unbalanced circuit, then this circuit is lucky. Otherwise, $|\mathcal{D}| \geq 4$, so we may choose $u, v \in \mathcal{D}$ and by (1) we may choose a circuit $C \subseteq G$ with $u, v \in V(C)$. Now C is a lucky circuit.

(3) G does not contain a lucky circuit with one of the following properties

(A) $G \setminus V(C)$ contains an unbalanced theta

(B) $G \setminus V(C)$ is completely balanced, and $\sigma(C) = \epsilon$

We may apply induction to choose a NZ watering ϕ of $G \setminus V(C)$. In case (A), we may choose ϕ so that $\sigma(supp(\phi_1)) = \epsilon\sigma(E(C))$. Now by Lemma 7.2, we may extend ϕ to a NZ watering ϕ' of G so that $supp(\phi'_1) = supp(\phi_1) \cup E(C)$. By construction we have that $\sigma(supp(\phi'_1)) = \epsilon$.

(4) G does not contain two unbalanced circuits C_1, C_2 so that $V(C_1) \cup V(C_2)$ contains all the vertices of degree three in G

If $G \setminus C_i$ contains an unbalanced theta, then C_i is a lucky circuit satisfying (3A), so we are finished. Thus, we may assume that every component of $G \setminus (E(C_1) \cup E(C_2))$ is a path with one end in $V(C_1)$ and the other end in $V(C_2)$. If $\epsilon = -1$, then we may choose an unbalanced circuit $C \subseteq G$ so that $G \setminus V(C)$ is a forest. This contradicts (B) of (3). If $\epsilon = 1$, then $G \setminus (V(C_1) \cup V(C_2))$ is a forest. Let ϕ be a nowhere-zero watering of $G \setminus (V(C_1) \cup V(C_2))$. By two applications of Lemma 7.2, we may extend ϕ to a watering ϕ' of G with $supp(\phi'_1) = E(C_1) \cup E(C_2)$. Now $\sigma(supp(\phi'_1)) = 1$ as required.

(5) There is no $X \subseteq V(G)$ so that $\delta(X)$ seperates cycles with $|\delta(X)| = 2$ so that $G[V(G) \setminus X]$ contains an unbalanced theta.

Choose a minimal set X with the above properties. Since $G[V(G) \setminus X]$ contains an unbalanced theta, every lucky circuit of $G[X]$ satisfies (3A), so it will suffice to show that $G[X]$ contains a lucky circuit. If $G[X]$ contains an unbalanced circuit C, then C is lucky. Otherwise, we have that $|X \cap \mathcal{D}| \geq 2$. By the minimality of X, $G[X]$ must be 2-connected. Thus, we may choose a circuit $C \subseteq G[X]$ with $|V(C) \cap \mathcal{D}| \geq 2$.

(6) There is no $X \subseteq V(G)$ with $\delta(X)$ seperating cycles with $|\delta(X)| = 2$ so that $G \setminus \delta(X)$ contains no unbalanced circuits.

Choose a minimal set X with the above properties, and let $\delta(X) = \{e_1, e_2\}$. By possibly replacing σ_G by an equivalent signature, we may assume that $\sigma_G(e_1) = -1$, and that $\sigma(e) = 1$ for every other edge $e \in E(G) \setminus \{e_1\}$. If $\epsilon = -1$, then let C be a circuit of G with $e_1 \in E(C)$. Then C is a lucky circuit contradicting (3B) so we are done. Thus, we may assume that $\epsilon = 1$. Now, $|X \cap \mathcal{D}| \geq 2$ and by the minimality of X, we have that $G[X]$ is 2-connected. Thus, we may choose a circuit $C \subseteq G[X]$ with $|V(C) \cap \mathcal{D}| \geq 2$. If e_1 is incident with a vertex of $V(C)$ or e_1 is a cut-edge of $G \setminus V(C)$, then C is a lucky circuit satisfying (3B). Otherwise, e_1 is in an unbalanced theta of $G \setminus V(C)$, so C is a lucky circuit satisfying (3A).

(7) G is cyclically 3-edge-connected

Let $e \in E(G)$ be an edge in a 2-edge-cut of G which seperates cycles, let $S = \{f \in E(G) \mid \{e, f\}$ is an edge-cut of G $\} \cup \{e\}$, and let $H_1, H_2, \ldots, H_m$ be the non-trivial components of $G \setminus S$. Note that $m \geq 2$. By (5), we have that every H_i is either completely balanced or it is an unbalanced circuit. By (6) we may assume that H_1 is an unbalanced circuit. Let $X_i = \{v \in V(H_i) \mid v \text{ is incident with an edge in } S\}$ for $1 \leq i \leq m$. Now for every $2 \leq i \leq m$ we will choose a path $P_i \subseteq H_i$ with $Ends(P_i) = X_i$ according to the following strategy: If H_i is completely balanced, then by Lemma 7.3 we may choose $P_i \subseteq H_i$ so that $|\mathcal{D} \cap Int(P_i)| \geq 2$. If H_i is an unbalanced circuit of size two, then let P_i be a single edge path in H_i. If H_i is an unbalanced circuit of size at least three, then choose $P_i \subseteq H_i$ so that $Int(P_i) \cap \mathcal{D} \neq \emptyset$. Finally, choose a path $P_1 \subseteq H_1$ so that $Ends(P_1) = X_1$ and so that $C = \cup_{i=1}^{m} P_i \cup S$ is a circuit with $\sigma(C) = \epsilon$. If one of $H_2, \ldots, H_m$ is completely balanced, then $|\mathcal{D} \cap V(C)| \geq 2$, so C is lucky. Otherwise, by (4) $m \geq 3$ so $|\mathcal{D} \cap V(C)| + |\mathcal{U}(C)| \geq 2$ and again C is lucky. In either case, C contradicts (3B).

(8) G does not contain two disjoint unbalanced cycles

If C_1 and C_2 are disjoint unbalanced cycles, then by (4) we may choose a vertex $v \in V(G) \setminus (V(C_1) \cup V(C_2))$ of degree three. By (9) we may choose 3 edge disjoint paths P_1, P_2, P_3 so that $Ends(P_i) = \{v, w_i\}$ for some $w_i \in V(C_1) \cup V(C_2)$. Without loss, we may assume that $w_1, w_2 \in V(C_2)$. Thus, $P_1 \cup P_2 \cup C_2$ is an unbalanced theta, so C_1 is a lucky circuit contradicting (3).

(9) $\epsilon = 1$

If $\epsilon = -1$, then by (2) we may choose an unbalanced circuit $C \subseteq G$. Now, $\sigma(C) = \epsilon$ and by (8) we have that $G \setminus C$ is balanced. Thus, C is a lucky circuit contradicting (3B).

(10) There is no edge $e = xy \in E(G)$ with $x, y \in V(G) \setminus \mathcal{D}$ so that $G' = G \setminus e$ is completely balanced.

Since G is a shrubbery, we have that $\mathcal{D} \neq \emptyset$, so we may choose $z \in \mathcal{D}$. Let $G' = G \setminus \{xy\}$. If G' contains a circuit C with $\{x, y, z\} \in V(C)$, then C is a lucky circuit of G contradicting (3B). Thus, we may choose a partition of $V(G')$ into $\{A_1, A_2, B_1, B_2, B_3\}$ as in Theorem 7.4. We will assume that $x \in B_1$, $y \in B_2$, $z \in B_3$. Note that by (7) $G[B_3]$ is a path. If there is a vertex $w \in \mathcal{D} \setminus \{z\}$, then any circuit $C \subseteq G'$ with $w, z \in V(C)$ is a lucky circuit of G contradicting (3B). Thus $\mathcal{D} = \{z\}$, and since $G[A_i], G[B_j]$ are completely balanced, we find that $|A_i| = 1 = |B_j|$ for $i = 1, 2$ and $1 \leq j \leq 3$. In this case, G contains a balanced circuit of length four, contradicting our assumption.

(11) There is no edge $e \in E(G)$ so that $G \setminus e$ is completely balanced

Let $P \subseteq G$ be a maximal path of G with $Int(P) \subseteq \mathcal{D}$ and with $e \in E(P)$. By (10) we may assume that $|E(P)| \geq 2$. Let $Ends(P) = \{x_1, x_2\}$ and let $G' = G \setminus Int(P)$. Now, G' is completely balanced, so we may choose $\{y_1, y_2\} \subseteq \mathcal{D} \setminus \{x_1, x_2\}$. If G' contains a circuit C so that $\{y_1, y_2, x_1\} \subseteq V(C)$, then this is a lucky circuit of G contradicting (3B). Otherwise, there is a partition of $V(G')$ into $\{A_1, A_2, B_1, B_2, B_3\}$ as in Theorem 7.4. We will assume that $y_1 \in B_1$, $y_2 \in B_2$, and $x_1 \in B_3$. By (7) we find that $x_2 \notin B_3$. Let $C \subseteq G'$ be a circuit with $y_1, y_2 \in V(C)$. Then $G \setminus V(C)$ is completely balanced, so C contradicts (3B).

Let H be the graph obtained from G by suppressing all of the vertices of G of degree two. Now, every edge $e \in E(H)$ is associated with a subpath P_e of G so that $Int(P_e) \subseteq \mathcal{D}$ and $Ends(P_e) \cap \mathcal{D} = \emptyset$. Define a signature σ_H of H by setting $\sigma_H(e) = \sigma(P_e)$ for every $e \in E(H)$. Now every subgraph $K \subseteq H$ is associated with a subgraph $K' \subseteq G$ of the same sign.

By (7) H is 3-connected, and by (11), $H \setminus e$ contains an unbalanced circuit for every $e \in E(H)$. Thus, by Lemma 6.3 we may choose a halo $C \subseteq H$ and a cross (P_1, P_2) of C. Let A_1, A_2, A_3, A_4 be the sides of C with respect to (P_1, P_2) and assume that A_1 and A_3 are opposite. Let A'_1, A'_2, A'_3, A'_4 be the corresponding paths of G and assume that $|\mathcal{D} \cap V(A_1 \cup A_3)| \leq |\mathcal{D} \cap V(A_2 \cup A_4)|$. Now, by Lemma 6.4 we may choose a cross (R_1, R_2) of H so that B_1, B_2, B_3, B_4 are the sides of C with respect to (R_1, R_2) and so that $B_i \subseteq A_i$ and $|E(B_i)| = 1$ for $i = 1, 3$. Let R'_1, R'_2 be the paths of G which correspond to R_1, R_2 and let B'_i be the path of G which corresponds to B_i for $1 \leq i \leq 4$. Note that $Int(B_1) \cup Int(B_3) \subseteq \mathcal{D}$. Now, consider the cycle $D = B'_2 \cup B'_4 \cup R'_1 \cup R'_2$. By construction, D is a balanced cycle and $G \setminus V(D)$ is completely balanced. If $|E(B'_i)| = 1$ for $i = 1$ or $i = 3$, then B'_i is a single edge which forms an unbalanced chord with respect to D. Since $|\mathcal{D} \cap V(B'_2 \cup B'_4)| \geq |\mathcal{D} \cap V(B'_1 \cup B'_3)|$ we find that $|\mathcal{D} \cap V(D)| + \mathcal{U}(D) \geq 2$. Thus D is a lucky circuit which contradicts (3B). This completes the theorem. □

Proof of Theorem 1.3 By Lemma 5.1, it suffices to prove that every cubic shrubbery has a balanced watering. This follows from Observation 7.5 and Lemma 7.6. □

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References

[1] A. Bouchet, *Nowhere-zero integral flows on a bidirected graph* J. Combinatorial Theory Ser. B 34 (1983), 279-292.

[2] J. Edmonds, *Minimum partition of a matroid into independent subsets*, J. Res. Nat. Bur. Standards 69B (1965), 67-72.

[3] J. Edmonds, *A combinatorial representation for polyhedral surfaces*, Notices Amer. Math. Soc. 7 (1960), 646.

[4] F. Jaeger, *Flows and generalized coloring theorems in graphs*, J. Combinatorial Theory Ser. B 26 (1979), 205-216.

[5] F. Jaeger, *Nowhere-zero flow problems. In Selected Topics in Graph Theory, volume 3*, edited by L.W.Beineke and R.J.Wilson, (1988), 71-95.

[6] T. Jensen and B. Toft, **Graph Coloring Problems** John Wiley & Sons, 1995

[7] A. Khelladi, *Nowhere-zero integral chains and flows in bidirected graphs*, J. Combinatorial Theory Ser. B 43 (1987), 95-115.

[8] P.A. Kilpatrick, *Tutte's first colour-cycle conjecture*, Ph.D. thesis, Cape Town. (1975)

[9] D.M. Mesner and M.E. Watkins, *Some theoreme about n-vertex connected graphs*, J. Math. Mech. 16 (1966) 321-326.

[10] B. Mohar and N. Robertson, *Planar graphs on nonplanar surfaces*, J. Combinatorial Theory Ser. B 68 (1996), 87-111.

[11] C.S.J.A. Nash-Williams, *Edge disjoint spanning trees of finite graphs*, J. London Math. Soc. 36 (1961), 445-450.

[12] P.D. Seymour, *Nowhere-zero 6-flows*, J. Combinatorial Theory Ser. B 30 (1981), 130-135.

[13] P.D. Seymour, Nowhere-Zero Flows, in *Handbook of Combinatoircs*, edited by R. Graham, M. Grotschel and L. Lovasz, (1995) 289-299.

[14] W.T. Tutte, *A contribution to the theory of chromatic polynomials*, Canad. J. Math. 6 (1954), 80-91.

[15] W.T. Tutte, *On the problem of decomposing a graph into n connected factors*, J. London Math. Soc. 36 (1961), 221-230.

[16] W.T. Tutte, *How to draw a graph*, Proc. London Math. Soc. 13 (1963), 743-767.

[17] D.J.A. Welsh, "Matriod Theory", Academic Press. London/New York. 1976.

[18] T. Zaslavsky, *Signed graphs*, Discrete Appl. Math. 4 (1982), 47-74.

[19] T. Zaslavsky, *Erratum* Discrete Appl. Math. 5 (1983), 248.

[20] T. Zaslavsky, *Orientations of signed graphs*, European J. Combin. 12 (1991), no. 4, 361-375.