Coloring-flow duality of graphs on surfaces

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Abstract

Let G be a directed graph embedded in a surface. A map $\phi : E(G) \to \mathbb{R}$ is a *tension* if for every circuit $C \subseteq G$, the sum of ϕ on the forward edges of C is equal to the sum of ϕ on the backward edges of C. If this rule is satisfied for every circuit of G which is a contractible curve in the surface, then ϕ is a *local-tension*. If $1 \leq |\phi(e)| \leq \alpha - 1$ holds for every $e \in E(G)$, we say that ϕ is an α-tension (or an α-local tension). The circular chromatic number of a graph G, denoted $\chi_c(G)$, is equal to $\inf\{\alpha \in \mathbb{R} \mid G \text{ has an } \alpha\text{-tension}\}$. We define $\chi_{lt}(G) = \inf\{\alpha \in \mathbb{R} \mid G \text{ has an } \alpha\text{-local tension}\}$.

It is an immedite consequence of these definitions that $\chi_{lt}(G) \leq \chi_c(G)$. The main result of this paper shows that for every surface $\mathcal{X}$ and every $\varepsilon > 0$, there exists an integer M so that $\chi_c(G) \leq \chi_{lt}(G) + \varepsilon$ holds for every graph embedded in $\mathcal{X}$ with edge-width at least M. In the final section, we characterize the embedded graphs with $\chi_{lt}(G) = 2$ and the triangulations with $\chi_{lt}(G) = 3$. Together with the main theorem, these characterizations yield new upper and lower bounds on the circular chromatic number of these graphs.

1 Introduction

We begin with motivation and background for the paper. Our discussion will be centered on tensions, local-tensions, and flows. Although rigorous definitions of these terms will be postponed until the following section. The reader may find an informal description of tensions (α-tensions) and local-tensions (α-local-tensions) in the abstract above. If G is an embedded directed graph, then a map $\phi : E(G) \to \mathbb{R}$ is a flow if at every vertex the sum of the values on the incoming edges is equal to the sum of the values on the outgoing edges. We say that ϕ is an α-flow if $1 \leq |\phi(e)| \leq \alpha - 1$ holds for every $e \in E(G)$.

If ϕ is an α-tension (local-tension, flow) of G and we switch the orientation of $e \in E(G)$, then replacing $\phi(e)$ by $-\phi(e)$ maintains the property that ϕ is an α-tension (local-tension, flow). Thus, the existence of an α-tension (local-tension, flow) depends only on the underlying undirected (embedded) graph and we say that an undirected embedded graph G admits an α-tension (local-tension, flow) if some (and thus every) orientation of it admits such a map. Extending the definitions given in the abstract, we have the following three parameters defined for an unoriented embedded graph G.

$$\begin{aligned}
\chi_{\mathrm{c}}(G) &= \inf\{\alpha \in \mathbb{R} \mid G \text{ admits an } \alpha\text{-tension}\} \\
\chi_{\mathrm{lt}}(G) &= \inf\{\alpha \in \mathbb{R} \mid G \text{ admits an } \alpha\text{-local-tension}\} \\
\phi_c(G) &= \inf\{\alpha \in \mathbb{R} \mid G \text{ admits an } \alpha\text{-flow}\}
\end{aligned}$$

A basic property of χ_{c} is that $\chi(G) = \lceil \chi_{\mathrm{c}}(G) \rceil$ for every graph G. Thus, we may view χ_{c}, the circular chromatic number, as a refinement of the usual notion of chromatic number. Based on this we have that $\chi(G) \geq \chi_{\mathrm{c}}(G) \geq \chi_{\mathrm{lt}}(G)$ for every embedded graph G. Our main result is the following upper bound on χ_{c}. The *edge-width* of G is the length of the shortest circuit of G which forms a non-contractible curve in the surface.

Theorem 1.1 *Let $\mathcal{X}$ be a surface and let $\varepsilon > 0$. There exists an integer M so that every graph embedded in $\mathcal{X}$ with edge-width $\geq M$ satisfies $\chi_{\mathrm{c}}(G) \leq \chi_{\mathrm{lt}}(G) + \varepsilon$.*

Once we have established this, we will use χ_{lt} to find both upper and lower bounds on χ_{c} for certain families of embedded graphs. Indeed one of the key themes in this paper is the utility of χ_{lt} in understanding χ_{c}, and indeed χ for embedded graphs.

Let G and G^* be unoriented dual graphs embedded in an orientable surface in the usual manner. Using a global orientation of the surface, we may direct the edges of G and G^* so that each edge $e^* \in E(G^*)$ crosses left to right over the corrseponding edge $e \in E(G)$.

⟦Perhaps it would be nice to have a small picture here depicting the crossing edges e and e^*.⟧

For every map $\phi : E(G) \to \mathbb{R}$, let $\phi^* : E(G^*) \to \mathbb{R}$ be given by the rule $\phi^*(e^*) = \phi(e)$ where e^* is the edge of G^* which corresponds to the edge e of G. It now follows from our definitions that ϕ is a local-tension if and only if ϕ^* is a flow. Thus, duality exchanges local-tensions with flows and we have the relation $\chi_c(G) \geq \phi_C(G^*)$. In the special case when our graphs are embedded in the sphere or the plane, every local-tension is also a tension, so duality exchanges flows and tensions and we have that $\chi_c(G) = \chi_{lt}(G) = \phi_c(G^*)$. Indeed this is an early observation of Tutte, usually stated as flow/coloring duality, which provided motivation for our work. A consequence of this duality is the following Corollary of our main theorem.

Corollary 1.2 *Let $\mathcal{X}$ be an orientable surface and let $\varepsilon > 0$. There exists an integer M so that every graph G embedded in $\mathcal{X}$ with edge-width $\geq M$ satisfies $\chi_c(G) \leq \phi_c(G^*) + \varepsilon$ (here G^* denotes the dual graph of G).*

When our dual graphs G and G^* are embedded in a surface which is not orientable, the duality between local-tensions of G and flows of G^* no longer holds. In order to work effectively with graphs embedded in non-orientable surfaces, we will therefore concentrate on tensions and local-tensions.

The study of α-flows on abstract graphs was initiated by Tutte, who conjectured that every graph with no cut-edge admits a 5-flow. If this conjecture is true, then the following conjecture holds by duality.

Conjecture 1.3 (Tutte) *Every graph embedded in an orientable surface with edge-width ≥ 2 admits a 5-local-tension.*

Seymour has proved that every graph with no cut-edge admits a 6-flow, so by duality the above conjecture holds with 5 replaced by 6. There is a related conjecture of Bouchet (usually called Bouchet's 6-flow conjecture) which would imply the following conjecture.

Conjecture 1.4 (Bouchet) *Every graph embedded in a surface with edge-width ≥ 2 admits a 6-local-tension.*

DeVos has shown that the above conjecture holds with 6 replaced by 12. Together with our main theorem, the partial results towards these two conjectures imply upper bounds on the circular chromatic number of graphs embedded in a fixed surface with sufficiently high edge-width. However, even if Tutte's and Bouchet's conjectures hold, the upper bounds on χ_c which would follow from our theorem are weaker than those already given by the following result of Thomassen.

Theorem 1.5 (Thomassen) *For every fixed surface $\mathcal{X}$ there are only finitely many 6-critical graphs which embed in $\mathcal{X}$. In particular, there exists an integer M so that $\chi(G) \leq 5$ for every graph embedded in $\mathcal{X}$ with edge-width $\geq M$.*

For every surface $\mathcal{X}$ other than the sphere (or plane) there exist graphs of arbitrarily high edge-width embedded in G with chromatic number 5. Thus, it is not possible to improve the parameter of 5 in the second sentence of the above theorem. It is natural to consider the behavior of χ_c for graphs on a fixed surface of high edge-width. By Thomassen's theorem, for every fixed surface $\mathcal{X}$, we have that $\chi_c(G) \leq 5$ for every graph G embedded in $\mathcal{X}$ with sufficiently high edge-width. In Section 5 we show for every non-orientable surface $\mathcal{X}$, the existence of a family of graphs embedded in $\mathcal{X}$ with unbounded edge-width and with circular chromatic number at least 5. Thus, together with Thomassen's theorem, this gives us a best possible bound on χ_c for graphs of large edge-width on a fixed non-orientable surface. No such family of graphs is known for any orientable surface. Indeed, if the following conjecture of Grunbaum is true, then together with our main theorem it would imply that for every fixed orientable surface $\mathcal{X}$ and every $\varepsilon > 0$, all graphs embedded in $\mathcal{X}$ with sufficiently high edge-width have circular chromatic number at most $4 + \varepsilon$.

Conjecture 1.6 (Grunbaum) *There exists a fixed integer k (or stronger that $k = 3$ suffices) so that every graph embedded in an orientable surface with edge-width $\geq k$ has a 4-local-tension.*

This rather shocking conjecture is a strengthening of the Four color theorem since every local-tension of a planar graph is also a tension. The following conjecture may be viewed as a non-orientable variant of Grunbaum's conjecture.

Conjecture 1.7 (DeVos) *There exists a fixed integer k (or stronger that $k = 4$ suffices) so that every graph embedded in a surface with edge-width $\geq k$ has a 5-local-tension.*

Note that by Thomassen's theorem the weak form of the above conjecture is true when we restrict to a fixed surface.

2 Basic definitions

All graphs considered in this paper are finite. Graphs may have multiple edges, but for convenience of notation, we will forbid loops. We will assume basic knowledge about (2-cell) embeddings of graphs in surfaces (see [11] for a good reference). However, we shall need a slightly less standard treatment of embedded graphs where each edge and each face is oriented.

An *embedded graph* G is a triple $(V(G), E(G), F(G))$ where $(V(G), E(G))$ is a directed graph and $F(G)$ is a finite set of *faces*. Associated with every face $R \in F(G)$ is a *boundary*, also called the *facial walk* of R, which is a cyclic list $v_1, e_1, \ldots, v_k, e_k$ of vertices and edges such that $e_i \in E(G)$ has distinct endpoints v_i and v_{i+1} (indices modulo k) for every $1 \leq i \leq k$. Moreover, every vertex is incident with at least one edge, and every edge occurs precisely twice in the boundaries of the faces, either once in two distinct boundaries, or twice in one. A vertex $v \in V(G)$ or an edge $e \in E(G)$ is *incident* with the face $R \in F(G)$ if v or e occurs in the boundary of R. If every edge of G is incident with two (distinct) faces, then G is *strongly* embedded.

For every embedded graph G, we construct a topological space $|G|$ as follows. If $R \in F(G)$ is a face with boundary $v_1, e_1, \ldots, v_k, e_k$, then R is associated with a regular unit edge-length k-gon $\pi(R) \subseteq \mathbb{R}^2$. For convenience we will assume that $\pi(R) \cap \pi(S) = \emptyset$ if $R \neq S$. The vertices and edges of $\pi(R)$ correspond to the boundary of R. We call the ith edge of $\pi(R)$ a *copy* of $e_i \in E(G)$. Now, we obtain $|G|$ from the disjoint union of these polygons by identifying both copies of every edge $e \in E(G)$ according to their orientations. The topological space $|G|$ is always a compact 2-manifold (without boundary) which is herafter called a *surface*. For any surface $\mathcal{X}$ we say that *G is embedded in $\mathcal{X}$* if $|G|$ is homeomorphic to $\mathcal{X}$.

For every vertex $v \in V(G)$, edge $e \in E(G)$, and face $R \in F(G)$ with boundary $v_1, e_1, \ldots, v_k, e_k$, we make the following definitions:

$$\langle v, e\rangle = \begin{cases} 0 & \text{if } v \text{ and } e \text{ are not incident} \\ -1 & \text{if } v \text{ is the tail of } e \\ 1 & \text{if } v \text{ is the head of } e. \end{cases}$$

$$\langle e, R\rangle = \sum_{\{1 \leq i \leq k | e = e_i\}} \langle v_i, e_i\rangle$$

We say that e is a *forward* edge of R if $\langle e, R\rangle = 1$ and that e is a *backward* edge of R if $\langle e, R\rangle = -1$. Next we define the function $\tau : E(G) \to \mathbb{Z}$ by the following rule.

$$\tau(e) = \frac{1}{2} \sum_{R \in F(G)} \langle e, R \rangle.$$

Note that $\tau(e) \in \{-1, 0, 1\}$ for every $e \in E(G)$ by our assumptions. Further, $\tau(e) = 1$ if e is a forward edge of two distinct faces, $\tau(e) = -1$ if e is a backward edge of two distinct faces, and $\tau(e) = 0$ if e is forward in one face and backward in another. Based on this, we define the *sign* of G to be the map $\sigma : E(G) \to \mathbb{Z}$ given by the rule

$$\sigma(e) = (-1)^{\tau(e)}$$

The notion of the sign is equivalent to what is known as the signature of the dual graph (cf., e.g., [11]).

For any embedded graph G, we may obtain a new embedded graph by switching the orientation of an edge $e \in E(G)$. This has the effect of changing $\langle e, R \rangle$ to $-\langle e, R \rangle$ for every $R \in F(G)$, but it has no effect on the sign or the space $|G|$. Similarly, we may also obtain a new embedded graph by replacing the boundary $v_1, e_1, \ldots, v_k, e_k$ of some face $R \in F(G)$ by $v_1, e_k, v_{k-1}, e_{k-1}, \ldots, v_2, e_1$. This operation is called *switching the orientation* of R. Switching the orientation of R changes $\langle e, R \rangle$ to $-\langle e, R \rangle$ for every $e \in E(G)$ and changes $\sigma(e)$ to $-\sigma(e)$ for every edge e with $\langle e, r \rangle = \pm 1$. However, this change also has no effect on $|G|$. We say that G is *orientable* if the map σ may be changed into the constant 1 map by switching the orientation of some faces. Although edge and face orientations are essential for our definitions, all of our results will be independent of these orientations.

For $i \in \mathbb{Z}^+$ and $j \in \mathbb{Z}^+ \setminus \{0\}$, let $\mathbb{S}_i$ be an orientable surface of genus $2i$ and let $\mathbb{N}_j$ be a nonorientable surface of genus j. It follows from the classification theorem that every surface is isomorphic to exactly one of these spaces. Further, G is orientable if and only if G is embedded in $\mathbb{S}_i$ for some $i \geq 0$.

Let G be a directed graph or an embedded graph and let Γ be a fixed abelian group. A 0-*chain* is a map from $V(G)$ to Γ, a 1-*chain* is a map from $E(G)$ to Γ, and a 2-*chain* is a map from $F(G)$ to Γ. For $i = 0, 1, 2$, the i-chains form a group under componentwise addition and we denote this group by $C_i(G, \Gamma)$.

If $c \in C_0(G, \Gamma)$, then we define the *coboundary* of c to be the map $\delta c \in C_1(G, \Gamma)$ given by the rule $\delta c(e) = \sum_{v \in V(G)} \langle v, e \rangle c(v)$. If $c \in C_1(G, \Gamma)$, then we define the *coboundary* of c to be the map $\delta c \in C_2(G,\Gamma)$ given by the rule $\delta c(R) = \sum_{e\in E(G)} \langle e, R\rangle c(e)$.

If $c \in C_1(G,\Gamma)$, then we define the *boundary* of c to be the map $\partial c \in C_0(G,\Gamma)$ given by the rule $\partial c(e) = \sum_{e\in V(G)} \langle v, e\rangle c(v)$. If $c \in C_2(G,\Gamma)$, then we define the *boundary* of c to be the map $\delta c \in C_1(G,\Gamma)$ given by the rule $\delta c(e) = \sum_{R\in F(G)} \langle e, R\rangle c(e)$.

If c^1 is a 1-chain and $c^1 = \delta c^0$ for some 0-chain c^0, then we call c^1 a *tension* or a Γ-*tension*. If $\partial c^1 = 0$ then we call c^1 a *flow* or a Γ-*flow*. If G is an embedded graph and $\delta c^1 = 0$, then we call c^1 a *local tension* or a Γ-*local tension*. If $c^1 = \partial c^2$ for some $c^2 \in C_2(G,\Gamma)$ then we call c^1 a *facial flow* or a Γ-*facial flow*.

We denote the set of tensions, local-tensions, flows, and facial-flows by $\mathcal{T}(G,\Gamma)$, $\mathcal{L}(G,\Gamma)$, $\mathcal{F}(G,\Gamma)$, and $\mathcal{K}(G,\Gamma)$. Note that all four of these sets are subgroups of $C_1(G,\Gamma)$. Further, $\delta\delta c = 0$ for every 0-chain c and $\partial\partial c = 0$ for every 2-chain c, so $\mathcal{T}(G,\Gamma)$ is a subgroup of $\mathcal{L}(G,\Gamma)$ and $\mathcal{F}(G,\Gamma)$ is a subgroup of $\mathcal{K}(G,\Gamma)$. The following proposition gives a well known relationship between these spaces.

Proposition 2.1 (a) $\mathcal{T}(G,\Gamma) = \{c \in C_1(G,\Gamma) \mid \sum_{e\in E(G)} d(e)c(e) = 0$ *for every* $d \in \mathcal{F}(G,\mathbb{Z})$.
(b) $\mathcal{L}(G,\Gamma) = \{c \in C_1(G,\Gamma) \mid \sum_{e\in E(G)} d(e)c(e) = 0$ *for every* $d \in \mathcal{K}(G,\mathbb{Z})$.

We define the homology group $H_1(G,\Gamma)$ to be the group $\mathcal{F}(G,\Gamma)/\mathcal{K}(G,\Gamma)$ and we define the cohomology group $H^1(G,\Gamma)$ to be the group $\mathcal{L}(G,\Gamma)/\mathcal{T}(G,\Gamma)$. These groups depend only on $|G|$ and not on the particular structure of the graph, so for a surface Σ we define $H_1(\Sigma,\Gamma)$ ($H^1(\Sigma,\Gamma)$) to be the homology (cohomology) group of every graph embedded in Σ. The group $H^1(G,\Gamma)$ may be viewed as a measure of the difference between the spaces of local tensions and tensions of G. The point of this paper is to show that under the right circumstances, it is possible to change a local tension of a graph embedded on a surface to a tension by small adjustments on each edge. As such, the group $H^1(G,\Gamma)$ will play a key role. For any abelian group Γ, we let $\mathrm{Inv}(\Gamma)$ denote the subgroup of Γ consisting of the elements of order ≤ 2. For convenience, we state here the cohomology groups of the surfaces [4]:

$$H^1(\mathbb{S}_i,\Gamma) \cong \Gamma^{2i} \qquad \text{and} \qquad H^1(\mathbb{N}_j,\Gamma) \cong \Gamma^{j-1} \times \mathrm{Inv}(\Gamma).$$

e will use Proposition 2.1 to detect if a local tension is a tension, so the homology group $H_1(G,\mathbb{Z})$ will also play a key role in our argument. For convenience, we list this group here.

$$H_1(\mathbb{S}_i, \mathbb{Z}) \cong \mathbb{Z}^{2i} \qquad \text{and} \qquad H_1(\mathbb{N}_j, \mathbb{Z}) \cong \mathbb{Z}^{j-1} \times \mathbb{Z}/2\mathbb{Z}.$$

It is not true in general that every local tension may be modified to a tension by means of small adjustments (in Section 5 we offer an example illustrating this). However, we can find a suitable adjustment of every local tension which satisfies an additional property. We say that a local tension ϕ is *strong* if it satisfies the following equation:

$$\sum_{e \in E(G)} \tau(e)\phi(e) = 0. \tag{1}$$

Note that if ϕ is a local tension and we adjust G by flipping the orientation of the face R, then the above sum is changed by $\sum_{e \in E(G)} \langle e, R \rangle \phi(e) = 0$. Thus, the left hand side in (1) is independent of the orientation of the faces. We highlight two more observations below.

Proposition 2.2 (a) *If $|G|$ is orientable, then every local tension is strong.*
(b) $\sum_{e \in E(G)} \tau(e)\phi(e) \in \mathrm{Inv}(\Gamma)$.

Proof. To see part (a), note that if $|G|$ is orientable, then we may switch the orientation of faces to arrange that τ is identically zero. Part (b) follows from the equation $0 = \sum_{R \in F(G)} \sum_{e \in E(G)} \langle e, R \rangle \phi(e) = 2 \sum_{e \in E(G)} \tau(e)\phi(e)$. □

Let $\mathcal{L}^+(G, \Gamma)$ denote the set of strong Γ-local tensions of G. It follows easily that $\mathcal{L}^+(G, \Gamma)$ is a group. Further, since $\tau \in \mathcal{F}(G, \mathbb{Z})$ we have by Proposition 2.1 that $\mathcal{T}(G, \Gamma)$ is a subgroup of $\mathcal{L}^+(G, \Gamma)$. We summarize the group containment relations below.

$$\mathcal{T}(G, \Gamma) < \mathcal{L}^+(G, \Gamma) < \mathcal{L}(G, \Gamma)$$

As was the case with local tensions, the group $H^+(G, \Gamma) = \mathcal{L}^+(G, \Gamma)/\mathcal{T}(G, \Gamma)$ depends only on the surface $|G|$ and not on the structure of the graph so for a surface Σ, we define $H^+(\Sigma, \Gamma)$ to be $H^+(G, \Gamma)$ for any graph G embedded in Σ. In comparison with the cohomology groups, we have

$$H^+(\mathbb{S}_i, \Gamma) \cong \Gamma^{2i} \qquad \text{and} \qquad H^+(\mathbb{N}_j, \Gamma) \cong \Gamma^{j-1}$$

3 Main results

For any abelian group Γ and any positive integer k, we say that a subset $Q \subseteq \Gamma$ is a *k-part* of Γ if $0 \in Q$, $Q = -Q$, and $\underbrace{Q + Q + \cdots + Q}_{k} = \Gamma$.

Theorem 3.1 *Let $\mathcal{X}$ be a surface and let Q be a k-part of the abelian group Γ. Then there exists an integer M, so that for every graph G embedded in Γ with edge-width $\geq M$ and every $\phi \in \mathcal{L}^+(G, \Gamma)$, there is a $\phi' \in \mathcal{T}(G, \Gamma)$ with $\phi(e) - \phi'(e) \in 2Q$ for every $e \in E(G)$.*

The main application of the above result is the following theorem restated from the introduction.

Theorem 1.1: *Let $\mathcal{X}$ be a fixed surface and let $\varepsilon > 0$. There exists an integer M so that $\chi_{\mathrm{c}}(G) \leq \chi_{\mathrm{lt}}(G) + \varepsilon$ for every graph embedded in $\mathcal{X}$ with edge-width $\geq M$.*

The relationship between tensions and local tensions turns out to be very useful in understanding the circular chromatic number of several families of embedded graphs. In the final section of the paper, we consider two such families. The main results are summarized below.

If the boundary of every face of G contains an even number of edges, then we say that G is *locally bipartite*. Note that in this case, $\prod_{e \in E(G)} \sigma(e)$ is invariant under switching the orientation of faces. We say that G is of *odd type* if $\prod_{e \in E(G)} \sigma(e) = -1$ and G is of *even type* otherwise. Using Theorem 1.1, we arrive at the following result concerning locally bipartite embedded graphs.

Theorem 3.2 *For every fixed surface S and every $\varepsilon > 0$, there exists a positive integer M such that $\chi_{\mathrm{c}}(G) \leq 2 + \varepsilon$ for every graph G embedded in S that is locally bipartite, of even type and has edge-width at least M.*

The restriction of this theorem to the case when S is orientable is a result of Hutchinson [7]. Note that in this case the assumption of even type is redundant. Indeed, a locally bipartite embedding is of odd type if and only if there exists a onesided cycle C in G of odd length such that after cutting the surface along C, an orientable surface is obtained. When G is of odd type and locally bipartite, there is a lower bound on $\chi_{\mathrm{lt}}(G)$ which is independent of the edge-width. This gives us the following lower bound on $\chi_{\mathrm{c}}(G)$.

Theorem 3.3 *If G is a locally bipartite embedded graph of odd type, then $\chi_c(G) \geq 2 + 4/(r-2)$ where r is the size of the largest face in G.*

A weaker form of Theorem 3.3 for the usual chromatic number was proved for the case of quadrangulations of the projective plane (i.e., the case when $r = 2$) by Youngs [14], and extended to arbitrary surfaces by Archdeacon, Hutchinson, Nakamoto, Negami, and Ota [2], and by Mohar and Seymour [10].

An embedded graph G is a triangulation if every face of G has exactly three edges in its boundary. Applying the same ideas as used to find bounds on the circular chromatic number of locally bipartite graphs, we establish two theorems concerning the circular chromatic number of triangulations. For simplicity, we state these theorems here only for orientable surfaces.

Theorem 3.4 *For every orientable surface $\mathcal{X}$ and every $\varepsilon > 0$, there exists a positive integer M with the property that $\chi_c(G) \leq 3+\varepsilon$ for every triangulation G of $\mathcal{X}$ with bipartite dual and with edge-width at least M.*

In the case when the dual is not bipartite, we find a lower bound on χ_{lt} which gives the following lower bound on χ_c.

Theorem 3.5 *If G is a triangulation of an orientable surface and the dual of G is not bipartite, then $\chi_c(G) \geq 4$.*

If G is a triangulation with a bipartite dual, then every vertex of G must have even degree. Thus, these two theorems are close relatives of the following result of Hutchinson, Richter, and Seymour [8]: If G is an Eulerian triangulation of an orientable surface and G has sufficiently large edge-width, then $\chi(G) \leq 4$.

4 From strong local tensions to tensions

Let $C \subseteq G$ be a circuit and let $f \in E(C)$. Every $f \in E(C)$ will either have the same or the opposite orientation as e (with respect to C). Define the map $I_C : E(G) \to \mathbb{Z}$ by the following rule:

$$I_C(e) = \begin{cases} 1 & \text{if } f \in E(C) \text{ has the same orientation as } e \\ -1 & \text{if } f \in E(C) \text{ has the opposite orientation as } e \\ 0 & \text{if } f \notin E(C) \end{cases}$$

Note that a different choice of e would either yield the same map I_C or yield the map $-I_C$. Since this difference will not be essential, we will frequently write I_C without specifying the choice of the edge e.

It follows immediately from the definitions that $I_C \in \mathcal{F}(G, \mathbb{Z})$. We say that a set of circuits Y *generates* $H_1(G, \mathbb{Z})$ if $\{I_C + \mathcal{K}(G, \mathbb{Z}) \mid C \in Y\}$ generates $H_1(G, \mathbb{Z})$. If $\phi \in \mathcal{L}(G, \Gamma)$ and $C \subseteq G$ is a circuit, then we say that C is *conservative* if $\sum_{e \in E(G)} \phi(e) I_C(e) = 0$. The following proposition states two key properties we will require in the main theorem.

Proposition 4.1 *Let $\phi \in \mathcal{L}(G, \Gamma)$, let Y be a set of circuits which generate $H_1(G, \mathbb{Z})$ and let C be a circuit with $2I_C \in \mathcal{K}(G, \mathbb{Z})$.*

(i) *$\phi \in \mathcal{T}(G, \Gamma)$ if and only if every circuit in Y is conservative.*

(ii) *If ϕ is a strong local tension, then C is conservative.*

Proof. Part (i) follows immediately from Proposition 2.1. For Part (ii), note that if $2I_C \in \mathcal{K}(G, \mathbb{Z})$, then we may orient the faces of G so that $e \in E(G)$ has $\sigma(e) = -1$ if and only if $e \in E(C)$. Now the assumption that ϕ is a strong local tension implies that C is conservative. □

Let G be an embedded graph, let $e \in E(G)$ have ends u, v, and let $R_1, R_2 \in F(G)$ be the faces incident with e. If there are no edges parallel to e, then we let G/e denote the embedded graph obtained from G by identifying u and v to a single new vertex, say w, removing e from $E(G)$, and replacing each occurrence of v, e, u or u, e, v in the boundary of R_1 and R_2 by w. We say that G/e is obtained from G by *contracting* the edge e. If e occurs in the boundary of distinct faces R_1 and R_2, and $\sigma(e) = 1$, then we let $G \setminus e$ denote the embedded graph obtained from G by removing e from $E(G)$, and identifying R_1 and R_2 in a natural way to a single new face whose boundary orientation is inherited from R_1 and R_2. We say that $G \setminus e$ is obtained from G by *deleting* the edge e. If $v \in V(G)$ appears at most once on the boundary of every face and every edge incident with v has signature 1, then we let $G \setminus v$ be the embedded graph obtained by deleting v and every edge incident with v, and then identifying the faces incident with v in the natural manner. We say that $G \setminus v$ is obtained from G by *deleting* the vertex v. Any embedded graph H obtained from G by a sequence of switching orientations of edges and faces and deleting and contracting edges and deleting vertices is called a *surface minor* of G.

The reader will observe that our definition of surface minor is somewhat more restrictive than the usual one. For instance, if e is an edge incident with two distinct faces, then we cannot delete e if $\sigma(e) = -1$. This type of restriction is inessential as we may simply flip the orientation of a face incident with e first and then delete e. A similar proceedure is required to delete a vertex. A second difference is that we are not permitted to delete an edge which is incident with only one face, so for instance a graph embedded in the torus will not contain any one edge graph as a surface minor. Fortunately, we are still able to apply Theorem 4.2, since no such deletions are made when using ordinary surface minors to create a 2-cell embedded graph.

We define the *face-width* of G to be the smallest integer k so that every non-contractible curve in $|G|$ intersects the graph in at least k points. It is easy to see that the face-width of G is always less than or equal to the edge-width of G. We shall begin this section by proving our main results for graphs of high face-width. In the last part of this section we provide a reduction showing that our results also extend to graphs of high edge-width.

Theorem 4.2 (Robertson, Seymour) *For every graph H embedded in $\mathcal{X}$, there exists an integer M such that every graph G embedded in $\mathcal{X}$ with face-width at least M contains a surface minor isomorphic to H.*

In [12], only the existence of the constant M in Theorem 4.2 is proved. Explicit upper bounds which are not too large can be derived from [3] and [6], see also [9].

⟦Define R_h, R_h^k, T_h, T_h^k. Make help of Figure 1].⟧

Figure 1: Embedded graphs R_h and T_h

A surface minor H of G is a *subdivision* of G if H can be obtained from G by contracting edges which are incident with a vertex of degree two. Since the graphs R_h and T_h have maximum degree three, any graph which has an R_h or T_h minor also has an R_h or T_h subdivision.

If C is a circuit of G/e, then exactly one of $E(C)$ or $E(C) \cup \{e\}$ is the edge set of a circuit C' in G. We say that this circuit C' *corresponds* to C. More generally, if H is a surface minor of G and $C \subseteq H$ is a circuit, then there is a circuit $C' \subseteq G$ which *corresponds* to C. If Y is a collection of circuits of H, then we say that $Y' = \{C' \subseteq G \mid C'$ corresponds to $C\}$ *corresponds* to Y.

Lemma 4.3 *Let G be a graph embedded in $\mathbb{N}_{2h+i}$ where $i = 0, 1$ ($\mathbb{S}_h$) and assume that G contains a subdivision of R_{2h+i} (T_h). Let $A'_1, B'_1, \ldots, A'_h, B'_h$ (and C' and D' if applicable) be the circuits of G which correspond to $A_1, B_1, \ldots, A_h, B_h$ (and C and D if applicable). Then we have:*

(i) $\{A'_1, B'_1, \ldots, A'_h, B'_h\}$ *(and C' and D' if applicable) generate $H_1(G, \mathbb{Z})$.*

(ii) $2I_{C'} \in \mathcal{K}(G, \mathbb{Z})$

Proof. Part (i) follows by construction. Part (ii) follows from the observation that we may orient the faces of G so that an edge $e \in E(G)$ has $\sigma(e) = -1$ if and only if $e \in E(C')$. □

A *face walk* W of G is a sequence $R_1, e_1, R_2, e_2, \ldots, e_k, R_{k+1}$ with the property that $R_i \in F(G)$ for $1 \leq i \leq k+1$, and $e_i \in E(G)$ is an edge incident with the distinct faces R_i and R_{i+1} for $1 \leq i \leq k$. We say that W is a *closed* face walk if $R_1 = R_{k+1}$. We define the *sign* of W to be $\sigma(W) = \prod_{i=1}^{k} \sigma(e_i)$. Suppose that W is a closed face walk and that $R_1, R_2, \ldots, R_{k+1}$ are all distinct. Form a topological space by identifying the disjoint union of discs corresponding to these faces along the edges $e_1, e_2, \ldots, e_k$. It follows from our definitions that this space is a cylinder if $\sigma(W) = 1$ and it is a Möbius band if $\sigma(W) = -1$.

Let G be an embedded graph, let x be an element of the abelian group Γ, and let W be the closed face walk $R_1, e_1, \ldots, e_k, R_{k+1}$. We define $\Omega^x_W : E(G) \to \Gamma$ by the rule

$$\Omega^x_W(e) = \begin{cases} \prod_{j=1}^{i-1} \sigma(e_j)\langle e_i, R_i \rangle x & \text{if } e = e_i \text{ for some } 1 \leq i \leq k \\ 0 & \text{otherwise} \end{cases}$$

Proposition 4.4 *Let G be an embedded graph and let Γ be an abelian group. Then for every closed face walk W with $\sigma(W) = 1$ and every $x \in \Gamma$, the map Ω^x_W is a strong local tension.*

Proof. Let $R \in F(G)$ be a face and $e \in E(G)$ an edge. If $\Omega^x_W \neq 0$, then $R = R_i$ is some face in the face walk W and $e \in \{e_{i-1}, e_i\}$. In this case, the contributions of e_{i-1} and e_i to $\Omega^x_W(R_i)$ cancel. ⟦Are the indices correct?⟧ It follows that Ω^x_W is a strong local tension. □

Lemma 4.5 *Let G be an embedded graph in the surface $\mathbb{N}_{2h+i}$ (where $i = 0, 1$) (or $\mathbb{S}_h$) and assume that G contains a subdivision of R^k_{2h+i} (or T^k_h). Let $A'_1, B'_1, \ldots, A'_h, B'_h$ (and C' if the surface is $\mathbb{N}_{2h+1}$) (and D' if the surface is $\mathbb{N}_{2h+2}$) be the circuits of G which correspond to $A_1, B_1, \ldots, A_h, B_h$ (and C and D if applicable) and let $Q = E(C) \cup E(D) \cup \bigcup_{i=1}^{h} (E(A'_i) \cup E(B'_i)$. Then there exist facial walks α^j_i, β^j_i (and γ^j if the surface is $\mathbb{N}_{2h+2}$) for $1 \leq i \leq h$ and $1 \leq j \leq k$ with the following properties:*

(i) $\sigma(\alpha^j_i) = \sigma(\beta^j_i) = \sigma(\gamma^j) = 1$ *for* $1 \leq i \leq h$ *and* $1 \leq j \leq k$.

(ii) *Every face occurs in at most two of the facial walks* $\alpha^j_i, \beta^j_i, \gamma^j$.

(iii) α^j_i, β^j_i *and* γ^j *have no repeated faces.*

(iv) α^j_i *contains exactly one edge of* A'_i *and is disjoint from* $Q \setminus E(A'_i)$.

(v) β^j_i *contains exactly one edge of* B'_i *and is disjoint from* $Q \setminus E(B'_i)$.

(vi) γ^j *contains exactly one edge of* D' *and is disjoint from* $Q \setminus E(D')$.

Proof. We proceed by induction on $|E(G)|$. The base case is when $G = R_{2h+i}$ or $G = T_h$ and here the walks are given in Figure **??**. For the inductive step, contract or delete an edge e forming G' so that G' still has R_{2h+i} or T_h as a minor. By induction, G' has face walks with the properties above. If e was contracted, then the face walks of G' are face walks of G with the required properties. If e was deleted from a face F, and adding e back splits F into the faces F_1, F_2, then we adjust each face walk containing F in the obvious manner. It is easy to verify that the face walks will still satisfy (i)-(vi) above. □

With this we are ready to prove the following lemma, which identical to the main theorem except that edge-width is replaced here by face-width.

Lemma 4.6 *For every surface $\mathcal{X}$ and positive integer k there exists a positive integer M with the following property. For every k-part Q of an abelian group Γ, every graph G embedded in $\mathcal{X}$ with face-width at least M, and every $\phi \in \mathcal{L}^+(G, \Gamma)$, there exists $\phi' \in \mathcal{T}(G, \Gamma)$ such that $\phi(e) - \phi'(e) \in 2Q$ for every $e \in E(G)$.*

Proof. Apply Theorem 4.2 to choose an integer M so that every graph embedded in Σ with face-width at least M contains R^k_{2h+i} as a minor if $\Sigma \cong \mathbb{N}_{2h+i}$ for $i = 1, 2$ and T^k_h as a minor if $\Sigma \cong \mathbb{S}_h$. Let G be a graph of face-width $\geq M$ embedded in Σ and let $\phi \in \mathcal{L}^+(G, \Gamma)$. By Theorem 4.2 G contains R^k_{2h+i} as a subdivision if $\Sigma \cong \mathbb{N}_{2h+i}$ and it contains T^k_h as a subdivision if $\Sigma \cong \mathbb{S}_h$. Let $A'_1, B'_1, \ldots, A'_k, B'_k$ and C' (if the surface is $\mathbb{N}_{2h+1}$) and D' (if the surface is $\mathbb{N}_{2h+2}$) be the circuits of G which correspond to $A_1, B_1, \ldots, A_k, B_k$ and C and D if applicable of R^k_{2h+i} or T^k_{2h}. Choose walks $\alpha^j_i, \beta^j_i, \gamma^j$ in accordance with Lemma 4.5. Let x^j_i, y^j_i, z^j be variables in Γ for $1 \leq i \leq h$ and $1 \leq j \leq k$ and consider the map

$$\phi' = \phi + \sum_{i,j} \Omega^{\alpha^j_i}_{x^j_i} + \sum_{i,j} \Omega^{\beta^j_i}_{y^j_i} + \sum_{j} \Omega^{\gamma^j}_{z^j}$$

It follows from this construction that $\phi' \in \mathcal{L}(G, \Gamma)$. Furthermore, by properties (iv)-(vi) and the assumption that Q is a k-part, we may choose x^j_i, y^j_i, and z^j in Q so that $A'_1, B'_1, \ldots, A'_h, B'_h$ and D' are all conservative with respect to ϕ'. By part (ii) of Proposition 4.1 we have that C' is a conservative circuit and by part (i) we have that that ϕ' is a tension as required. □

Let R be a face of the embedded graph G which is bounded by a circuit C_0 of length $k \geq 4$. Next we shall define an embedded graph from G. Throughout this description we will work with the toplogical space $|G|$. We start by drawing k nested circuits $C_1, C_2, \ldots, C_k$ inside the disc $\pi(R)$ (i.e. inside the face R). For $1 \leq i \leq k$, we draw a perfect matching between $V(C_{i-1})$ and $V(C_i)$. The orientation of R gives an orientation of the disc $\pi(R)$, and we give this orientation to each new face formed by our drawing. We then orient the new edges arbitrarily. We say that this new embedded graph is obtained from G by adding a *chimney* to R.

⟦a figure depicting a chimney would be good here⟧

The following proposition is easy to verify, a proof can also be found in [1]

Proposition 4.7 *Let G be an embedded graph and modify G to form the embedded graph G' by adding a chimney to every face of G which is bounded by a circuit of length ≥ 4. Then the edge-width of G' is at least the face-width of G.*

With the help of this reduction we can now prove Theorem 3.1

Proof of Theorem 3.1: Form the embedded graph G' as in Proposition 4.7 by adding a chimney to every face which is bounded by a circuit of length ≥ 4. Now, it is straightforward to find $\psi \in \mathcal{L}(G', \Gamma)$ so that $\psi|_{E(G)} = \phi$. It follows immediately from our construction that ψ is a strong local tension. Thus by Lemma 4.6 we may choose $\psi' \in \mathcal{T}(G', \Gamma)$ so that $\psi'(e) - \psi(e) \in 2Q$ for every $e \in E(G')$. It now follows that $\phi' = \psi'|_{E(G)}$ is a tension of G and the proof is complete. $\square$

Let H be a directed graph, let $C \subseteq H$ be a circuit, and let (A, B) be a partition of $E(G)$ so that every edge in A and every edge in B have opposite orientations relative to C. If A or B is empty, then we define the *imbalance* of C, denoted by $\iota(C)$ to be ∞. Otherwise, we define the *imbalance* of C to be $\iota(C) = \min\{|E(C)|/|A|, |E(C)|/|B|\}$. We define the *imbalance* of H, denoted by $\iota(H)$, to be the maximum of $\iota(C)$ over all circuits $C \subseteq H$. If G is an unoriented graph, then we let $\mathcal{O}(G)$ denote the set of orientations of G.

Let $\mathbf{S} = \mathbb{R}/\mathbb{Z}$ denote the circle group. We identify $\mathbf{S}$ with the half open interval $[0, 1)$ in the usual way. For every $x \in \mathbf{S}$, we let $\|x\| = \min\{x, 1 - x\}$.

We now state three equivalent expressions for the circular chromatic number [5].

Theorem 4.8 *Let G be an (undirected) graph and let $\alpha \geq 2$ be a real number. Then the following are equivalent.*

(i) $\chi_c(G) = \alpha$.

(ii) $\min\{\iota(G_0) \mid G_0 \in \mathcal{O}(G)\} = \alpha$.

(iii) $\max_{\phi \in \mathcal{T}(G_0, \mathbf{S})} \min_{e \in E(G_0)} \|\phi(e)\| = \frac{1}{\alpha}$ *for every $G_0 \in \mathcal{O}(G)$*

With the help of the equivalence of (i) and (iii) in Theorem 4.8, we are now ready to prove the following theorem from the introduction.

Proof of Theorem 1.1: It is a consequence of Euler's formula that we may choose an integer M_0 so that every graph H embedded in Σ with edge-width at least M_0 has $\chi(H) \leq 6$. Note that this also implies $\chi_c(H), \chi_{lt}(H) \leq 6$. Choose an integer n such that $\varepsilon > \frac{72}{n}$ and apply Lemma 4.6 to choose a positive integer M_1 based on the surface $\mathcal{X}$ and the parameter n. Let $M = \max\{M_0, M_1\}$. We claim that $\chi_c(G) \leq \chi_{lt}(G) + \varepsilon$ for every graph G embedded in $\mathcal{X}$ with edge-width at least M.

Let $\psi : E(G) \to \mathbb{R}$ be an α-local tension of G such that $\frac{1}{\alpha} \geq \frac{1}{\chi_{lt}(G)} - \frac{1}{n}$. Define $\phi : E(G) \to \mathbf{S}$ by the rule $\phi(e) = \psi(e)/\alpha$. Now ϕ is a $\mathbf{S}$-local tension with $\min_{e \in E(G)} \|e\| \geq \frac{1}{\alpha}$. Furthermore, ψ is a strong local tension (by Proposition 2.2 and the fact that $\mathrm{Inv}(\mathbb{R}) = \{0\}$), so ϕ is also a strong local tension. The interval $[-\frac{1}{2n}, \frac{1}{2n}]$ is a n-part of $\mathbf{S}$, so by the main theorem, we may choose a tension $\phi' : E(G) \to \mathbf{S}$ such that $\phi'(e) - \phi(e) \in [-\frac{1}{n}, \frac{1}{n}]$. Now ϕ' is a tension of G so we have that

$$\begin{aligned}
\chi_{\mathrm{c}}(G) - \chi_{\mathrm{lt}}(G) &\leq \frac{36}{\chi_{\mathrm{c}}(G)\chi_{\mathrm{lt}}(G)}(\chi_{\mathrm{c}}(G) - \chi_{\mathrm{lt}}(G)) \\
&= 36(\frac{1}{\chi_{\mathrm{lt}}(G)} - \frac{1}{\chi_{\mathrm{c}}(G)}) \\
&\leq 36(\frac{1}{\chi_{\mathrm{lt}}(G)} - \min_{e \in E(G)} \|\phi'(e)\|) \\
&\leq 36(\frac{1}{\chi_{\mathrm{lt}}(G)} - \min_{e \in E(G)} \|\phi(e)\| + \frac{1}{n}) \\
&\leq 36(\frac{1}{\chi_{\mathrm{lt}}(G)} - \frac{1}{\alpha} + \frac{1}{n}) \\
&\leq \varepsilon
\end{aligned}$$

as required. □

5 Families of embedded graphs

In this section, we study two families of embedded graphs, locally bipartite graphs and triangulations. Recall that an embedded graph is *locally bipartite* [10] if every facial walk has even length. For every locally bipartite graph G we find necessary and sufficient conditions for $\chi_{\mathrm{lt}}(G) = 2$. For every triangulation G we find necessary and sufficient conditions for $\chi_{\mathrm{lt}}(G) = 3$. In both cases, we obtain lower bounds on the circular chromatic number of graphs which do not satisfy the conditions, and (using our main theorem) we obtain upper bounds on the circular chromatic number of graphs which do satisfy the conditions.

The following lemma gives a characterization of when the edges of a locally bipartite embedded graph may be oriented such that every face has the same number of forward edges as backward edges.

Lemma 5.1 *Let G be a locally bipartite embedded graph. It is possible to switch the orientation of some edges in G so that every face has the same number of forward edges as backward edges if and only if G is of even type.*

Proof. Consider the following equation

$$\sum_{R \in F(G)} \sum_{e \in E(G)} \langle e, r \rangle = 2 \sum_{e \in E(G)} \tau(e) \tag{2}$$

Every $e \in E(G)$ with $\sigma(e) = 1$ contributes zero to the right hand side and every edge $e \in E(G)$ with $\sigma(e) = -1$ contributes either 2 or -2 to the right hand side. It follows that the total number of forward edges minus the total number of backward edges is congruent to 2 modulo 4 if G is of odd type and it is congruent to 0 modulo 4 if G is of even type. In particular, if G can be oriented so that every face has the same number of forward edges as backward edges, then G must be even.

To prove the if direction we will require the dual graph. Construct the graph G^* with vertex set $F(G)$ by joining two vertices of G^* if and only if these faces are adjacent in G. Since G^* is connected and G is locally bipartite, G^* is Eulerian. Choose an Euler tour $v_1, e_1^*, \ldots, e_k^*, v_{k+1}$ of G^* and let $R_1, e_1, \ldots, e_k, R_{k+1}$ be the corresponding face walk of G. Orient e_1 arbitrarily, and for e_i with $1 < i < k$, give e_i the opposite orientation as e_{i-1} relative to the face R_i (if e_{i-1} is forward in R_i make e_i backward and vice versa). It follows immediately from this construction that the faces $R_2, R_3, \ldots, R_k$ have the same number of forward edges as backward edges and that the number of forward edges and backward edges of $R_1 = R_{k+1}$ differ by at most 2. Now, by our earlier observation that the total number of forward edges minus the total number of backward edges is congruent to 0 mod 4, we find that the face R_1 is oriented properly, so G is oriented as desired. □

The above lemma combined with Theorem 4.8 allows us to prove Theorem 3.2 and Theorem 3.3 from the introduction.

Proof of Theorem 3.2: If G is a locally bipartite graph of even type, then by the above lemma we may choose an orientation of G so that every face has the same number of forward edges as backward edges. Now the map $\phi : E(G) \to \mathbb{R}$ given by the rule $\phi(e) = 1$ for every $e \in E(G)$ is a 2-local-tension of G. The result now follows immediately from our main theorem. □

Proof of Theorem 3.3: By Theorem 4.8, we have that $\chi_c(G) = \min_{G_0 \in \mathcal{O}(G)} \iota(G_0)$. If there is no orientation of G which balances every facial circuit, then every orientation must have imbalance at least $2+4/(r-2)$ where r is the size of the largest face. □

As noted in the introduction, this implies that every non-bipartite quadrangulation of the projective plane has circular chromatic number at least 4 (to see that such a graph G is odd, note that we may orient the faces so that the edges with $\sigma(e)=-1$ are precisely the edges in some odd circuit $C \subseteq G$). The following proposition is well known and easy to verify.

Proposition 5.2 *For every non-orientable surface $\mathcal{X}$, there exists a collection of quadrangulations of $\mathcal{X}$ odd-type which have unbounded edge-width.*

With the help of the above proposition, we can finally demonstrate that Theorem 3.1 would not hold true under the weaker assumption that $\phi \in \mathcal{L}(G,\Gamma)$. To see this, let G be a quadrangulation of odd type. Now the map $\phi : E(G) \to \mathcal{S}$ given by the rule $\phi(e) = 1/2$ is a local-tension, but by Theorem 4.8 and Theorem 3.3 $\min_{e \in E(G)} \|\psi(e)\| \leq 1/4$ for every tension $\psi : E(G) \to \mathcal{S}$.

Mohar proved that the graph obtained from a non-bipartite quadrangulation of the projective plane by adding a new vertex in every face adjacent to all four vertices on the boundary has chromatic number at least 5. This property also extends to circular chromatic number as follows.

Proposition 5.3 *Let G be a quadrangulation of a surface of odd type and let G' be the embedded graph obtained from G by adding a new vertex in each face adjacent to all four vertices on the boundary. Then $\chi_c(G') \geq 5$.*

Proof. It will suffice to show that $\iota((V(G'), E(G'))) \geq 5$. By Lemma 5.1, we may choose a face R of G such that the circuit C bounding R is unbalanced. Let $C' \subseteq G'$ be the subgraph of G' consisting of C together with the vertex added in face R and all of its incident edges. It is now easy to verify that any acyclic orientation of C' contains a 5-circuit with either only one forward edge or only one backward edge. Thus $\iota(G_0) \geq 5$ as required. □

Propositions 5.2 and 5.3 establish for every non-orientable surface $\mathcal{X}$, the existence of a collection of graphs of unbounded edge width and circular chromatic number at least 5. Next we turn our attention to triangulations.

Lemma 5.4 *Let G be a triangulation. Then the following are equivalent.*

(i) *G has a 3-local tension.*

(ii) *G has a $(4-\varepsilon)$-local tension for some $\varepsilon > 0$.*

(iii) *The faces of G may be oriented such that $\sigma(e) = -1$ for every $e \in E(G)$.*

(iv) *There does not exist a closed face walk $R_1, e_1, \dots, e_k, R_{k+1}$ such that $|\{1 \le i \le k \mid \sigma(e_i) = 1\}|$ is odd.*

Proof. We prove that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i), and that (iii) $\Leftrightarrow$ (iv).

Clearly, (i) implies (ii). To prove that (ii) implies (iii), let $\phi : E(G) \to \mathbb{R}$ be a $(4-\varepsilon)$-local tension of G for some $\varepsilon > 0$. By possibly reversing the orientation of the edge e and replacing $\phi(e)$ by $-\phi(e)$, we may assume that $\phi(e) > 0$ for every $e \in E(G)$. Let $S = \{e \in E(G) \mid \phi(e) < 2\}$ and let $B = \{e \in E(G) \mid \phi(e) \ge 2\}$. Since every face of G is a triangle and $\phi(e) < 3$ for every $e \in E(G)$, it follows that every face R is incident with one edge in B and two edges in S. Furthermore, the two edges in S have the opposite orientation with respect to R as the edge in B. Now, by possibly flipping the orientation of some faces, we may assume that every face is oriented such that the unique edge in B incident with it is a forward edge. Now, every edge e has $\sigma(e) = -1$ as desired.

To prove that (iii) implies (i), suppose that G has an orientation as described. If H is the (unoriented) geometric dual graph, then H is cubic and since G is loopless, H must be 2-edge-connected. It now follows from Petersen's theorem that we may choose a perfect matching $M \subseteq E(H)$. Let M' be the corresponding set of edges in G. Now, by possibly changing the orientation of some edges, we may assume that every edge in M' is an up edge and that every edge in $E(G) \setminus M'$ is a down edge. Now the map $\phi : E(G) \to \mathbb{R}$ given by the rule

$$\phi(e) = \begin{cases} 2 & \text{if } e \in M' \\ 1 & \text{otherwise} \end{cases}$$

is a 3-local tension.

To see that (iii) implies (iv) note that the parity of $|\{1 \le i \le k \mid \sigma(e_i) = 1\}|$ is unaffected by flipping the orientation of faces. If this set has odd size, then no reorientation of the faces can satisfy (iii).

Next we prove that (iv) implies (iii). Choose a face $R \in F(G)$ and let X be the set of faces $Q \in F(G)$ such that there exists a face walk $R_1, e_1, \dots, e_k, R_{k+1}$ with $R_1 = R$, $R_{k+1} = Q$, and $|\{1 \le i \le k \mid \sigma(e) = 1\}|$ odd. Let $Y = F(G) \setminus X$. Since no closed face walk has an odd number of edges e with $\sigma(e) = 1$, it follows that an edge $e \in E(G)$ has $\sigma(e) = 1$ if and only if e is incident with one face in X and one face in Y. Thus, the orientation obtained by reversing the faces in X has sign which is the constant -1. □

As was the case with the orientation lemma for locally bipartite graphs, the above lemma yields the following lower bound on the circular chromatic number of triangulations.

Theorem 5.5 *Let G be a triangulation. If G contains a closed face walk with an odd number of edges e with $\sigma(e) = 1$, then $\chi_c(G) \geq 4$.*

Proof. If $\chi_c(G) < 4$, then G has a $(4 - \varepsilon)$-tension for some $\varepsilon > 0$. Since this map is also a $(4 - \varepsilon)$-local tension, this contradicts the above lemma. □

Theorem 3.5 from the introduction is the specialization of the above theorem to the case when the surface is orientable. Also as was the case earlier, the above lemma in conjunction with our main theorem now yields an upper bound on the circular chromatic number. Theorem 3.4 from the introduction is the specialization of the following result to orientable surfaces.

Theorem 5.6 *For every surface $\mathcal{X}$ and every $\varepsilon > 0$, there exists an integer M such that $\chi_c(G) \leq 3 + \varepsilon$ for every triangulation G of $\mathcal{X}$ with the property that no closed face walk of G contains an odd number of edges e with $\sigma(e) = 1$.*

Proof. This follows immediately from Theorem 1.1 and from Lemma 5.4. □

6 Conclusion

bordered surfaces

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