A short proof of Kneser's addition theorem for abelian groups

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Abstract

Martin Kneser proved the following addition theorem for every abelian group G. If $X, Y \subseteq G$ are finite and nonempty, then $|X+Y| \geq |X+K| + |Y+K| - |K|$ where $K = \{g \in G \mid g + X + Y = X + Y\}$. The purpose of this note is to give a short proof of this fact. Our proof relies only upon a simple intersection union argument.

Throughout we shall assume that G is an additive abelian group. If $X, Y \subseteq G$ and $g \in G$, then $X + Y = \{x + y \mid x \in X \text{ and } y \in Y\}$ and $X + g = g + X = \{x + g \mid x \in X\}$. We define the *stabilizer* of X to be $stab(X) = \{g \in G \mid X + g = X\}$. Note that $stab(X) \leq G$.

Theorem 1 (Kneser) *If $X, Y \subseteq G$ are finite and nonempty and $K = stab(X+Y)$, then $|X+Y| \geq |X+K| + |Y+K| - |K|$.*

Proof. We proceed by induction on $|X+Y| + |X|$. Suppose that $K \neq \{0\}$ and let $\phi : G \to G/K$ be the canonical homomorphism. Then $stab(\phi(X+Y))$ is trivial, so by applying induction to $\phi(X), \phi(Y)$ we have that $|X+Y| = |K|(|\phi(X)+\phi(Y)|) \geq |K|(|\phi(X)|+|\phi(Y)|-1) = |X+K| + |Y+K| - |K|$. Thus, we may assume that $K = \{0\}$. If $|X| = 1$, then the result is trivial, so we may assume $|X| > 1$ and choose distinct $x, x' \in X$. Since $x' - x \notin stab(Y) \subseteq stab(X+Y) = \{0\}$, we may choose $y \in Y$ so that $y + x' - x \notin Y$. Now by replacing Y by $Y - y + x$ we may assume that $\emptyset \neq X \cap Y \neq X$.

Let $Z \subseteq X + Y$ and let $H = stab(Z)$. We say that Z is a *portion* if

$$|Z| + |H| \geq |X \cap Y| + |(X \cup Y) + H|.$$

Set $Z_0 = (X \cap Y) + (X \cup Y)$ and observe that $Z_0 \subseteq X + Y$. Since $0 < |X \cap Y| < |X|$, we may apply induction to $X \cap Y$ and $X \cup Y$ to conclude that Z_0 is a portion. Thus a portion exists, and we may now choose a portion Z with $H = stab(Z)$ minimal. If $H = \{0\}$ then $|X+Y| \geq |Z| \geq |X \cap Y| + |X \cup Y| - |\{0\}| = |X| + |Y| - 1$ and we are finished. Therefore, we may assume (for a contradiction) that $H \neq \{0\}$. Since $stab(X+Y) = \{0\}$ and $stab(Z) = H$, we may choose $a \in X$ and $b \in Y$ so that $(a + b + H) \not\subseteq X + Y$. Let $X_1 = X \cap (a + H)$, $X_2 = X \cap (b + H)$, $Y_1 = Y \cap (b + H)$, and $Y_2 = Y \cap (a + H)$ and note that $X_1, Y_1 \neq \emptyset$. For $i = 1, 2$ let $Z_i = Z \cup (X_i + Y_i)$ and let $H_i = stab(X_i + Y_i)$. Observe that if $X_i, Y_i \neq \emptyset$, then $H_i = stab(Z_i) < H$. The following equation holds for $i = 1$, and it also holds for $i = 2$ if $X_2, Y_2 \neq \emptyset$. It follows from the fact that Z_i is not a portion (by the minimaity of H), and induction applied to X_i, Y_i.

$$\begin{aligned} |(X \cup Y) + H| - |(X \cup Y) + H_i| &< (|Z| + |H| - |X \cap Y|) - (|Z_i| + |H_i| - |X \cap Y|) \\ &= |H| - |X_i + Y_i| - |H_i| \\ &\leq |H| - |X_i + H_i| - |Y_i + H_i| \qquad (1) \end{aligned}$$

If $Y_2 = \emptyset$, then $|(X \cup Y) + H| - |(X \cup Y) + H_1| \geq |H| - |X_1 + H_1|$ which contradicts equation 1 for $i = 1$. We get a similar contradiction under the assumption that $X_2 = \emptyset$. Thus $X_2, Y_2 \neq \emptyset$ and equation 1 holds for $i = 1, 2$. If $a + H = b + H$, then $X_1 = X_2$ and $Y_1 = Y_2$ and we find that $|(X \cup Y) + H| - |(X \cup Y) + H_1| \geq |H| - |(X_1 \cup Y_1) + H_1| \geq |H| - |X_1 + H_1| - |Y_1 + H_1|$ which contradicts equation 1. Therefore, $a + H \neq b + H$. The following inequality follows from the observation that the left hand side of equation 1 is nonnegative, and all terms on the right hand side are multiples of $|H_i|$.

$$|H| \geq |X_i| + |Y_i| + |H_i| \qquad (2)$$

Let $A = (a + H) \setminus (X_1 \cup Y_2)$ and let $B = (b + H) \setminus (X_2 \cup Y_1)$. Note that A and B are disjoint. The following equation follows from the fact that $X + Y$ is not a portion (by the minimality of H), and induction applied to X_i, Y_i.

$$\begin{aligned} |H| &\geq |(X \cup Y) + H| + |X \cap Y| - |Z| \\ &\geq |A| + |B| + |X \cup Y| + |X \cap Y| - |X + Y| + |X_i + Y_i| \\ &> |A| + |B| + |X_i| + |Y_i| - |H_i| \qquad (3) \end{aligned}$$

Summing the four inequalites obtained by taking equations 2 and 3 for $i = 1, 2$ and then dividing by two yields $2|H| > |X_1|+|Y_2|+|A|+|X_2|+|Y_1|+|B|$. However, $a+H = A \cup X_1 \cup Y_2$ and $b + H = B \cup X_2 \cup Y_1$. This final contradiction completes the proof. $\square$

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References

[1] M.B. Nathanson, *Additive Number Theory: Inverse Problems and the Geometry of Sumsets*, GTM 165, Springer, 1996.