On Packing T-Joins

Matt DeVos Paul Seymour*

mdevos@sfu.ca pds@math.princeton.edu

Abstract

A *graft* is a graph $G = (V, E)$ together with a set $T \subseteq V$ of even cardinality. A *T-cut* of G is an edge cut $\delta(X)$ for which $|X \cap T|$ is odd. A *T-join* of G is a set of edges $S \subseteq E$ with the property that a vertex of the graph (V, S) has odd degree if and only if it is in T. A *T-join packing* of G is a set of pairwise disjoint T-joins.

Let $\tau(G)$ be the size of the smallest T-cut of G and let $\nu(G)$ be the size of the largest T-join packing of G. It is an easy fact that every T-cut and every T-join intersect. Thus, $\nu(G) \leq \tau(G)$.

In this paper, we prove that $\nu(G) \geq \lfloor \frac{1}{6}\tau(G) \rfloor$. In the specific case that G is eulerian, or $T = \{v \in V \mid \ deg(v) \text{ is odd}\}$, we prove that $\nu(G) \geq \lfloor \frac{1}{2}\tau(G) \rfloor$. This resolves conjecture of Zhang.

1 Introduction

In this paper, all graphs are finite, but may have loops or multiple edges. If G is a graph and $X \subseteq V(G)$, we let $\delta_G(X)$ denote the set of edges of G with one end in X and one end in $V(G) \setminus X$. If the underlying graph is clear from context, we drop the subscript and write $\delta(X)$. If $X = \{v\}$, we will abbreviate the notation by writing $\delta_G(v)$ or $\delta(v)$.

A *graft* is a graph G together with a set $T \subseteq V(G)$ of even cardinality. A *T-cut* of G is an edge-cut $\delta(X)$ with the property that $|X \cap T|$ is odd. A *T-join* is a set of edges $S \subseteq V(G)$

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with the property that a vertex of the graph (V, S) has odd degree if and only if it is in T. A *T-join packing* of G is a set of pairwise disjoint T-joins. We let $\tau(G)$ denote the size of the smallest T-cut of G and we let $\nu(G)$ denote the size of the largest T-join packing of G. An *r-graph* is a graft G for which G is r-regular, $T = V(G)$, and $\tau(G) \geq r$.

Every T-join must contain an odd number of edges from every T-cut, so in particular, every T-join intersects every T-cut. Thus $\nu(G) \leq \tau(G)$. The purpose of this paper is to establish the following two theorems which give a lower bound on $\nu(G)$ in terms of $\tau(G)$. The first theorem resolves a conjecture of Zhang [8].

Theorem 1.1 *If G is a graft and either G is eulerian, or $T = \{v \in V(G) \mid \ deg(v) \text{ is odd}\}$, then $\nu(G) \geq \lfloor \frac{1}{2}\tau(G) \rfloor$.*

Theorem 1.2 *$\nu(G) \geq \lfloor \frac{1}{6}\tau(G) \rfloor$ for every graft G.*

One may ask what the best possible parameters are in the above two theorems. There is a sequence of r-graphs $\{H_k\}_{k=2}^{\infty}$ with the property that $\tau(H_k) = k$ and $\nu(H_k) \leq k - 2$. To build the sequence, we let H_k be an r-graph (with $r = k$) which is not k-edge-colorable. If there was a T-join packing of H_k of size $k - 1$, then each of the T-joins in this packing would have to have degree one at every vertex. In other words, each T-join would have to be a perfect matching. Since H_k is not k-edge-colorable, we conclude that $\nu(H_k) \leq k - 2$. The following conjecture of Rizzi asserts that this example is essentially the worst case.

Conjecture 1.3 (Rizzi [5]) *If G is an r-graph, then $\nu(G) \geq \tau(G) - 2$.*

For the general case, we have a sequence of grafts F_k with the property that $\nu(F_k) = 2k$ and $\tau(F_k) = 3k$. Let G be the graph of the cube and let $\{U, V\}$ be a bipartition of $V(G)$. Now, we obtain F_k by adding $k-1$ new copies of each vertex in U, and then setting $T = V$. It is not difficult to verify that $\tau(F_k) = 3k$. Now, if $\{R_1, R_2, \ldots, R_t\}$ is a T-join packing in F_k, then every vertex of U is incident with either zero or two edges of R_i for every $1 \leq i \leq t$. It follows that at least one edge incident with every $u \in U$ is not contained in any of $R_1, R_2, \ldots, R_t$. Thus $|\bigcup_{i=1}^{t} R_i| \leq 2|U| = 8k$. Since every T-join of F_k has size at least 4, it follows that $t \leq 2k$ and we have that $\nu(F_k) \leq 2k$ as desired. Rizzi has constructed a more complicated family of grafts G_k for which $\tau(G_k) = k$ and $\nu(G_k) \leq \lceil \frac{2}{3}k \rceil - 1$.

Let $\rho(G)$ be the size of the smallest T-join and let $\mu(G)$ be the maximum number of pairwise disjoint T-cuts contained in G. Since every T-join intersects every T-cut, we have as before that $\mu(G) \leq \rho(G)$. One may ask if it is possible to give a lower bound on μ in terms of ρ. However, this question has a negative answer. For example, if G is the complete graph on $2n$ vertices and $T = V(G)$, then $\mu(G) = 1$, and $\rho(G) = n$. However, there are several intersting results concerning the packing of T-cuts.

We mention here a theorem of Seymour.

Theorem 1.4 (Seymour [6]) *If G is a bipartite graft, then $\mu(G) = \rho(G)$.*

Corollary 1.5 *If G is a graft, then there exist T-cuts $F_1, F_2, \ldots, F_{2\rho(G)}$ so that every edge of G is in at most two members of this list.*

2 Packing T-joins I

The purpose of this section is to prove the following theorem, already stated in the introduction.

Theorem 1.1 *Let G be a graft and assume that either G is eulerian or that $T = \{v \in V(G) \mid deg(v)$ is odd$\}$. Then $\nu(G) \geq \lfloor \frac{1}{2}\tau(G) \rfloor$.*

It will be helpful for us to define a few basic operations which produce new grafts from old ones. Let G be a graft and let $X \subseteq V(G)$. Form a new graft G' by deleting every edge with both ends in X, identifying X to a single new vertex x, and then adding x to T if $|X \cap T|$ is odd. We say that G' is obtained from G by *identifying* X. If $e \in E(G)$ is an edge, then we let G/e denote the graph obtained from G by deleting e and then identifying the ends of e.

If $x, y \in V(G)$, then we let $\lambda(x, y)$ denote the size of the smallest edge-cut of G which separates x and y. If G is a graft, a partition $\mathcal{P}$ of T is called a *pairing* if every $X \in \mathcal{P}$ has size two. Let $|T| = 2h$ and let $\mathcal{P} = \{\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_h, y_h\}\}$ be a pairing of T. Now, every T-cut of G must separate x_i and y_i for some $1 \leq i \leq h$. It follows that $\tau(G) \geq \min_{1 \leq i \leq h} \lambda(x_i, y_i)$. If $\min_{1 \leq i \leq h} \lambda(x_i, y_i) = \tau(G)$, then we say that $\mathcal{P}$ is a *τ-certificate* for G. Next we state a helpful lemma of Rizzi.

Lemma 2.1 (Rizzi - personal communication) *Every graft has a τ-certificate.*

Proof: Let F be a Gomory-Hu tree for G with edge weights given by $\lambda : E(F) \to \mathbf{Z}$ (see [2] for definitions of these terms). Let $S = \{e \in E(F) \mid \lambda(e) < \tau(G)\}$. Note that the fundamental cut of every $e \in S$ is not a T-cut. It follows from this that every component of $F \setminus S$ contains an even number of vertices of T. Let $2h = |T|$ and let $\mathcal{P} = \{\{x_1, y_1\}, \ldots, \{x_h, y_h\}\}$ be a partition of T with x_i and y_i in the same component of $F \setminus S$ for every $1 \leq i \leq h$. By construction, $\lambda(x_i, y_i) \geq \tau(G)$ for every $1 \leq i \leq h$ so $\mathcal{P}$ is a τ-certificate as required. □

Let G be a graph and let $v \in V(G)$ be a vertex. Let $e, f \in \delta(v)$ and let u, v and v, w be the ends of e and f respectively. Modify G to form a new graph G' by deleting the edges e and f and then adding a new edge h with ends u, w. We say that G' is obtained from G by making a *split* at u. The following theorem of Mader will be used in our reductions.

Theorem 2.2 (Mader [3]) *Let G be a graph, let $u \in V(G)$ be a vertex with $deg(u) > 3$ and assume that u is not incident with a cut-edge. Then we may modify G to form a new graph G' by making a split at u so that $\lambda_{G'}(v, w) = \lambda_G(v, w)$ for every $v, w \in V(G) \setminus \{u\}$.*

The following proposition shows how we will use Mader's splitting theorem.

Proposition 2.3 *Let G be a graft and assume that G does not have any cut-edges. Let $u \in V(G) \setminus T$ be a vertex with $deg(u) > 3$. Then, we may alter G to form a new graft G' by making a split at u so that $\tau(G') = \tau(G)$.*

Proof: It is obvious that $\tau(G') \leq \tau(G)$. Let $2h = |T|$, and apply Lemma 2.1 to choose a τ-certificate $\mathcal{P} = \{\{x_1, y_1\}, \ldots, \{x_h, y_h\}\}$ of G. Now, apply Theorem 2.2 to split u forming the graft G'. Then we have that

$$\tau(G') \geq \min_{1 \leq i \leq h} \lambda_{G'}(x_i, y_i) = \min_{1 \leq i \leq h} \lambda_G(x_i, y_i) \geq \tau(G)$$

as required. □

If G is a graft, a subgraph $F \subseteq G$ is a T-*connector* if every component of F contains an even number of vertices of T. The following lemma shows that every T-connector contains a T-join.

Lemma 2.4 *Let G be a graft and let F be a T-connector of G. Then there is a T-join of G contained in $E(F)$.*

Proof: Let F' be a component of F, and let H be a spanning tree of H. We will think of H as a graft for the set $T \cap V(F')$. Now, let $S = \{e \in E(H) \mid \{e\}$ is a T-cut of H $\}$. It follows easily that S is a T-join of H. Repeating this construction for each component of F gives us a T-join contained in F. □

The following theorem is essential to our proof.

Theorem 2.5 (Nash-Williams [4] and Tutte [7]) *Let G be a graph. Then G contains k-edge-disjoint spanning trees if and only if $e(\mathcal{P}) \geq k(|\mathcal{P}|-1)$ for every partition $\mathcal{P}$ of $V(G)$.*

Corollary 2.6 *Every $2k$-edge-connected graph contains k edge-disjoint spanning trees.*

Now we are ready to prove the workhorse lemma of this section.

Lemma 2.7 *Let G be a graft and assume that $\tau(G) \geq 2k$. Then G contains k disjoint T-connectors.*

Proof: We proceed by induction on $|E(G)|$. We may assume that $k > 0$ (otherwise the lemma is trivial).

If e is a cut-edge of G, then e cannot be contained in any T-cut of size $2k$. In this case, the lemma follows by applying induction to the graph $G \setminus e$. Thus, we may assume that G has no cut-edges.

Let $u \in V(G) \setminus T$. If $deg(u) = 2$, then the Lemma follows by applying induction to the graft G/e for some edge $e \in \delta(u)$. Thus, we may assume that $deg(u) > 2$. Since $|deg(u)|$ is even, we have that $deg(u) \geq 4$. Now, by Lemma 2.3, we may form a new graft G' by splitting off a pair of edges e, f at u so that $\tau(G') = \tau(G) \geq 2k$. Let h be the edge formed by this split. Now the Lemma follows by applying induction to G' and then replacing the edge h by e and f in the component which contains h. Thus, we may assume that $V(G) = T$.

Suppose that G is not $2k$-edge-connected, and let $X \subseteq V(G)$ be a minimal set with $|\delta(X)| < 2k$. Let $\mathcal{P} = \{X_1, X_2, \ldots, X_t\}$ be a nontrivial partition of X (by nontrivial, we mean that $|\mathcal{P}| \geq 2$). Now

$$\begin{aligned} e(\mathcal{P}) &= \sum_{1 \leq i < j \leq t} e(X_i, X_j) \\ &= \frac{1}{2}\sum_{i=1}^{t} e(X_i, X \setminus X_i) \\ &= \frac{1}{2}\sum_{i=1}^{t}(|\delta(X_i)| - e(X_i, V(G) \setminus X)) \\ &= \frac{1}{2}(\sum_{i=1}^{t} |\delta(X_i)|) - \frac{1}{2}|\delta(X)| \\ &\geq kt - k \end{aligned}$$

Thus, it follows from Theorem 2.5 that we may choose k edge-disjoint spanning trees $F_1, F_2, \ldots, F_k$ of the graph $G[X]$. Modify G to form the graft G' by contracting X to a single new node x. By applying the lemma inductively, we may choos k edge-disjoint T-connectors $R_1, R_2, \ldots, R_k$ of G'. Since $|\delta(X)| < 2k$, it must be that $|X \cap T|$ is even. It follows from this that $R_1 \cup F_1, R_2 \cup F_2, \ldots, R_k \cup F_k$ is a list of pairwise edge-disjoint T-connectors for G.

By the above argument, we may now assume that G is $2k$-edge-connected. Now, by Corollary 2.6, G contains k edge-disjoint spanning trees. Since each spanning tree is a T-connector, this completes the proof. □

Now we are ready to prove the main theorem of this section.

Proof of Theorem 1.1 By the above lemma, we may choose $\lfloor \frac{\tau(G)}{2} \rfloor$ disjoint T-connectors of G. By Lemma 2.4, each one contains a T-join. □

3 Hypergraphs

In order to extend the proof techniques used in the previous section to the general case, we will need to prove some properties of spanning subgraphs in hypergraphs.

We allow hypergraphs to have multiple edges and treat them as graphs. If H is a hypergraph, $e \in E(H)$ and $v \in V(H)$, then we write $v \in e$ if e contains the vertex v, and we let $|e|$ denote the number of vertices contained in e. If $F \subseteq H$ is a subgraph, $V(F) = V(H)$ and F is connected, then we say that F is *spanning*.

As in the case of ordinary graphs, if $X \subseteq V(H)$, then we define $\delta_H(X) = \{e \in E(H) \mid e \cap X \neq \emptyset \neq e \cap V(G) \setminus X\}$. and $H[X] = (X, \{e \in E(H) \mid e \subseteq X\})$.

The following Lemma is a naive hypergraph extension of the Nash-Williams/Tutte theorem on disjoint spanning trees.

Lemma 3.1 *Let $H = (V, E)$ be a hypergraph and assume that every edge of H has size $\leq k$ and that $e(\mathcal{P}) \geq k(|\mathcal{P}| - 1)$ for every partition $\mathcal{P}$ of $V(H)$. Then H contains k disjoint spanning subgraphs.*

Proof: Let G be the graph $(V, \{e \in E(H) \mid |e| \leq 2\}$. Now, we will adjust G by the following process: For every $e \in E(H)$ with $|e| > 2$, let v_e be a new vertex, and add an edge from v_e to v for every $v \in e$. Then, choose one such newly added edge e, and add $k + 1 - |e_i|$ new copies of this edge to G. Let $U = \{v_e \mid |e| > 2\}$. Then $V(G)$ is the disjoint union of U and V, U is an independent set of G, and every $v_e \in U$ has degree $k + 1$. Let $\mathcal{Q} = \{Y_1, Y_2, \ldots, Y_s, Z_1, Z_2, \ldots, Z_t\}$ be a partition of $V(G)$ and assume that $Y_i \cap V \neq \emptyset$ and $Z_j \cap V = \emptyset$ for $1 \leq i \leq s$ and $1 \leq j \leq t$. Let $Z = \cup_{i=1}^t Z_i$ and let $Y_i' = Y_i \cap V$ for $1 \leq i \leq s$. Now

$$\begin{aligned} e(\mathcal{Q}) &= \sum_{1 \leq i < j \leq s} e(Y_i, Y_j) + \sum_{i=1}^{s} e(Y_i, Z) \\ &\geq (e(\{Y_1', Y_2', \ldots, Y_s'\}) - |Z|) + (k+1)|Z| \\ &\geq k(s-1) + kt \\ &= k(s + t - 1) \end{aligned}$$

It follows from this equation and Theorem 2.5 that G contains k edge-disjoint spanning subgraphs $F_1, F_2, \ldots, F_k$. We may assume without loss that $\cup_{i=1}^k E(F_i) = E(G)$. Now, every vertex $v_i \in U$ has degree $k + 1$. It follows from this that $deg_{F_j}(v_i) = 1$ for all but one subgraph F_j. Next we will partition the edges of H into k subsets $R_1, R_2, \ldots, R_k$ as follows. If $e_i \in H$ has size ≤ 2, then put e_i in R_j where $e_i \in E(F_j)$. If $e_i \in H$ has size > 2, then put e_i in R_j where $deg_{F_j}(v_i) > 1$. It follows from this construction that the hypergraphs (V, R_j) are spanning as required. $\Box$

Let H be a hypergraph and let $X \subseteq V(H)$. For every nonnegative integer i, we define $d^i(X) = |\{e \in \delta(X) \mid |e \cap (V(G) \setminus X)| = i\}|$. If H is a 3-hypergraph and k is a nonnegative integer, we say that a set $X \subseteq V(H)$ is *k-troublesome* if $\frac{1}{2}d^1(X) + \frac{1}{3}d^2(X) < 2k$.

Proposition 3.2 *Let H be a 3-hypergraph and let k be a nonnegative integer. Let $X \subseteq V(H)$, let $H' = H[X]$, and assume that no proper subset of X is k-troublesome. Then $e_{H'}(\mathcal{P}) \geq 2k|\mathcal{P}| - \frac{2}{3}d^1_H(X) - \frac{1}{3}d^2_H(X)$ for every nontrivial partition $\mathcal{P}$ of X,*

Proof: Let $\mathcal{P} = \{X_1, X_2, \ldots, X_t\}$. In order to count the edges in $e_{H'}(\mathcal{P})$, it will be important to keep track of the different possible behaviors of edges of size two and three. To assist in this, we define four new sets. We let $B_2 = \{e \in \delta(X) \mid |e| = 2\}$, $B_3^2 = \{e \in \delta(X) \mid |e \cap (V(G) \setminus X)| = 2\}$, $B_3^1 = \{e \in \delta(X) \mid |e \cap X_i| = 2 \text{ for some } 1 \leq i \leq t\}$, and $Q = \{e \in \delta(X) \mid e \cap X_i \neq \emptyset \neq e \cap X_j \text{ for some } 1 \leq i < j \leq t\}$. Then

$$\begin{aligned}
e_{H'}(\mathcal{P}) &\geq \sum_{i=1}^{t} \tfrac{1}{2}d^1_{H'}(X_i) + \tfrac{1}{3}d^2_{H'}(X_i) \\
&= \sum_{i=1}^{t} \tfrac{1}{2}d^1_H(X_i) + \tfrac{1}{3}d^2_H(X_i) - \tfrac{1}{2}|\delta_H(X_i) \cap (B_2 \cup B_3^1)| - \tfrac{1}{3}|\delta_H(X_i) \cap (B_3^2 \cup Q)| \\
&\geq 2kt - \tfrac{1}{2}|B_2| - \tfrac{1}{2}|B_3^1| - \tfrac{1}{3}|B_3^2| - \tfrac{2}{3}|Q| \\
&\geq 2kt - \tfrac{2}{3}d^1_H(X) - \tfrac{1}{3}d^2_H(X).
\end{aligned}$$

□

With the help of the above proposition, we are now ready to prove the main lemma of this section, which will later be required for packing T-joins.

Lemma 3.3 *Let H be a 3-hypergraph on at least four vertices with minimum degree $6k$, and let $u \in V(H)$ be a vertex with degree $6k$. Then there is a subset $Y \subseteq V(H) \setminus u$ of size ≥ 2 so that $H[Y]$ contains k edge-disjoint spanning subgraphs.*

Proof: First, we consider the case that there is a subset $X \subseteq V(H) \setminus u$ which is k-troublesome. Let X be a minimal set with these properties (note that $|X| \geq 2$). Let $H' = H[X]$ and let $\mathcal{P} = \{X_1, X_2, \ldots, X_t\}$ be a nontrivial partition of X (so $t \geq 2$). Then we by the above proposition, we have that

$$e(\mathcal{P}) \geq 2kt - \tfrac{2}{3}d^1_H(X) - \tfrac{1}{3}d^2_H(X)$$

$$\begin{aligned} &\geq 2kt - \tfrac{4}{3}(\tfrac{1}{2}d_H^1(X) + \tfrac{1}{3}d_H^2(X)) \\ &\geq 2kt - \tfrac{8}{3}k \\ &= k(t-1) + k(t - \tfrac{5}{3}) \\ &\geq k(t-1) \end{aligned}$$

It follows from Lemma 3.1 that H' contains k edge-disjoint spanning subgraphs. Thus, we may assume that no subset of $V(H) \setminus u$ is k-troublesome.

Let $X = V(H) \setminus \{u\}$ and let $H' = H[X]$.

Claim: $e_{H'}(\mathcal{P}) \geq k(|\mathcal{P}| - 1)$ for every partition $\mathcal{P}$ of X with $|\mathcal{P}| \geq 3$.
Proof: Let $\mathcal{P}$ be a partition of X with $|\mathcal{P}| = t \geq 3$. Then by Proposition 3.2,

$$\begin{aligned} e_{H'}(\mathcal{P}) &\geq 2kt - \tfrac{2}{3}d_H^1(X) - \tfrac{1}{3}d_H^2(X) \\ &= 2kt - \tfrac{2}{3}|\delta(u)| \\ &= 2kt - 4k \\ &= k(t-1) + k(t-3) \\ &\geq k(t-1) \end{aligned}$$

If $e_{H'}(\mathcal{P}) \geq k(|\mathcal{P}| - 1)$ holds for every partition $\mathcal{P}$ of X, then H' contains k edge-disjoint spanning subgraphs, and we are finished. Thus, we may choose a partiton $\mathcal{P}$ of X for which $e_{H'}(\mathcal{P}) < k(|\mathcal{P}| - 1)$. By the above claim, we may assume that $\mathcal{P} = \{X_1, X_2\}$. Since $|V(H)| \geq 4$, we may assume without loss that $|X_1| \geq 2$. Let $H_1 = H[X_1]$ and let $\mathcal{Q}$ be a nontrivial partition of X_1. Let $Q' = Q \cup \{X_2\}$ and note that $|Q'| \geq 3$. Now, by Proposition 3.2

$$\begin{aligned} e_{H_1}(\mathcal{Q}) &= e_{H'}(\mathcal{Q}') - e(X_1, X_2) \\ &\geq k(|\mathcal{Q}'| - 1) - k \\ &= k(|\mathcal{Q}| - 1). \end{aligned}$$

It follows that H_1 contains k edge-disjoint spanning subgraphs, and this completes the lemma. □

4 Packing T-joins II

The purpose of this section is to prove the following theorem, stated in the introduction.

Theorem 1.2 $\nu(G) \geq \lfloor \frac{1}{6}\tau(G) \rfloor$ *for every graft* G

A *rooted graft* is a graft G with a distinguished vertex u, called the *root* which must be a member of T. We say that a subgraph F of G is a *rooted* T-*connector* if every component of F contains an even number of vertices of T and u is not a cut-vertex of F.

Lemma 4.1 *Let* G *be a rooted graft and let* H *be a rooted T-connector of* G*. If* $|\delta_H(u)|$ *is odd, then* H *contains a T-join* R *with* $\delta_H(u) \subseteq R$*.*

Proof: Let $\delta_H(u) = \{e_1, e_2, \ldots, e_t\}$. We modify H to form a new grapht H' as follows. Add t new vertices $v_1, v_2, \ldots, v_t$ to H and for every edge e_i, change its ends so that it is incident with v_i instead of v. Now, delete v and add all of the vertices $v_1, v_2, \ldots, v_t$ to T. Note that since t is odd, this maintains the parity of $|T|$. Now, H' is an (ordinary) T-connector, so we may choose a T-join R of H'. In the original graph H, this is a T-join with $\delta_H(u) \subseteq R$ as desired. □

Now we are ready to prove the main lemma.

Lemma 4.2 *Let* k *be a positive integer, let* G *be a rooted graft, and assume that* $\tau(G) \geq 6k$ *and that* $deg(u) = 6k$*. Let* $f : \delta(u) \to \{0, 1, \ldots, k\}$ *be a map with* $f^{-1}(\{i\}) \neq \emptyset$ *for* $1 \leq i \leq k$*. Then* G *contains* k *edge-disjoint rooted T-connectors* $F_1, \ldots, F_k$ *so that* $E(F_i) \cap \delta(u) = f^{-1}(\{i\})$ *for* $1 \leq i \leq k$*.*

Proof: We proceed by induction on $|E(G)|$. The lemma is trivial if $|V(G)| = 2$, so we may assume that $|V(G)| \geq 3$. Clearly, we may also assume that G is connected. For convenience, we set $O = V(G) \setminus T$.

(1) every edge of G is in a T-cut of size $6k$

If $e \in E(G)$ is not in a T-cut of size $6k$, then the lemma follows by applying induction to the graph $G \setminus e$ (note that in this case $e \notin \delta(u)$).

(2) G does not have a cut-edge

If $e \in E(G)$ is a cut-edge, then by assumption, $\{e\}$ is not a T-cut. In this case, e cannot occur in a T-cut of minimum size, and we have a contradiction to (1).

(3) Every T-cut of G of size $6k$ is a vertex star

Assume that S is a T-cut of G of size $6k$ and that $S \neq \delta(v)$ for every $v \in T$. Let $\{A_1, A_2\}$ be the partition of $V(G)$ induced by S and assume that $u \in A_1$. For $i = 1, 2$, let G_i be the graft obtained from G by contracting A_{3-i} to a single new node u_{3-i}. By induction, we may choose a rooted T-connector $H_1, H_2, \ldots, H_k$ of G_1. Now, let $f' : \delta(u_1) \to \{0, 1, \ldots, k\}$ be given by the rule

$$f'(e) = \begin{cases} i & \text{if } e \in E(H_i) \\ 0 & \text{if } e \notin \cup_{i=1}^{k} E(H_i) \end{cases}$$

Now, we may apply the lemma inductively to G_2 for the root u_1 and the map f'. This gives us a rooted T-connecter $H'_1, H'_2, \ldots, H'_k$. Now $H_1 \cup H'_1, H_2 \cup H'_2, \ldots, H_k \cup H'_k$ is a rooted T-connector as required.

(4) Every vertex in O has degree three

Suppose that $v \in O$ has degree two and let $\delta(v) = \{e, f\}$. Since $\delta(u)$ is a minimum size T-cut, $\{e, f\} \not\subseteq \delta(u)$ and we may assume that $e \notin \delta(u)$. Now the lemma follows by applying induction to the graft G/e and then adding the edge e back to the T-connector containing f.

Suppose that $v \in O$ has degree > 3. By Proposition 2.3 we may form a new graft G' by splitting off a pair of edges e, f at v so that $\tau(G') \geq 6k$. Now, the lemma follows by applying induction to G'.

(5) O is independent

By (1), every edge of G is in a T-cut of minimum size, and by (3), every T-cut of G of minimum size is the star of a vertex in T. Thus, every edge must have at least one end in T. It follows that O is independent.

(6) every $v \in O$ has three distinct neighbors

Suppose that (6) does not hold and let $v \in O$ have ≤ 2 neighbors. Let $\delta(v) = \{e_1, e_2, f\}$ and assume that e_1 and e_2 are parallel. In this case, e_1 is not in a T-cut of size $6k$. This contradicts (1).

Now, we will form a hypergraph H with vertex set T as follows. If $e \in E(G)$ is an edge with both ends in T, then we add e to H. If v is a vertex in O, then we add an edge e_v to H of size three containing the neighbors of v. By construction, H has minimum degree $6k$ and $deg_H(u) = 6k$. If $|T| = 2$, then by (6) we have that $O = \emptyset$, and we find that $|V(G)| = 2$. This contradicts our assumption. Thus, $|V(H)| = |T|$ is an even number > 2, so $|V(H)| \geq 4$, and we may apply Lemma 3.3 to choose a subset $X \subseteq V(H)$ so that the hypergraph $H[X]$ contains k edge-disjoint spanning subgraphs. $F_1, F_2, \ldots, F_k$. Let $Y = \{v \in O \mid \text{every neighbor of } v \text{ is in } X\}$ and let $Y_i = \{v \in Y \mid e_v \in F_i\}$. For $1 \leq i \leq k$, let F'_i be the subgraph of G with vertex set $X \cup Y_i$ and edge set $\cup_{v \in Y_i} \delta(v) \cup \{e \in F_i \mid |e| = 2\}$. Now, $F'_1, F'_2, \ldots, F'_k$ are all edge-disjoint subgraphs of G and each F'_i spans X.

Now, let G' be the graft obtained from G by identifying $X \cup Y$ to a single new vertex z. By induction, we may choose k edge-disjoint T-connectors $J_1, J_2, \ldots, J_k$ of G' so that $J_i \cap \delta(u) = f^{-1}(\{i\})$ for $1 \leq i \leq k$. Now, by construction $E(F'_1) \cup J_1, E(F'_2) \cup J_2, \ldots, E(F'_k) \cup J_k$ is a list of T-connectors satisfying the lemma. □

Finally, we apply this lemma to prove a slightly stronger version of Theorem 1.2.

Theorem 1.2$^+$ *Let G be a graft with $\tau(G) \geq 6k$. Then $\nu(G) \geq k$. Furthermore, if S is a T-cut of G of size $6k$ and $f : S \to \{0, 1, \ldots, k\}$ is a map with $|f^{-1}(\{i\})|$ odd for $1 \leq i \leq k$, then G contains k edge-disjoint T-joins $F_1, F_2, \ldots, F_k$ with $F_i \cap \delta(u) = f^{-1}(\{i\})$ for $1 \leq i \leq k$.*

Proof: If every T-cut of G has size $> 6k$, then we may remove edges one by one until some T-cut has size $6k$. Thus, to prove the theorem, it suffices to consider the case where S is a T-cut of G of size $6k$ and the map f has specified the behavior on S. Let $\{X_1, X_2\}$ be the partition of $V(G)$ induced by S and for $i = 1, 2$, let G_i be the graft obtained from G by identifying X_i to a single vertex u_i. Now, by applying the above lemma to G_i with root u_i for the map f, we may choose k edge-disjoint rooted T-connectors $F^i_1, F^i_2, \ldots, F^i_k$ of G_i with $F^i_j \cap S = f^{-1}(\{j\})$. By Lemma 4.1, we may choose T-joins $R^i_j \subseteq F^i_j$ for $i = 1, 2$ and $1 \leq j \leq k$ so that $\delta(u) \cap F^1_j = \delta(u) \cap F^2_j = f^{-1}(\{j\})$ for $1 \leq j \leq k$. Now $F^1_1 \cup F^2_1, F^1_2 \cup F^2_2, \ldots F^1_k \cup F^2_k$ is a list of T-joins which satisfies the theorem. □

The above theorem is a slight strengthening of the usual packing T-joins problem, since the behavior on a minimum size T-cut is prespecified. One may ask how good a T-join packing is it possible to attain, with this additional requirement. The following family of examples shows that in this case, it is not in general possible to find more than $\tau(G) - 1$ disjoint T-joins. Let F_k be the family of grafts from the introduction (F_k is obtained from the cube with bipartition (U, V) by adding $k - 1$ additional copies of each vertex in U and then setting $T = V$). Now, let $v \in V$ and let $\delta(v) = \{e_1, e_2, \ldots, e_{3k}\}$. Define $f : \delta(v) \to \{0, 1, \ldots, k + 2\}$ by the rule

$$f(e_i) = \begin{cases} i & \text{if } i < k+2 \\ k+2 & \text{if } i \geq k+2 \end{cases}$$

Note that $|f^{-1}(\{i\})|$ is odd for $1 \leq i \leq k + 2$ since $|f^{-1}(\{k + 2\})| = 3k - (k + 1) = 2k - 1$. Now, suppose that F_k contained $k+2$ edge-disjoint T-joins $R_1, R_2, \ldots, R_{k+2}$ with $\delta(u) \cap R_i = f^{-1}(\{i\})$ for $1 \leq i \leq k+2$. Then, $|R_i| \geq 4$ for $1 \leq i \leq k+1$ and $|R_{k+2}| \geq 2(2k-1) = 4k-2$. Thus $| \cup_{i=1}^{k} R_i| \geq 4(k + 1) + 4k - 2 = 8k + 2$. But this contradicts the fact that one edge incident with each vertex of U must not be contained in any R_i, so the number of edges used in any packing of T-joins is at most $8k$.

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