Nowhere-zero $\mathbb{Z}_3$-flows through $\mathbb{Z}_3$-connectivity

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Abstract

Jaeger, Linial, Payan, and Tarsi established for every abelian group Γ a class of graphs which they call Γ-connected. The main interest in Γ-connected graphs is that every Γ-connected graph has a nowhere-zero flow in the group Γ. The goal of this paper is to establish some conditions which imply that a graph is $\mathbb{Z}_3$-connected. Our techniques lead to a generalization of a theorem of Lai on nowhere-zero $\mathbb{Z}_3$-flows in locally connected graphs, and to a simplified proof of a theorem of Xu and Zhang on nowhere-zero $\mathbb{Z}_3$-flows in squares of graphs.

1 Introduction

Throughout this article, graphs and directed graphs may have multiple edges and loops. Let G be a directed graph, let Γ be an abelian group and let $\phi : E(G) \to \Gamma$ be a map. We say

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that ϕ is *nowhere-zero* if $0 \notin \phi(E(G))$. The *boundary* of ϕ is the map $\partial\phi : V(G) \to \Gamma$ given by the rule $\partial\phi(v) = \sum_{e \in E^-(v)} \phi(e) - \sum_{e \in E^+(v)} \phi(e)$ (note that $\sum_{v \in V(G)} \partial\phi(v) = 0$). If $\partial\phi$ is identically zero, then ϕ is said to be a *flow* or a Γ-*flow*. We say that G is Γ-*connected* if for every $p : V(G) \to \Gamma$ with $\sum_{v \in V(G)} p(v) = 0$ there exists a nowhere-zero map $\phi : E(G) \to \Gamma$ with boundary p.

If we change the orientation of the edge e and switch $\phi(e)$ to $-\phi(e)$, then the boundary is preserved, and the new map is nowhere-zero if and only if the original was. Thus the existence of a nowhere-zero map with a specified boundary depends only on the underlying undirected graph and not on the orientation of the edges. Accordinly, we say that an undirected graph *admits a nowhere-zero* Γ-*flow* (is Γ-*connected*) if some (and thus every) orientation of it admits a nowhere-zero Γ-flow (is Γ-connected). In the remainder of the paper, we will restrict our attention to the group $\mathbb{Z}_3 = \mathbb{Z}/3\mathbb{Z}$. The main interest in nowhere-zero $\mathbb{Z}_3$-flows stems from the following conjecture of Tutte.

Conjecture 1.1 (The three flow conjecture; Tutte) *Every 4-edge-connected graph admits a nowhere-zero $\mathbb{Z}_3$-flow.*

In their article on group connectivity, Jaeger *et al* make the following conjecture on $\mathbb{Z}_3$-connected graphs.

Conjecture 1.2 (Jaeger Linial Payan and Tarsi [4]) *Every 5-edge connected graph is $\mathbb{Z}_3$-connected.*

Recently M. Kochol [5] has shown that Conjecture 1.1 can be reduced to 5-edge-connected graphs. It follows from this that Conjecture 1.2 implies Conjecture 1.1. Thus, it may be possible to achieve the three flow conjecture by instead proving a result on $\mathbb{Z}_3$-connectivity. Although we appear to be quite far from a resolution to either of these conjectures, the idea of using $\mathbb{Z}_3$-connectivity to establish results on $\mathbb{Z}_3$-flows is quite useful. Indeed, this is the main theme of our paper. Before stating our main theorems, we shall introduce a definition and an easy observation which is also suggestive of this theme.

If H is a connected subgraph of the graph G, then G *contract* H, denoted G/H, is defined to be the graph obtained from G by deleting all edges in H and then identifying $V(H)$ to a single new vertex. The following observation follows easily from the definitions.

Observation 1.3 *Let H be a $\mathbb{Z}_3$-connected subgraph of the graph G.*

(i) If G/H admits a nowhere-zero $\mathbb{Z}_3$-flow, then so does G.

(ii) If G/H is $\mathbb{Z}_3$-connected, then so is G.

Let us call a $\mathbb{Z}_3$-connected graph on > 1 vertex *reducible*. Now suppose that $\mathcal{F}$ is a class of graphs which is closed under contracting subgraphs (for instance the class of all 5-edge-connected graphs). Then to prove that every graph in in $\mathcal{F}$ has a nowhere-zero $\mathbb{Z}_3$-flow, it suffices to consider only those graphs which do not contain a reducible subgraph. Similarly, to prove that every graph in $\mathcal{F}$ is $\mathbb{Z}_3$-connected, it suffices to show that every graph in $\mathcal{F}$ contains a reducible subgraph. This is suggestive of a kind of reducibility/unavoidability approach to Conjecture 1.2. An obvious difficulty in this approach is that for any finite list of reducible graphs $H_1, H_2, \ldots, H_k$, there exist 5-edge-connected graphs containing no subgraph isomorphic to H_i for $1 \leq i \leq k$ (this follows from the existence of 5-edge-connected graphs of high girth and the fact that every reducible graph contains a circuit).

It is possible to prove that graphs with some added structure are $\mathbb{Z}_3$-connected. We define a connected graph G to be *neighborhood connected* if $G[N(v)]$ is connected for every $v \in V(G)$. We define G to be *triangle connected* if for every pair of non-loop edges $e, f \in E(G)$, there exists a sequence of circuits $C_1, C_2, \ldots, C_k$ so that $e \in E(C_1)$, $f \in E(C_k)$, $|E(C_i)| \leq 3$ for $1 \leq i \leq k$, and $E(C_j) \cap E(C_{j+1}) \neq \emptyset$ for $1 \leq j \leq k-1$. Every neighborhood connected graph on at least three vertices is triangle connected. To see this, first observe that if e, f are adjacent edges, say $e = uv$ and $f = uw$, then a sequence of circuits $C_1, C_2, \ldots, C_k$ as in the definition of triangle connected can be found using a path from v to w in $N(u)$. The following theorem gives a sufficient condition for a graph to be $\mathbb{Z}_3$-connected.

Theorem 1.4 *If G is a loopless triangle connected graph with minimum degree ≥ 4 then G is $\mathbb{Z}_3$-connected.*

This theorem is a generalization of the following result of Lai.

Theorem 1.5 (Lai) *If G is 2-edge-connected and $G[N(v)]$ is 3-edge-connected for every $v \in V(G)$ then G has a nowhere-zero $\mathbb{Z}_3$-flow.*

To see that Theorem 1.4 implies the Theorem 1.5, note that every graph G on > 2 vertices satisfying the input condition of Theorem 1.5 is 2-connected and triangle connected.

If $v \in V(G)$ and $u \in N(v)$, then $deg_{G[N(u)]}(v) \geq 3$ so $deg_G(v) \geq 4$. Thus the result follows from Theorem 1.4.

If G is a simple graph, then G^2 is the simple graph with $V(G^2) = V(G)$ so that $u, v \in V(G^2)$ are adjacent (in G^2) if and only if they are distance ≤ 2 in G. Using our techniques, we establish necessary and sufficient conditions on G for G^2 to be $\mathbb{Z}_3$-connected. As a consequence of this, we obtain a simplified proof of a theorem of Xu and Zhang which gives necessary and sufficient conditions on G for G^2 to have a nowhere-zero $\mathbb{Z}_3$-flow. Before stating this results, we require the following definition.

Definition 1.6

$\mathcal{T}_{1,3} =$ $\{T \mid T$ *is a tree and* $d_T(v) \in \{1,3\}$ *for every* $v \in V(T)\}$.

$\bar{\mathcal{T}}_{1,3} =$ $\{T \mid T \in \mathcal{T}_{1,3}$ *or* T *is a simple 4-vertex graph contiaining a 4-circuit or* T *is obtained from some* $T' \in \mathcal{T}_{1,3}$ *by adding some edges which join two distance 2 leaves of* $T'\}$.

$\mathcal{T}_{1,2,3} =$ $\{T \mid T$ *is a tree and* $d_T(v) \in \{1,2,3\}$ *for every* $v \in V(T)\}$.

$\bar{\mathcal{T}}_{1,2,3} =$ $\{T \mid T \in \mathcal{T}_{1,2,3}$ *or* T *is a simple 4-vertex graph contiaining a 4-circuit or* T *is obtained from some* $T' \in \mathcal{T}_{1,2,3}$ *by adding some edges which join two distance 2 leaves of* $T'\}$.

Theorem 1.7 *Let* G *be a simple connected graph. Then* G^2 *is* $\mathbb{Z}_3$*-connected if and only if* $G \notin \bar{\mathcal{T}}_{1,2,3}$.

Theorem 1.8 (Xu, Zhang) *Let* G *be a simple connected graph. Then* G^2 *admits a nowhere-zero* Z_3*-flow if and only if* $G \notin \bar{\mathcal{T}}_{1,3}$.

2 Proofs

We begin with the following observation which follows easily from the definitions.

Observation 2.1 *Let* H *be a* $\mathbb{Z}_3$*-connected subgraph of the graph* G.

(i) If $V(G) = V(H)$*, then* G *is* $\mathbb{Z}_3$*-connected.*

(ii) If $V(G) \setminus V(H) = \{u\}$ *and there are at least two edges from* u *to* $V(H)$*, then* G *is* $\mathbb{Z}_3$*-connected.*

For every integer $k \geq 2$, let C_k denote a graph which is a circuit on k vertices. A special case of (ii) above is that the graph C_2 is $\mathbb{Z}_3$-connected. The proof of Theorem 1.4 requires the following easy proposition.

Proposition 2.2 *If G is triangle connected and $H \subseteq G$ is a $\mathbb{Z}_3$-connected graph on at least two vertices, then G is $\mathbb{Z}_3$-connected.*

Proof: Choose a maximal $\mathbb{Z}_3$-connected subgraph H' of G with $H \subseteq H'$. If $H' = G$ then we are done. Otherwise, there must exist an edge $e \in E(H')$ and a triangle $C \subseteq G$ so that $e \in E(H')$ and $V(C) \not\subseteq V(H')$. It now follows from (iv) of Observation 2.1 that $H' \cup C$ is $\mathbb{Z}_3$-connected, contradicting our choice of H'. □

Proof of Theorem 1.4: If $|V(G)| \leq 3$ then the theorem is obvious, so we shall assume that $|V(G)| \geq 4$. Choose a maximal triangle connected subgraph $H \subseteq G$ with $V(H) \neq V(G)$. It follows (as in the proof of Proposition **??**) that $V(G) \setminus V(H)$ contains a single vertex; We shall call this vertex u. Choose an edge $e \in E(H)$ and a triangle $C \subseteq G$ so that $u \in V(C)$ and $e \in E(C)$. Modify the graph G to form the graph G' by deleting $E(C) \setminus \{e\}$ and then adding a new edge e' parallel to e. Let $H' \subseteq G'$ be obtained from H by adding the edge e'. If G' is $\mathbb{Z}_3$-connected, then we find that G is also $\mathbb{Z}_3$-connected. Thus, the theorem follows from the fact that H' is $\mathbb{Z}_3$-connected (by Proposition 2.2 and the fact that C_2 is $\mathbb{Z}_3$-connected) and part (ii) of Observation 2.1. □

Corollary 2.3 *If $n \geq 5$, then C_n^2 is $\mathbb{Z}_3$-connected.*

Next we will prove that another family of graphs is $\mathbb{Z}_3$-connected. For every $n \geq 3$, we define W_n to be a simple graph obtained from a circuit of length n by adding a new vertex adjacent to all existing vertices.

Proposition 2.4 *For every integer $k \geq 2$, the graph W_{2k} is $\mathbb{Z}_3$-connected.*

Proof: Let G be a graph obtained from a circuit of length $2k$ by adding a new vertex x adjacent to all existing vertices. Let $p : V(G) \to \mathbb{Z}_3$ be a map with $\sum_{v \in V(G)} p(v) = 0$. If p is identically zero, then it is straightforward to show that G admits a nowhere-zero map with boundary p. Otherwise, we may choose a vertex $y \in V(G) \setminus \{x\}$ with $p(y) \neq 0$. Let $N(y) = \{x, z_1, z_2\}$, let G' be the graph obtained from $G \backslash y$ by adding a new edge (in parallel) from z_1 to x, and let $p' : V(G') \to \mathbb{Z}_3$ be given by the rule $p'(v) = p(v)$ if $v \in V(G') \setminus \{z_2\}$ and $p'(z_2) = p(y) + p(z_2)$. It follows from Proposition 2.2 (and the fact that C_2 is $\mathbb{Z}_3$-connected) that G' admits a nowhere-zero map with boundary p'. It follows easily from this that G admits a nowhere-zero map with boundary p. □

We shall require an additional definition and two lemmas before proving Theorems 1.7 and 1.8. Let G_1, G_2 be graphs, let $u_1v_1 \in E(G_1)$ and let $u_2v_2 \in E(G_2)$. Form a new graph from the disjoint union of $G_1 \setminus \{uv\}$ and G_2 by identifying u_1 and u_2 and identifying v_1 and v_2. The resulting graph is a *two sum* of G_1 and G_2 *over the edges* u_1v_1 *and* u_2v_2.

Lemma 2.5 *Let H be a two sum of the graphs G_1 and G_2.*

(i) if neither G_1 or G_2 admits a nowhere-zero $\mathbb{Z}_3$-flow then H does not admit a nowhere-zero $\mathbb{Z}_3$-flow.

(ii) if neither G_1 or G_2 is $\mathbb{Z}_3$-connected, then H is not $\mathbb{Z}_3$-connected.

Proof: Let H be a two sum of G_1 and G_2 over the edges u_1v_1 and u_2v_2. Let $u, v \in V(H)$ be the vertices obtained by identifying u_1 and u_2 and identifying v_1 and v_2 respectively. In case (i) let $p_i : V(G_i) \to \mathbb{Z}_3$ be given by $p_i(v) = 0$ for $i = 1, 2$. In case (ii) for $i = 1, 2$ we choose $p_i : V(G) \to \mathbb{Z}_3$ so that $\sum_{v \in V(G_i)} p_i(v) = 0$ and so that there does not exist a nowhere-zero map $\phi : E(G_i) \to \mathbb{Z}_3$ with boundary p_i. Define $q : V(H) \to \mathbb{Z}_3$ by the following rule

$$q(w) = \begin{cases} p_1(u_1) + p_2(u_2) & \text{if } w = u \\ p_1(v_1) + p_2(v_2) & \text{if } w = v \\ p_1(w) & \text{if } w \in V(G_1) \setminus \{u_1, v_1\} \\ p_2(w) & \text{if } w \in V(G_2) \setminus \{u_2, v_2\} \end{cases}$$

To resolve cases (i) and (ii) it suffices to prove that there does not exist a nowhere-zero map on $E(H)$ with boundary q. Suppose (for a contradiction) that there does exist such a map $\psi : E(H) \to \mathbb{Z}_3$. Let $\phi_i = \psi|_{E(G_i) \setminus \{u_iv_i\}}$ for $i = 1, 2$. By construction, ϕ_i is a nowhere-zero map of $G_i \setminus u_iv_i$ and $\partial\phi_i(w) = p_i(w)$ for every $w \in V(G_i) \setminus \{u_i, v_i\}$. Since $\partial\phi_i(u_i) + \partial\phi_i(v_i) = p_i(u_i) + p_i(v_i)$ and G_i does not have a nowhere-zero map with boundary p_i it follows that $\partial\phi_i = p_i$. But then we have a contradiction to the assumptions that $\psi(u_2v_2) \neq 0$ and $\partial\psi = q$. This completes the proof. □

Lemma 2.6

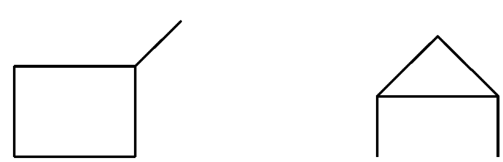
(i) If $G \in \bar{\mathcal{T}}_{1,2,3}$ then G^2 is not $\mathbb{Z}_3$-connected.

(ii) If $G \in \bar{\mathcal{T}}_{1,3}$ then G^2 does not admit a nowhere-zero $\mathbb{Z}_3$-flow.

(iii) If $G \in \bar{\mathcal{T}}_{1,2,3} \setminus \bar{\mathcal{T}}_{1,3}$ then G^2 admits a nowhere-zero $\mathbb{Z}_3$-flow.

Proof: We shall prove (i) and (ii) simultaneously by induction on $|V(G)|$. In both cases, the statement is trivial if $|V(G)| \leq 4$. Otherwise we may choose a cut-edge $uv \in E(G)$ so that $deg(u) > 1$ and $deg(v) > 1$. Let X, Y be the vertex sets of the two components of $G \setminus uv$ and assume that $u \in X$ and $v \in Y$. Let $G_1 = G \setminus (X \setminus \{u\})$ and let $G_2 = G \setminus (Y \setminus \{v\})$. Now G^2 is a two sum of G_1^2 and G_2^2, so the result follows by Lemma 2.5 and by induction on G_1 and G_2.

To prove (iii) we will proceed by induction on $|V(G)|$. The statement is trivial if $|V(G)| \leq 4$. Suppose that there exist $x, y, z \in V(G)$ so that z is adjacent to x and y, $deg(z) = 3$, and either $deg(x) = deg(y) = 1$ or $xy \in E(G)$ (note that in the latter case $deg(x) = deg(y) = 2$). It follows by induction that $(G \setminus \{x, y\})^2$ has a nowhere-zero $\mathbb{Z}_3$-flow. Since the edges in G^2 but not in $(G \setminus \{x, y\})^2$ form a graph with a nowhere-zero $\mathbb{Z}_3$-flow (they form a graph isomorphic to K_4 minus an edge) we find that G^2 has a $\mathbb{Z}_3$-flow as desired. Thus we may assume that such vertices x, y, z do not exist. Note that this implies in particular that G is a tree. Let $v_1, v_2, \ldots, v_{k-1}, v_k$ be the vertex sequence of a longest path of G. It follows from our assumptions that v_2 and v_{k-1} are distinct vertices, $deg(v_2) = deg(v_{k-1}) = 2$ and $deg(v_1) = 1$. Now, by induction $(G \setminus v_1)^2$ has a nowhere-zero $\mathbb{Z}_3$-flow. By altering this flow on the edge $v_1 v_2$ it may be extended to a nowhere-zero $\mathbb{Z}_3$-flow of G^2. This completes the proof. $\square$



H_1 H_2

For positive integers n, m let $K_{n,m}$ denote the complete bipartite graph on n opposite m vertices. Let H_1 be a graph obtained from a circuit of length four by adding a new vertex adjacent to exactly one of the existing vertices. Let H_2 be a graph obtained from a circuit of length three by adding two new vertices, each adjacent to a distinct point on the original circuit. The above figure depicts H_1 and H_2.

Proof of Theorem 1.7: First we shall establish the following claim.

Claim: if $G \notin \bar{\mathcal{T}}_{1,2,3}$ then G has a subgraph isomorphic to one of $K_{1,4}$, H_1, H_2, or C_n for some $n \geq 5$.

Proof: We shall prove the contrapositive, so we assume that G does not contain any of the graphs listed above. If G has a 4-circuit, then $|V(G)| = 4$ so $G \in \bar{\mathcal{T}}_{1,2,3}$. If it has no 4-circuit, then G has maximum degree ≤ 3 and every circuit of G is a triangle which contains two degree two vertices, so we have that $G \in \bar{\mathcal{T}}_{1,2,3}$.

If $G \in \bar{\mathcal{T}}_{1,2,3}$, then it follows from part (i) of Lemma 2.6 that $G \notin \langle \mathbb{Z}_3 \rangle$. If $G \notin \bar{\mathcal{T}}_{1,2,3}$, then G contains one of the graphs listed in the above claim as a subgraph. If G contains C_n for some $n \geq 5$ as a subgraph, then G^2 is $\mathbb{Z}_3$-connected by Proposition 2.2 and Corollary 2.3. If G contains $K_{1,4}$, H_1, or H_2, then G^2 is $\mathbb{Z}_3$-connected by Proposition 2.2, Proposition 2.4 and the observation that the square of each of these three graphs contains W_4. □

Proof of Theorem 1.8: This theorem follows immediately from Theorem 1.7, and parts (ii) and (iii) of Lemma 2.6. □

Acknowledgement. Special thanks goes to Dr. H.-J. Lai for helpful discussion.

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