The Finiteness of Fano Combinatorial Divisorial Polytopes

by

Charlotte Trainor

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Name: **Charlotte Trainor**

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Examining Committee:

Chair: Marni Mishna
Associate Professor

Nathan Ilten
Supervisor
Assistant Professor

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Abstract

Through fundamental connections and correspondences, combinatorics acts as a powerful tool for informing algebraic geometry. One example is the correspondence between reflexive polytopes and Fano toric varieties, which generalizes to a correspondence between Fano Combinatorial Divisorial Polytopes (CDPs) and Fano complexity-1 T-varieties. There are only finitely many reflexive polytopes in any dimension, motivating the conjecture that a certain class of Fano CDPs is finite in any dimension. Previous work on this conjecture has focused on the geometric side of the correspondence, and, while successful in 2-dimensions, the arguments have not generalized to a complete proof in higher dimensions. We take a combinatorial approach to the problem. We first recover the results in the 2-dimension case, and generalize key parts of our arguments to higher dimensions, including bounds on some properties of Fano CDPs. Once complete, we are hopeful that our strategies will extend to algorithms to classify Fano CDPs, and hence Fano complexity-1 T-varieties.

Keywords: algebraic geometry; combinatorics; reflexive polytopes; Fano complexity-1 T-varieties

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Chapter 1

Introduction

There are fundamental connections and correspondences between algebraic geometry and combinatorics. These correspondences are a powerful tool: geometric problems can be posed as combinatorial problems, and hence combinatorial solutions and results can extend to geometric solutions and results, and vice versa.

One such example is the correspondence between a class of geometric objects called Fano toric varieties and a class of combinatorial objects called reflexive polytopes. There is a finite number of isomorphism classes of reflexive polytopes in any fixed dimension [5]. Consequently, a particularly fruitful use of the described correspondence is that known algorithms to enumerate reflexive polytopes allow one to enumerate Fano toric varieties [1].

Fano toric varieties naturally generalize to Fano complexity-1 T-varieties, the geometric object of primary focus in this work. Naturally, the finiteness result for Fano toric varieties motivates a conjecture concerning the finiteness of Fano complexity-1 T-varieties in any fixed dimension. Moreover, this conjecture has already been proven for 2-dimensions by Huggenberger in [2], who also provides a classification of 2-dimensional Fano complexity-1 T-varieties. However, the arguments used to prove the conjecture in 2-dimensions, which are rooted in algebraic geometry, did not fully establish the conjecture in higher dimensions. We take a combinatorial approach to the problem by utilizing a correspondence between complexity-1 T-varieties and *combinatorial divisorial polytopes (CDPs)*, defined by Ilten and Süß in [3]. Such a combinatorial object consists of two components, a base lattice polytope, say P, and a finite collection of suitable functions defined over P.

Ilten and Süß define the *Fano* property of a CDP so that under the correspondence with complexity-1 T-varieties, the Fano property is preserved. Just as Fano complexity-1 T-varieties generalize Fano toric varieties, we have that Fano CDPs generalize reflexive polytopes. In particular, there is a correspondence between reflexive polytopes and Fano CDPs with exactly two functions.

We reformulate the conjecture concerning the finiteness of Fano complexity-1 T-varieties into a conjecture concerning the finiteness of Fano CDPs. It turns out that the most naive interpretation is not correct; it is *not* the case that there are only finitely many Fano CDPs in any fixed dimension. We construct infinite families of inequivalent Fano CDPs in Chapter 2, but with a key observation: each of these CDPs consist of only two functions. Hence, the conjecture requires the following precision:

Conjecture 1.1. *In any fixed dimension and up to an equivalence relation, there are only finitely many Fano CDPs with more than two functions.*

Through the work of Huggenberger, we know that Conjecture 1.1 holds for 2-dimensional Fano CDPs. Through combinatorial techniques that are described in Chapter 3, we recover this result. Furthermore, in analogy to Huggenberger's classification of 2-dimensional Fano complexity-1 T-varieties, we complete a classification of 2-dimensional Fano CDPs with more than two functions, which is given in Appendix A.

We generalize key elements of our arguments from the 2-dimension case to obtain results for Fano CDPs in higher dimensions. For a fixed base polytope, we give an upper bound on the number of functions in a Fano CDP with this fixed base. It turns out that we need only look at the upper bounds given by finitely many bases, and hence we have the following theorem, which is given as Theorem 4.11 in Chapter 4:

Main Theorem 1. *In any fixed dimension, there is a uniform upper bound on the number of functions in a Fano CDP.*

Next we show that fixing a base polytope and number of functions yields only finitely many possible Fano CDP. Combining this result with Main Theorem 1 gives:

Main Theorem 2. *For any fixed lattice polytope in any dimension, there are only finitely many Fano CDPs with this lattice polytope as its base, up to equivalence.*

This result is given as Theorem 5.2 in Chapter 5. To prove Conjecture 1.1 we would need to bound the set of inequivalent permissible base polytopes, that is, those yielding Fano CDPs, to a finite set.

Chapter 2

Background

In Sections 2.1 and 2.2 we define and highlight key properties of reflexive polytopes and Fano combinatorial divisorial polytopes, respectively.

2.1 Reflexive Polytopes

We proceed first with the necessary definitions toward defining a reflexive polytope, for which we refer to [1], and follow with examples giving some intuitive understanding of these definitions.

Definition 2.1. A d-dimensional *lattice polytope* is the convex hull of a finite collection of points $S = \{v_1, \dots, v_n\} \subset \mathbb{Z}^d$, where the convex hull of S, denoted $\mathrm{Conv}(S)$, is given by

$$\mathrm{Conv}(S) = \left\{ \sum_{i=1}^{n} c_i v_i : \forall i \ c_i \geq 0 \text{ and } \sum_{i=1}^{n} c_i = 1 \right\}.$$

In this document, every reference to a polytope actually refers to a lattice polytope, unless otherwise specified.

Definition 2.2. We say that F is a *face* of a lattice polytope $P \subset \mathbb{R}^d$ if

$$F = \mathrm{face}_v(P) := \{u \in P : \langle u, v \rangle \text{ is minimal}\}$$

for some $v \in \mathbb{Z}^d$.

A codimension one face of a polytope is called a *facet* of the polytope, while a 0-dimensional face of a polytope is called a *vertex* of the polytope.

Definition 2.3. A facet F of a lattice polytope $P \subset \mathbb{R}^d$ is *at height 1* if there is some $v \in \mathbb{Z}^d$ whose components have gcd 1, so that $F = \mathrm{face}_v(P)$ and $\langle u, v \rangle = -1$ for all $u \in F$.

For a facet F at height 1, the vector v given in Definition 2.3 is in fact the inward normal vector of F. Consider the equation $\langle u, v \rangle = c$. Ranging c from -1 to 0 encodes all of the lines parallel to F that lie between F and the origin. Since v is integral, there is no integral point lying on one of these lines strictly between F and the origin. Thus this gives us the intuition that if a facet F is at height 1, then as F is slid towards the origin, F does not hit an integral point before reaching the origin.

Definition 2.4. A lattice polytope is *reflexive* if all of its facets are at height 1.

(a) Non-reflexive

(b) Reflexive

Figure 2.1: Examples of 2-dimensional polytopes

Example 2.5. In Figure 2.1 the point emphasized is the origin. The polytope given in Figure 2.1(a), say P, is the convex hull of $\{(-2,0),(0,-1),(1,1)\}$, but it is also the convex hull of $\{(-2,0),(0,-1),(1,1),(-1,0)\}$. Intuitively, in 2-dimensions, the convex hull of a set of points can be found by stretching an elastic bound around all of the points in the set and releasing. Imagine the elastic band catching on the points as if they are pegs — the interior and boundary of this shape is the convex hull of the points.

Note that P is not reflexive, as the facet F given by the line segment from $(-2,0)$ to $(1,1)$ is not at height 1: geometrically, if slid toward the origin, F passes the point $(-1,0)$ before reaching the origin. Meanwhile, every facet of the polytope, say P', given in Figure 2.1(b) is at height 1, and so P' is reflexive. For example, if we denote the facet given by the line segment from $(-2,-1)$ to $(1,1)$ by F', then taking $v = (2,-3)$ satisfies Definition 2.3 applied to F'. Note that P' has no interior lattice point except for the origin. These examples illustrate that a necessary condition of a polytope P being reflexive is that the origin must be the only interior lattice point of P.

An alternative and equivalent definition of a reflexive polytope can be made through the *dual* of a polytope.

Definition 2.6. Let P be a lattice polytope having 0 as an interior point. The *dual* polytope of P is

$$P^\circ = \{u \in \mathbb{R}^d : \langle u, v\rangle \geq -1 \text{ for all } v \in P\}.$$

In general the dual of a lattice polytope is not also a lattice polytope. Reflexive polytopes are precisely those lattice polytopes whose dual is also a lattice polytope. Furthermore, the dual of a reflexive polytope is also a reflexive polytope [1, p. 380].

Example 2.7. In Figure 2.2(a) we give a reflexive polytope P, and then its dual in Figure 2.2(b), which is also a reflexive polytope. If we apply Definition 2.6 to the vertex $(-1,1)$

of P, we see that every point $(x, y) \in P^\circ$ needs to satisfy $-x + y \geq -1$. That is, the point (x, y) is above the line $y = x - 1$, as illustrated in the figure with the red arrow. We can make similar conclusions with the other vertices of P to fully construct the dual polytope P°.

(a) P (b) P°

Figure 2.2: A reflexive polytope and its dual, which is also a reflexive polytope

Definition 2.8. Two lattice polytopes $P, P' \subset \mathbb{R}^d$ are *lattice equivalent* if there is an invertible linear map $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ and $v \in \mathbb{Z}^d$ such that $\varphi(\mathbb{Z}^d) = \mathbb{Z}^d$ and $\varphi(P) + v = P'$.

The following result concerning reflexive polytopes acts as strong motivation toward Conjecture 1.1:

Theorem 2.9 (Lagarias and Ziegler, 1991)**.** *In any fixed dimension, there are only finitely many equivalence classes of reflexive polytopes.*

Recent work has focused on the classification of reflexive polytopes, as the classification of reflexive polytopes is equivalent to the classification of Fano toric varieties [1, Theorem 8.3.4]. There are 16 isomorphism classes of 2-dimensional reflexive polytopes [1, Theorem 8.3.7]. Kreuzer and Skarke determined that there are over 400 isomorphism classes of reflexive polytopes in 3-dimensions, and over four million in 4-dimensions [4].

2.2 Fano Combinatorial Divisorial Polytopes

In Section 2.2.1, we introduce the notion of a combinatorial divisorial polytope, and then in Section 2.2.2 we describe the Fano property of a CDP.

2.2.1 Combinatorial Divisorial Polytopes

We can now give the precise definition of a combinatorial divisorial polytope as determined by Ilten and Süß [3].

Definition 2.10. A d-dimensional *combinatorial divisorial polytope (CDP)* Ψ consists of a lattice polytope $P \subset \mathbb{R}^{d-1}$ and functions $\psi_1, \ldots, \psi_n : P \to \mathbb{R}$ such that

1. each ψ_i is piecewise linear, continuous, and concave;

2. the vertices of the graph of each ψ_i are integral;

3. $\sum_{i=1}^{n} \psi_i$ is strictly positive on the interior of P.

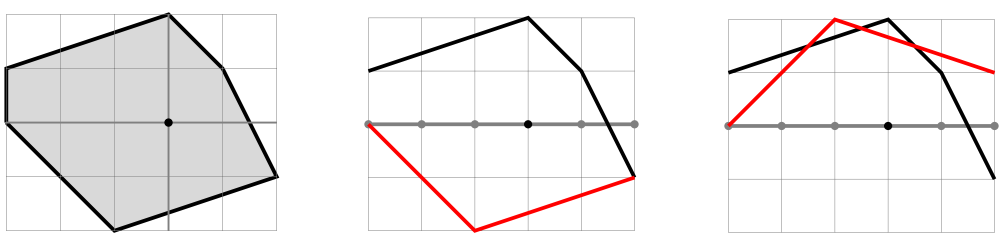
We frequently refer to the lattice polytope P as simply the *base polytope* or *base* of the CDP. Moreover, we denote the graph of a function ψ_i by $\Gamma(\psi_i)$. We give an example of the vertices of some $\Gamma(\psi_i)$ in Example 2.11.

We have a correspondence between d-dimensional polytopes and d-dimensional CDPs with exactly two functions. Starting with a lattice polytope $P \subset \mathbb{R}^d$, let $\pi_1 : \mathbb{R}^d \to \mathbb{R}^{d-1}$ be the projection to the first $d-1$ coordinates and $\pi_2 : \mathbb{R}^d \to \mathbb{R}$ be the projection to the last coordinate. Set $R = \pi_1(P)$, and define $\psi_1, \psi_2 : R \to \mathbb{R}$ by $\psi_1(u) = \max(\pi_2(\pi_1^{-1}(u) \cap P))$ and $\psi_2(u) = -\min(\pi_2(\pi_1^{-1}(u) \cap P))$. Then the base R with the functions ψ_1, ψ_2 is a CDP. Moreover, this process can be reversed to obtain a polytope from a CDP with two functions, by reflecting one of the functions and taking the convex hull of the vertices of the graph of the CDP.

Example 2.11. An example of this process is illustrated for a 2-dimensional polytope P in Figure 2.3(a). The polytope P is the convex hull of the points

$$\{(-3, 1), (0, 2), (1, 1), (2, -1), (-1, -2), (-3, 0)\}.$$

We project down one dimension to obtain the 1-dimensional polytope $R = [-3, 2]$, given in Figure 2.3(b). We encode the missing information by defining two functions ψ_1 and $-\psi_2$ over R, where ψ_1 gives the "upper" facets of P, and $-\psi_2$ gives the "lower" facets of P. These functions are also pictured in Figure 2.3(b). Finally, we reflect $-\psi_2$ so that both of the functions are concave. The resulting CDP is given in Figure 2.3(c).



(a) $P \subset \mathbb{R}^2$ (b) $R = \pi_1(P) \subset \mathbb{R}^1$ (c) Corresponding CDP

Figure 2.3: Example of correspondence between polytopes and CDPs with two functions: (a) the lattice polytope; (b) the lattice polytope projected down one dimension; and (c) the corresponding CDP.

The vertices of $\Gamma(\psi_2)$, the graph of the function given in red in Figure 2.3(c), are the endpoints $(-3, 0)$ and $(2, 1)$, and also the point $(-1, 2)$. Note that these points correspond to 0-dimensional faces of the original polytope P. Since P is a *lattice* polytope, the vertices of both $\Gamma(\psi_1)$ and $\Gamma(\psi_2)$ are indeed integral, as we require in Definition 2.10.

Before introducing the concept of the *regions of linearity* of a function ψ_i in a CDP, we define a *subdivision* of a polytope, to which we refer to [7].

Definition 2.12. A *subdivision* of a lattice polytope P is a finite collection $\mathcal{C}$ of lattice polytopes such that

1. $P = \cup_{R\in\mathcal{C}} R$, as sets of points;
2. the empty polytope is in $\mathcal{C}$;
3. if $R \in \mathcal{C}$, then all of its faces are also in $\mathcal{C}$;
4. if $Q, R \in \mathcal{C}$, then $Q \cap R$ is a face of both Q and R.

Recall that the functions ψ_i in a CDP are piecewise linear. Because ψ_i is concave, the facets of $\Gamma(\psi_i)$ are convex, and hence project down to a decomposition of the base polytope P into polytopes. Moreover, as $\Gamma(\psi_i)$ has integral vertices, the vertices of the polytopes in the decomposition are integral, and hence the graph of $\Gamma(\psi_i)$ corresponds to a subdivision of the base polytope. We say that a d-dimensional polytope in this subdivision is a *region of linearity* of the function ψ_i, where d is the dimension of P. Examples of possible regions of linearity of a function ψ_i defined over a 2-dimensional base are given in Figure 2.4.

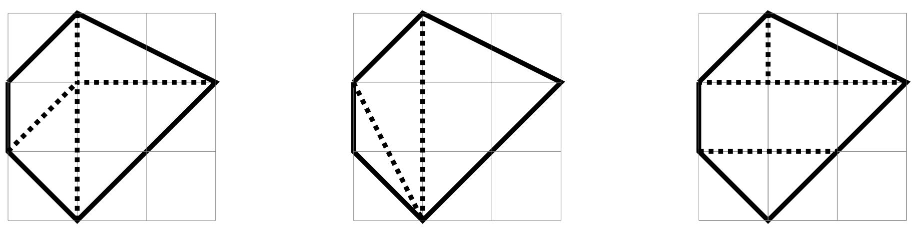


Figure 2.4: Example of possible regions of linearity

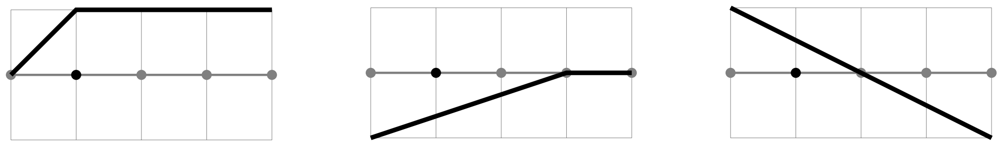
2.2.2 The Fano Property

Since CDPs can contain any finite number of functions, we can think of CDPs as a generalization of polytopes. The *Fano* property of a CDP generalizes the reflexive property of polytopes, as determined by Ilten and Süß [3].

Definition 2.13. A CDP Ψ consisting of base polytope P and functions $\psi_1, \ldots, \psi_n$ is said to be *Fano* if $0 \in P^\circ$ and there are integers $a_1, \ldots, a_n$ such that

1. $\sum_{i=1}^n a_i = -2$;
2. for all $i = 1 \ldots n$, $\psi_i(0) + a_i + 1 > 0$, and each facet of $\Gamma(\psi_i + a_i + 1)$ is at height 1;
3. for any facet F of P not at height 1, $\sum_{i=1}^n \psi_i$ is identically 0 on F.

When each facet of $\Gamma(\psi_i + a_i + 1)$ is at height 1, we say that $\psi_i + a_i + 1$ is *at height 1.*



(a) ψ_1 (b) ψ_2 (c) ψ_3

Figure 2.5: Example of Fano CDP

Example 2.14. Consider the Fano CDP with base polytope $P = [-1, 3]$ and the functions ψ_1, ψ_2, and ψ_3, depicted in Figure 2.5. Taking $a_1 = -1$, $a_2 = 0$, and $a_3 = -1$, the functions $\psi_i + a_i + 1$ are at height 1. Moreover, note that the point 3 is a facet of P which is not at height 1, but that $\sum_{i=1}^{3} \psi_i(3) = 0$, as required by Definition 2.13.

Remark 2.15. *Under the correspondence between CDPs with two function and polytopes, we have that the CDP is Fano if and only if the corresponding polytope can be translated so that it contains the origin as an interior point and is reflexive.*

We also have a notion of equivalence of CDPs. We may translate and shear the functions, or map to an equivalent base polytope, to obtain an equivalent CDP. More precisely: Let Ψ be a CDP as before. Suppose we have another CDP Φ consisting of functions $\gamma_1, \ldots, \gamma_n$ and base polytope $P' \subset \mathbb{R}^{d-1}$. Then Ψ and Φ are *equivalent* if there is a lattice equivalence $\varphi : P \to P'$ such that $\varphi(0) = 0$, a permutation $\sigma \in S_n$, a point $v \in \mathbb{Z}^{d-1}$, and integers $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ with $\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \beta_i = 0$, so that

$$\gamma_{\sigma(i)}(u) = \psi_i(\varphi(u)) + \alpha_i + (\beta_i)\langle \varphi(u), v \rangle \tag{2.1}$$

for all $u \in P$, $i = 1 \ldots n$. Furthermore, we can add the constant zero function to Ψ, and this resulting CDP is equivalent to Ψ.

Remark 2.16. *Equivalences between CDPs preserve the Fano property.*

Remark 2.17. *If Ψ and Φ are two equivalent CDPs with exactly 2 functions, then their corresponding polytopes are also equivalent.*

Example 2.18. Consider the CDP, say Ψ, given in the upper left hand corner of Figure 2.6. Its base polytope is $P = [-1, 1]$, and let ψ_1 be the red function, and ψ_2 be the black function. Then Ψ is Fano. Moreover, the corresponding polytope of Ψ is depicted below it, and it indeed is reflexive.

Returning to the CDP, we shear ψ_1 by a factor of -1 and ψ_2 by a factor of 1 to obtain an equivalent Fano CDP, which is depicted to its right. We can also translate the functions, and then map to an equivalent base, which in this case is given by the reflection of P through the origin. Note that the corresponding polytopes of the last two CDPs need to be translated down by 1 so that the origin is interior to the polytope and so that they are reflexive.

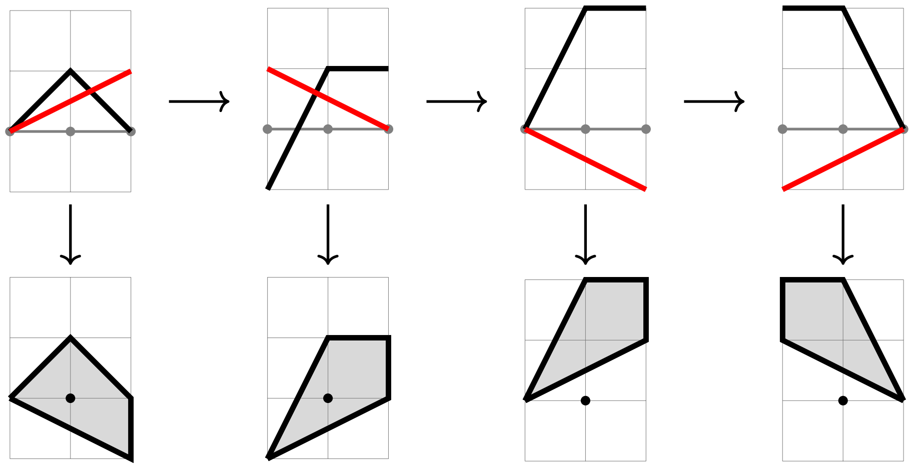


Figure 2.6: Equivalent CDPs and their corresponding polytopes

As illustrated in Example 2.18, shearing a CDP with exactly two functions corresponds to shearing its corresponding polytope in the vertical direction, that is, in the direction which we project along to obtain its corresponding CDP. However, polytopes can be sheared in other directions, implying that the converse of Remark 2.17 is not true. In fact, infinite families of inequivalent Fano CDPs can be constructed from an isomorphism class of reflexive polytopes. Thus the naive generalization of Theorem 2.9 to Fano CDPs, that is, there being finitely many isomorphism classes of Fano CDPs in any fixed dimension, does not hold. However, the Fano CDPs arising from reflexive polytopes each consist of only two functions. Hence Conjecture 1.1 generalizes Theorem 2.9 to Fano CDPs, except those with exactly two functions.

Example 2.19. Consider the reflexive polytope, say P_1, given in the upper left hand corner of Figure 2.7. It corresponds to a Fano CDP with two functions, depicted below it. We may shear P_1 horizontally to obtain another reflexive polytope, say P_2, depicted to the right of P_1, and which also corresponds to a Fano CDP. We could repeat this process, shearing P_2 in the same direction as before. Since these reflexive polytopes are obtained through shearing, they are all equivalent. However, the Fano CDPs they correspond to are not equivalent: the bases of the Fano CDPs have different lengths, and hence are not equivalent. Since this process could be repeated indefinitely, we may construct an infinite number of inequivalent Fano CDPs, all with exactly two functions.

The equivalence relations among CDPs allow us to make some key assumptions. Let Ψ be a Fano CDP with base polytope P and functions $\psi_1, \ldots, \psi_n$. Then:

1. We need only consider the translated functions $\psi_i' := \psi_i + a_i + 1$, which are at height 1.
2. We may assume that each $\Gamma(\psi_i')$ is not an integral plane, as otherwise the function could be sheared and translated to the zero function and hence discarded.

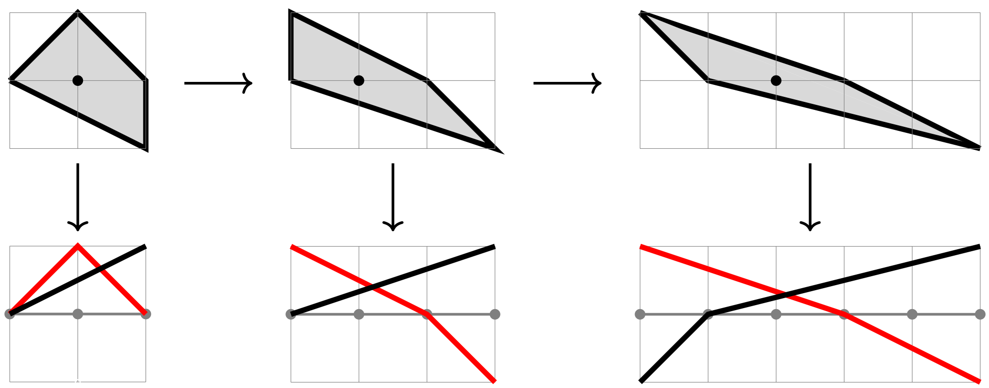


Figure 2.7: Equivalent reflexive polytopes can correspond to inequivalent Fano CDPs.

3. For the purposes of enumeration, we may shear $\psi'_1, \ldots, \psi'_{n-1}$ to a useful position, and then shear ψ'_n counter actively, so that the net shearing is zero. For example, in some cases we may shear each of $\psi'_1, \ldots, \psi'_{n-1}$ to zero at a particular point in the base polytope.

Indeed these three steps form the basis of our strategy. Applying Property 3 of Definition 2.10 to the translated functions ψ'_i yields

$$\sum_{i=1}^{n} \psi'_i > n-2 \quad \text{on the interior of } P\text{, and} \tag{2.2}$$

$$\sum_{i=1}^{n} \psi'_i \geq n-2 \quad \text{on the boundary of } P. \tag{2.3}$$

Furthermore, applying Property 3 of Definition 2.13 to the translated functions ψ'_i yields

$$\sum_{i=1}^{n} \psi'_i = n-2 \tag{2.4}$$

on any facet F of P which is not at height 1.

Finally, the following definitions offer a way to classify the functions ψ'_i, and they frequently appear as separate cases when determining properties of Fano CDPs.

Definition 2.20. Let ψ'_i be a function in a CDP with base polytope P. We say ψ'_i is *integral* if $\psi'_i(v)$ is integral for all integral points $v \in P$. Otherwise, we say that ψ'_i is *non-integral.*

A direct consequence of Lemma 4.1, which is proved in Chapter 4 for d-dimensional Fano CDPs, is the following proposition, which offers an equivalent way to classify the functions ψ'_i.

Proposition 2.21. *Let ψ'_i be a function in a Fano CDP with base polytope P. Then ψ'_i is integral if and only if $\psi'_i(0) = 1$.*

Chapter 3

Finiteness and Enumeration in 2-Dimensions

In this chapter we highlight the key ideas in our enumeration of all 2-dimensional Fano CDPs with more than two functions. These ideas inspire our arguments in the general case, described in Chapters 4 and 5.

3.1 Bounds on the Number of Functions

We begin by establishing bounds on the number of functions in a 2-dimensional Fano CDP.

Proposition 3.1. *Any 2-dimensional Fano CDP Ψ consists of at most 4 functions. Moreover, if the base polytope of Ψ has exactly one of ± 1 as a facet, then Ψ consists of at most 3 functions.*

The proof of Proposition 3.1 is given in a more general setting in Section 4, and hence omitted for now. The idea of the proof is to bound the sum $\psi_i'(-1) + \psi_i'(1)$ above by concavity arguments, and use the lower bound given in Inequality (2.3). A more subtle argument is needed to establish Proposition 3.5.

Proposition 3.5. *There are no 2-dimensional Fano CDPs with more than two functions and a base polytope having both -1 and 1 as interior points.*

The main idea of the proof is to find bounds on the values of the first $n-1$ functions, $\psi_1', \ldots, \psi_{n-1}'$, at the points -1 and 1, through shearing. Then, we can use the preliminary results established below and the bound $n-2 < \sum \psi_i'$ given in Inequality (2.2) to determine bounds on the final function, ψ_n', at -1 and 1. The proof formally appears at the end of this subsection, as we first establish some lemmas.

Lemma 3.2. *Suppose ψ_i' is a function in a 2-dimensional Fano CDP. If $\psi_i'(-1) = 1 = \psi_i'(1)$, then ψ_i' is identically 1.*

Proof. Suppose $\psi_i'(-1) = 1 = \psi_i'(1)$. Then by the concavity of ψ_i' and the fact that ψ_i' is at height 1, $\psi_i'(0)$ is at least and at most 1, respectively. Thus $\psi_i'(0) = 1$, and moreover, $\psi_i' \equiv 1$ on $[-1, 1]$. If the graph of ψ_i' had a distinct facet, say F, on the left hand side of the origin, it would necessarily have positive slope due to concavity, and so the point $(0, 1)$ would be in between F and the origin. Then F could not be at height 1, a contradiction. Similarly there could be no distinct facet to the right of the origin. Hence $\psi_i' \equiv 1$.

□

Lemma 3.3. *Suppose ψ_i' is a function in a 2-dimensional Fano CDP. If $\Gamma(\psi_i')$ is not an integral line, then $\psi_i'(-1) \leq 0$ or $\psi_i'(1) \leq 0$.*

Proof. Let $a = \psi_i'(-1)$ and $b = \psi_i'(1)$. Assume, toward a contradiction, that $a > 0$ and $b > 0$.

Suppose $\psi_i'(0)$ is non-integral. Then $(-1, a)$ and $(1, b)$ lie on the same facet F of the graph ψ_i', and so, there are $u, v \in \mathbb{Z}$ such that $-u + av = -1$ and $u + bv = -1$. The solution of this system gives $u = \frac{b-a}{a+b}$. As $a, b > 0$, we have that $|b - a| < |a + b|$ and so u is an integer if and only if $a = b$. Since ψ_i' is non-integral, $\psi_i' \equiv a$ on $[-1, 1]$ (as it could have no vertex inside this range). Since $a = \psi_i'(0)$ is non-integral, then $\Gamma(\psi_i')$ would have a non-integral vertex, which contradictions the definition of a CDP.

Suppose $\psi_i'(0)$ is integral. Then $\psi_i'(0) = 1$. As both $a, b > 0$ and are integers, $a = b = 1$. By Lemma 3.2, $\psi_i' \equiv 1$, which we assumed was not the case.

□

Lemma 3.3 is often used in the form of the following corollary:

Corollary 3.4. *Suppose ψ_i' is a function in a 2-dimensional Fano CDP and $\Gamma(\psi_i')$ is not an integral line. If $\psi_i'(1) > -1$, then $\psi_i'(-1) \leq 1$. Similarly, if $\psi_i'(-1) > -1$, then $\psi_i'(1) \leq 1$.*

Proof. Let ψ_i' be as above, with $\psi_i'(-1) > -1$. Suppose, towards a contradiction, that $\psi_i'(1) > 1$. Then we may shear ψ_i' so that $\psi_i'(-1)$ moves up by one and $\psi_i'(1)$ moves down by one. Both resulting values would be positive, contradicting Lemma 3.3. A symmetric argument gives the result when $\psi_i(1) > -1$ and $\psi_i(-1) > 1$. □

We can use these preliminary results and the bound given in Inequality (2.2), which is $n - 2 < \sum_{i=1}^{n} \psi_i'$ on the interior of the base polytope, to deduce the following enumeration steps, which we reference as E1, E2, E3, and E4:

E1. $0 \leq \psi_i'(-1) < 1,\ i = 1 \ldots n-1$

Shear by taking $b_i = \lfloor \psi_i'(-1) \rfloor$ for $i = 1 \ldots n-1$, and $b_n = -\sum_{i=1}^{n-1} b_i$ in Equation (2.1).

E2. $\psi_i'(1) \leq 1,\ i = 1 \ldots n-1$

Apply the lower bound given in E1 to Corollary 3.4.

E3. $-1 < \psi'_n(-1)$ and $-1 < \psi'_n(1)$

Apply the upper bounds given in E1 and E2 to Inequality (2.2).

E4. $\psi'_n(-1) \leq 1$ and $\psi'_n(1) \leq 1$

Apply E3 to Corollary 3.4.

We are now able to give the proof of Proposition 3.5.

Proposition 3.5. *There are no 2-dimensional Fano CDPs with more than two functions and a base polytope having both -1 and 1 as interior points.*

Proof. Suppose $\psi'_1, \ldots, \psi'_n$ are the functions of such a Fano CDP. Then we may assume E1, E2, E3, and E4. By Lemma 3.3, either $\psi'_n(-1) \leq 0$ or $\psi'_n(1) \leq 0$. Without loss of generality, assume that $\psi'_n(-1) \leq 0$ (as otherwise, we may reflect the base polytope through the origin to obtain an equivalent CDP with $\psi'_n(-1) \leq 0$).

We will show that $\psi'_i(1) \leq 0$ for all $i = 1 \ldots n-1$, which enforces that $\psi'_n(1) > n-2$. This bound, together with the upper bound on $\psi'_n(1)$ given in E4, implies that $n < 3$.

Indeed, Inequality (2.2) requires that $n - 2 < \sum_{i=1}^{n-1} \psi'_i(-1)$, as $\psi'_n(-1) \leq 0$. Since $\psi'_i(-1) < 1$ for $i = 1 \ldots n-1$ by E1, if there is some j such that $\psi'_j(-1) \leq 0$, then the inequality cannot be satisfied. Hence we have that $\psi'_i(-1) > 0$ for all $i = 1 \ldots n-1$. Moreover, since only one of $\psi'_i(-1)$ and $\psi'_i(1)$ can be positive by Lemma 3.3, we see that $\psi'_i(1) \leq 0$ for all $i = 1 \ldots n-1$. Then again by Inequality (2.3), $\psi'_n(1) > n-2$.

We also have an upper bound on $\psi'_n(1)$ given by E4: $\psi'_n(1) \leq 1$. Putting the bounds on $\psi'_n(1)$ together, we have $n - 2 < \psi'_n(1) \leq 1$, and thus $n < 3$.

$\square$

3.2 Enumeration

Proposition 3.5 allows us to assume in the remainder of this chapter that the base polytope P has at least one of -1 and 1 as a facet. Through equivalences of the base polytope, we may assume that -1 is a facet.

3.2.1 Enumeration of Fano CDPs with base polytope of length 2

Since -1 is a facet of P, we know that $\psi'_i(-1)$ is integral, by the definition of a CDP. With this extra knowledge, we can amend the enumeration steps E1, E2, E3, and E4 to the following enumeration steps F1, F2, F3, and F4:

F1: $\psi'_i(-1) = 0$

Shear by taking $b_i = \psi'_i(-1)$ for $i = 1 \ldots n-1$ and $b_n = -\sum_{i=1}^{n=1} b_i$ in Equation (2.1).

F2: $\psi'_i(1) \leq 1$, $i = 1 \ldots n-1$

Apply the lower bound given in F1 to Corollary 3.4.

F3: $-1 \leq \psi'_n(1)$

Apply F2 to Inequality (2.3).

F4: $n-2 \leq \psi'_n(-1)$

Apply F1 to Inequality (2.3).

Moreover, a special case of Lemma 4.6, later proved in arbitrary dimensions, is:

Lemma 3.6. *Suppose ψ'_i is a function in a 2-dimensional Fano CDP. Then*

$$\psi'_i(-1) + \psi'_i(1) \leq 1.$$

Again, the proof of this lemma is given in a more general setting in Section 4.1, so we omit it for now. We can apply the lower bounds given in F3 and F4 to obtain upper bounds for $\psi'_n(1)$ and $\psi'_n(-1)$, respectively:

F5: $\psi'_n(1) \leq 1-(n-2)$

Apply F4 to Lemma 3.6.

F6: $\psi'_n(-1) \leq 1-\psi'_n(1)$

We can apply F1,. . . ,F6 to conclude the following proposition:

Proposition 3.7. *There are only finitely many 2-dimensional Fano CDPs with $n \geq 3$ functions and base polytope $P = [-1, 1]$.*

Proof. Suppose Ψ is a Fano CDP with n functions, $\psi'_1, \ldots, \psi'_n$, and base polytope $P = [-1, 1]$. By Proposition 3.1 we need only consider the cases $n = 3$ and $n = 4$. We will use F1,. . . ,F6 to provide upper bounds and lower bounds for the function values over the points -1 and 1.

Since 1 is a facet of P, $\psi'_n(1)$ is integral. Moreover, F3 and F5 give a lower bound and upper bound for $\psi'_n(1)$, respectively. Thus there are only finitely many choices for $\psi'_n(1)$. Fix a value for $\psi'_n(1)$. Then we can similarly bound $\psi'_n(-1)$ below and above by F4 and F6, leaving only finitely many choices for $\psi'_n(-1)$. Consider the inequality $n-2-\psi'_n(1) \leq \sum_{i=1}^{n-1} \psi'_i(1)$ which is given by Inequality (2.3). For a fixed $j \in \{1, \ldots, n-1\}$, we can bound this inequality above by $\psi'_j(1)+n-2$ by applying the upper bounds on $\psi'_i(1)$ given in F2. This yields:

F7: $-\psi'_n(1) \leq \psi'_i(1)$, $i = 1 \ldots n-1$

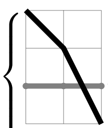
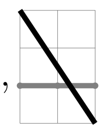
Thus $\psi'_i(1)$ is bounded above and below for all i, and hence can take only finitely many values. Thus we see that, for all $i = 1 \ldots n$, there are only finitely many possible choices of values for the pair $\psi'_i(-1)$ and $\psi'_i(1)$. Moreover, for any fixed pair of values for $\psi'_i(-1)$ and $\psi'_i(1)$, there are only finitely many ways to define ψ'_i. This is because there are two options for $\psi'_i(0)$: either ψ'_i is linear, or $\psi'_i(0) = 1$. The values of $\psi'_i(-1)$, $\psi'_i(0)$, and $\psi'_i(1)$ certainly determine the entire function. □

We can use the steps F1,...,F7 to explicitly enumerate all Fano CDPs with base polytope $P = [-1, 1]$ and 3 or 4 functions. We provide a classification of all such Fano CDPs in Appendix A.

Example 3.8. As an example of how to use F1,...,F7 for enumeration, we apply them to find all Fano CDPs with base function $P = [-1, 1]$ and $n = 4$ functions. Let ψ'_i, $i = 1 \dots 4$, be the four functions in such a Fano CDP. Assume they are sheared so that $\psi'_i(-1) = 0$, $i = 1 \dots 3$ as in F1. We can first establish bounds on $\psi'_4(1)$: by F3 and F5, we have $-1 \leq \psi'_4(1) \leq 1 - (n - 2) = -1$. Thus $\psi'_4(1) = -1$. Next we can use F4 and F6 to bound $\psi'_4(-1)$: we have

$$2 = n - 2 \leq \psi'_4(-1) \leq 1 - \psi'_4(1) = 2$$

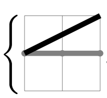
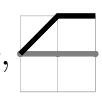
and so $\psi'_4(-1) = 2$. Thus

$\psi'_4 \in \Big\{$ ,  $\Big\}$.

Now that we have determined the possible values for ψ'_4, we can bound $\psi'_i(1)$ for $i = 1 \dots 3$. By F2 and F7 we know that

$$1 = -\psi'_4(1) \leq \psi'_i(1) \leq 1$$

for all $i = 1 \dots 3$, and so $\psi'_i(1) = 1$ for all $i = 1 \dots 3$. As we also assume $\psi'_i(-1) = 0$, we see that

$\psi'_1, \psi'_2, \psi'_3 \in \Big\{$ ,  $\Big\}$.

3.2.2 Enumeration of Fano CDPs with base polytope of length at least 3

In Proposition 3.1, we eliminated the possibility of a Fano CDP Ψ consisting of a base polytope P with exactly one facet at height 1, provided that Ψ consists of at least 4 functions. We now enumerate all possible Fano CDPs with such a base polytope P, consisting of exactly 3 functions. The enumeration will allow us to conclude the following:

Proposition 3.17. *There are only finitely many 2-dimensional Fano CDPs with exactly 3 functions and base polytope P such that exactly one of ± 1 is a facet of P. In these cases, P has length at most 6.*

Before returning to Proposition 3.17, we prove Lemma 3.12, which determines the possible forms of the functions in a Fano CDP with such a base P having exactly one facet at height 1. As before we assume that -1 is a facet of P, and so 1 is an interior point of P. Let m be the length of P, and thus we may write $P = [-1, m - 1]$. Note that $m - 1 > 1$ since 1 is interior to P. Thus $m \geq 3$.

Let ψ'_1, ψ'_2, ψ'_3 be the functions of Ψ at height 1. First we characterize what the ψ'_i can look like, and in particular, what values ψ'_i can take at the facets of P. Since we assume that the facet $m-1$ of P is not at height 1, we must satisfy Equation (2.4), that is,

$$\psi'_1(m-1) + \psi'_2(m-1) + \psi'_3(m-1) = n-2.$$

Hence, after identifying all possible values for $\psi'_i(m-1)$, we seek all integer solutions to Equation (2.4) to enumerate all possible such Fano CDPs. First we establish the following two lemmas, which help to identify the possible forms of the ψ'_i. Lemma 3.9 will be used to describe the forms of non-integral ψ'_i, and Lemma 3.11 will be used to describe the forms of integral ψ'_i.

Lemma 3.9. *Let ψ'_i be a function in a 2-dimensional Fano CDP with base polytope $P = [-1, m-1]$. Suppose that $\Gamma(\psi'_i)$ has a vertex over a point $p \geq 1$, and*

$$F = \{(x, \psi'_i(x)) : -1 \leq x \leq p\}$$

is a facet of $\Gamma(\psi'_i)$. Then ψ'_i can be sheared so that $\psi'_i(-1) = 0$ and $\psi'_i(x) = 1$ for all $x \in [p, m-1]$.

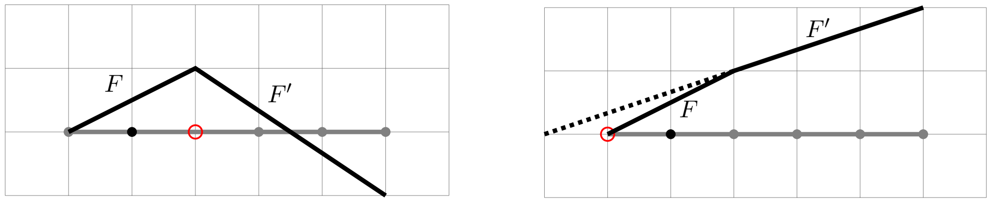
Proof. Let ψ'_i, p, and F be as above. By shearing, we may assume that $\psi'_i(-1) = 0$.

Suppose F meets another facet of the graph, F', at p. If F' has negative slope, then the point $(1, 0)$ would lie between the origin and F', contradicting it being at height 1. Suppose F' has positive slope. If we extend F', its value over the point -1 would have to be positive, due to the concavity of ψ'_i. Then the point $(-1, 0)$ would lie between the origin and the extension of F', again contradicting it being at height 1.

Thus we have established that F' is horizontal, and hence, to be at height 1, it lies one unit above the base P. A similar argument as above demonstrates that there is no vertex of $\Gamma(\psi'_i)$ over any point $q > p$, giving that $\psi'_i(x) = 1$ for all $x \in [p, m-1]$.

□

Example 3.10. In Figure 3.1, we give an example of the cases where F' has negative and positive slope for the base polytope $P = [-1, 4]$. In each case, the black point is the origin, and the red open circle is the lattice point that lies between F' (or the extension of F') and the origin.



(a) F' has negative slope (b) F' has positive slope

Figure 3.1: Example of the cases given in the proof of Lemma 3.9

Lemma 3.11. *Let ψ_i' be a function in a 2-dimensional Fano CDP. If ψ_i' is integral, then it is either linear or has a single vertex at the origin.*

Proof. As ψ_i' is integral, it may be sheared so that $\psi_i' \equiv 1$ on $[-1, 0]$. By the exact argument as given in Lemma 3.2, $\Gamma(\psi_i')$ has no vertex to the left of the origin. We have the same result for the right of the origin. Hence $\Gamma(\psi_i')$ has at most one vertex, and this vertex would be above the origin. $\square$

With these lemmas we can establish the possible values of $\psi_i'(m-1)$.

Lemma 3.12. *Assuming that $\psi_1'(-1) = \psi_2'(-1) = 0$,*

$$\psi_1'(m-1), \psi_2'(m-1) \in \left\{\frac{m}{d}, 1, -k(m-1)+1\right\}$$

and

$$\psi_3'(m-1) \in \left\{-m+2, -m\frac{d'}{d'+1}+1\right\}$$

where $k, d, d' \in \mathbb{N}$, d and $d'+1$ divide m, and $1 < d \leq \frac{m}{2}$.

Proof. First we will describe the possible forms of ψ_1'. Since these forms are determined as a consequence of the fact that $\psi_1'(-1) = 0$, all of the arguments also apply to ψ_2'. We split our arguments into two cases: one where $\psi_1'(0)$ is non-integral, and one where $\psi_1'(0)$ is integral.

Case 1. *The value $\psi_1'(0)$ is non-integral.*

Let d be the smallest number so that $\psi_1'(d-1) = 1$. Since $\psi_1'(-1) = 0$ but $0 < \psi_1'(0) < 1$, such a d exists, and $d - 1 > 0$. Moreover, since $\psi_1'(-1) = 0$ and ψ_1' is at height 1, d is an integer. By Lemma 3.9, $\Gamma(\psi_i')$ either has no vertices, or only one vertex. If $\Gamma(\psi_i')$ has no vertices, then ψ_i' is linear with slope d. Since $\psi_i'(m-1)$ is integral, d must divide m.

Example 3.13. If $m = 6$, we consider the divisors of 6 to get the following functions given in Figure 3.2:

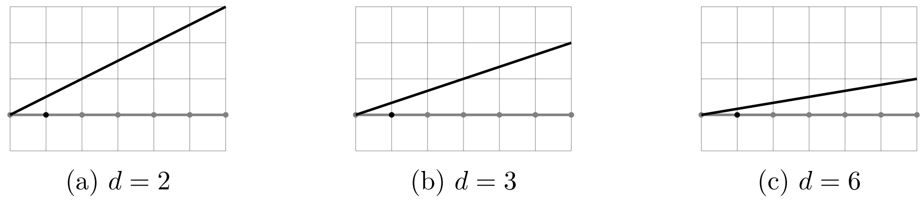


Figure 3.2: Example of non-integral linear ψ_i' defined over base $P = [-1, 5]$

If $\Gamma(\psi_i')$ has exactly one vertex, then by Lemma 3.9, one of the facets is identically 1. Hence this unique vertex is at $d - 1 \in P$.

Example 3.14. If $m = 4$, then we would have the following functions:

Figure 3.3: Example of non-integral ψ_i' which have one vertex and are defined over base $P = [-1, 3]$

Thus, if $\psi_1'(0)$ is non-integral, then $\psi_1'(m-1) = 1$ or $\psi_1'(m-1) = \frac{m}{d}$, where we may assume that $1 < d \leq \frac{m}{2}$.

Case 2. *The value $\psi_1'(0)$ is integral.*

By Lemma 3.11, the graph of ψ_i' only has a vertex at the point $(0, 1)$. Let $-k$ be the slope of ψ_i' on $[0, m-1]$. In order for ψ_i' to be concave and not linear with integral slope, k is non-negative. Thus $\psi_1'(m-1) = -k(m-1) + 1$ for some $k \in \mathbb{Z}_{\geq 0}$.

Example 3.15. If $m = 3$, some of the functions we could have are given in Figure 3.4:

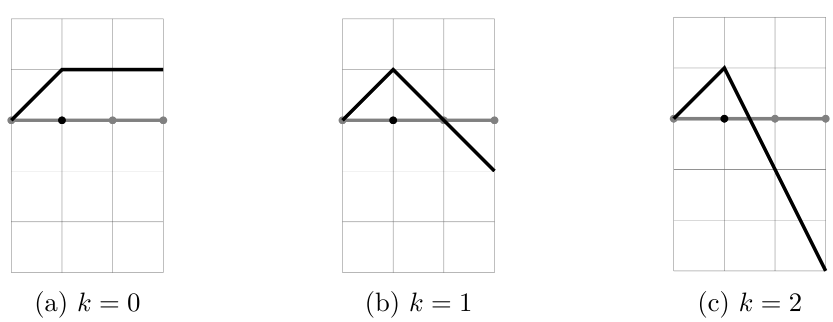


Figure 3.4: Example of integral ψ_i' over base $P = [-1, 2]$

Thus in any case

$$\psi_1'(m-1), \psi_2'(m-1) \in \left\{\frac{m}{d}, 1, -k(m-1)+1\right\},$$

where $k \in \mathbb{N}$ and $1 < d \leq \frac{m}{2}$, as claimed.

Next we determine the possible functions for ψ_3'. First note that $\psi_3'(-1) = 1$: this is an immediate consequence of E4 (which applies because 1 is an interior point of the base polytope) and F4.

The permissible functions for ψ_3' can be found by shearing any permissible function for ψ_1' by a factor of -1 (since the value of the function over -1 will go from 0 to 1). Hence we subtract $m-1$ from the possible values of $\psi_1'(m-1)$ and only keep the values that imply $\psi_3'(1) > -1$ (which is necessary by E3) to see that

$$\psi_3'(m-1) \in \left\{-m+2, -m\frac{d'}{d'+1}+1\right\}$$

where $d' \in \mathbb{N}$ and $d'+1 \mid m$.

Example 3.16. If $m = 4$, then ψ_3' would be one of the black functions in Figure 3.5, which are found by shearing the red function:

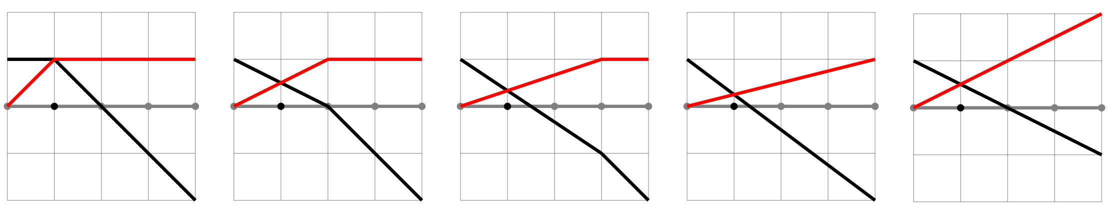


Figure 3.5: Example of functions at height 1 defined over the base $P = [-1, 3]$

□

Proposition 3.17. *There are only finitely many 2-dimensional Fano CDPs with exactly 3 functions and base polytope P such that exactly one of ± 1 is a facet of P. In these cases, P has length at most 6.*

Proof. As before, we may assume that $P = [-1, m-1]$ and $\psi_1'(-1) = \psi_2'(-1) = 0$. As determined in the proof of Lemma 3.12, this implies that $\psi_3'(-1) = 1$. By Lemma 3.12, there are only finitely many choices for the values $\psi_i'(m-1)$, and as seen in the proof, these choices yield only finitely many possible functions. We can enumerate all possible combinations of the ψ_i' by substituting the values for $\psi_i'(m-1)$ in Equation (2.4) and finding the integral solutions. These details are omitted. □

We provide a classification of all derived Fano CDPs in Appendix A.

Chapter 4

Bounds on the Number of Functions

As discussed in the introduction, one of our main results for an arbitrary dimension d is the existence of a uniform upper bound on the number of functions in a d-dimensional Fano CDP. Before we establish this theorem, we determine an upper bound on the number of functions in a Fano CDP that depends on its base polytope P.

This bound is established by first picking a point f_j so that f_j and its reflection through the origin $-f_j$ both lie on the j^{th} coordinate axis and lie in the base polytope P. Moreover, we pick the points f_j so that for each j these points have a common norm. Then we can give, through the concavity of the functions ψ_i', an upper bound on the sum $\psi_i'(-f_j) + \psi_i'(f_j)$, which is dependent on the norm $|f_j|$. This process gives an upper bound on

$$\sum_{i=1}^{n} \left(\psi_i'(-f_j) + \psi_i'(f_j)\right). \tag{4.1}$$

Recall that Inequality (2.3) gives a lower bound on this sum as well. We sum (4.1) over all $j = 1 \ldots d$. By arguing that the upper bound on $\psi_i'(-f_j) + \psi_i'(f_j)$ can only be achieved $d-1$ times for fixed i (as otherwise $\Gamma(\psi_i')$ would be an integral plane, a case discarded), the bound given in the theorem statement is implied.

The necessary lemmas used in the proof of this theorem are given in Section 4.1. In particular, Lemma 4.6 gives an upper bound on the sum $\psi_i'(-f_j)+\psi_i'(f_j)$, while Lemma 4.3 limits the number of times that this upper bound can be achieved. A rigorous version of the proof sketch given above is given in Section 4.2, as well as some subsequent corollaries.

4.1 Preliminary Results

Let Ψ be a $(d+1)$-dimensional CDP with base polytope P and functions $\psi_1, \ldots, \psi_n$. Let $\psi_1', \ldots, \psi_n'$ denote the translated functions that are at height 1, as before.

Lemma 4.1. *Suppose $\psi_i'(0) = 1$. Then ψ_i' is linear along the line segment from 0 to v for all $v \in \partial P$.*

Proof. Restrict the base polytope P to the given line, which we denote by L, and let F be the facet of $\Gamma(\psi_i')$ above L that includes the point above the origin. Suppose F meets another facet, say F', along L. Extending F', its value over the origin would be larger than 1 by the concavity of ψ_i'. Then the point $(0, \dots, 0, 1)$ lies between the extension of F' and the origin, contradicting it being at height 1.

□

A direct consequence of this lemma is that if ψ_i' is integral, then the only vertex in the interior of its graph is over the origin. Moreover, we can also determine the possible regions of linearity of some integral ψ_i': they are polytopes contained in the base polytope having the origin as a vertex.

Example 4.2. Figure 4.1 illustrates, with the dotted lines, possible regions of linearity of the integral functions of a Fano CDP with the depicted 2-dimensional base polytope:

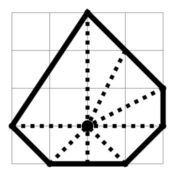


Figure 4.1: Example of regions of linearity of an integral ψ_i'

Lemma 4.3. *If ψ_i' is identically one across every coordinate axis, then $\psi_i' \equiv 1$.*

Proof. Let $v \in \partial P$, such that its components $v_i = 0$ for all $i = k+1 \dots d$, and $v_k \neq 0$. We establish that $\psi_i' \geq 1$ along the line from 0 to v by induction on k. Choose a constant $\gamma \neq 0$ so that $1 - \frac{v_k}{\gamma} \neq 0$, and write $\alpha v = \left(1 - \frac{v_k}{\gamma}\right) u + \frac{v_k}{\gamma} w$, where

$$u = \alpha \begin{pmatrix} \frac{v_1}{1-\frac{v_k}{\gamma}} & \cdots & \frac{v_{k-1}}{1-\frac{v_k}{\gamma}} & 0 & \cdots & 0 \end{pmatrix}, \quad w = \alpha \begin{pmatrix} 0 & \dots & 0 & \gamma & 0 & \dots & 0 \end{pmatrix}$$

and the constant $\alpha > 0$ is chosen so that $\alpha v, u, w \in P$. By induction, $\psi_i'(u) \geq 1$, and by assumption, $\psi_i'(w) = 1$. Thus, by the concavity of ψ_i', we have that $\psi_i'(\alpha v) \geq 1$, and so by Lemma 4.1, $\psi_i' \geq 1$ along the line from the origin to v. Hence $\psi_i' \geq 1$ on P.

Suppose that there is some point $v \in P$ such that $\psi_i'(v) > 1$. Choosing $\alpha > 0$ sufficiently small so that $-\alpha v \in P$, the concavity of ψ_i' would imply that $\psi_i'(-\alpha v) < 1$, a contradiction. Hence $\psi_i' \equiv 1$.

□

Next we work toward Lemma 4.6, which provides a bound on the sum of the function values $\psi_i'(-f_j) + \psi_i'(f_j)$, where f_j and $-f_j$ are as defined in the introduction of this chapter.

Lemma 4.4. *If ψ_i' is non-integral and at height 1, then $\psi_i'(0) \leq \frac{1}{2}$.*

Proof. The point $p = (0, \ldots, 0, \psi_i'(0)) \in \mathbb{R}^{d+1}$ is on the graph of ψ_i'. Since the graph is at height 1, there is some $v \in \mathbb{Z}^{d+1}$ such that $\langle p, v\rangle = -1$, that is,

$$\psi_i'(0) = \frac{-1}{v_{d+1}},$$

where v_{d+1} is the $(d+1)^{th}$ component of v. Since $0 < \psi_i'(0) < 1$ by assumption and $v_{d+1} \in \mathbb{Z}$, we have the desired result.

□

As a momentary aside, we note that the above lemma yields a bound on the number of non-integral functions in a Fano CDP:

Corollary 4.5. *We can have at most 3 non-integral functions in a Fano CDP.*

Proof. Let m be the number of non-integral functions and let l be the number of integral functions. Then, by Inequality (2.2) and Lemma 4.4, we have

$$m + l - 2 < \sum_{i=1}^{m+l} \psi_i'(0) \leq \frac{m}{2} + l \implies m < 4.$$

□

As alluded to previously, Lemma 4.4 can be used to bound the sum of the function values at two points centered about the origin.

Lemma 4.6. *Let $|f_j| = f$ for some $f > 0$. Suppose $\pm f_j \in P$. If ψ_i' is non-integral, then*

$$\psi_i'(-f_j) + \psi_i'(f_j) \leq 1.$$

If ψ_i' is integral but non-linear along the line segment from $-f_j$ to f_j, then

$$\psi_i'(-f_j) + \psi_i'(f_j) \leq 2 - f.$$

Finally, if ψ_i' is integral and linear along the line segment from $-f_j$ to f_j, then

$$\psi_i'(-f_j) + \psi_i'(f_j) = 2.$$

Proof. First assume that ψ_i' is non-integral. Then, by the concavity of ψ_i' and by Lemma 4.4, we have

$$\frac{\psi_i'(-f_j) + \psi_i'(f_j)}{2} \leq \psi_i'(0) \leq \frac{1}{2}$$

and the result follows.

Suppose ψ'_i is integral. By shearing we may assume that $\psi'_i(-f_j) = 1$. If ψ'_i is linear along the line segment from $-f_j$ to f_j, then $\psi'_i(f_j) = 1$, and the result in this case holds. Otherwise suppose ψ'_i is not linear along this line. Then the slope of ψ'_i along the line segment from the origin to f_j is at most -1, which gives $\psi'_i(f_j) \leq 1 - f$. Since the bound on the sum $\psi'_i(-f_j) + \psi'_i(f_j)$ is invariant under shearing, the result follows.

□

4.2 Main Results

With the preliminary results of the previous subsection, we are now equipped to generalize key parts of our arguments from our work in 2-dimensions. We begin by giving upper bounds on the number of functions in a Fano CDP.

Theorem 4.7. *Let Ψ be a $(d+1)$-dimensional Fano CDP with base polytope $P \subset \mathbb{R}^d$ and functions $\psi'_1, \ldots, \psi'_n$. Then $n \leq \frac{1}{c}d$ for some constant $0 < c \leq 1$, only depending on P.*

Proof. For each of the d coordinate axes, let $f_{j,1}$ and $f_{j,2}$ be the points on the j^{th} coordinate axis and the boundary of P. Take $c_j = \min\{|f_{j,1}|, |f_{j,2}|, 1\}$. Let $c = \min_j c_j$ and let $f_j = ce_j$. Define

$$R_j := \{i : 1 \leq i \leq n, \psi'_i(-f_j) + \psi'_i(f_j) = 2\}$$

and $r_j := \#R_j$ for each $j = 1 \ldots d$. Note that, for each $i \notin R_j$, we have, by Lemma 4.6,

$$\psi'_i(-f_j) + \psi'_i(f_j) \leq 2 - c.$$

This bound works for non-integral ψ'_i because $c \leq 1$. Using this and the lower bound given in Inequality (2.3), we have

$$2n - 4 \leq \sum_{i=1}^{n}(\psi'_i(-f_j) + \psi'_i(f_j)) \leq 2r_j + (n - r_j)(2 - c) = 2n - cn + cr_j$$

and hence $n - \frac{4}{c} \leq r_j$. Moreover, by Corollary 4.3, each ψ'_i is not an integral line across at least one coordinate axis, and so for each i, we have that $i \notin R_j$ for some j. Thus

$$\sum_{j=1}^{d} r_j = \sum_{i=1}^{n} \#\{j : i \in R_j\} \leq \sum_{i=1}^{n}(d-1) = (d-1)n.$$

Applying this to the sum of $n - \frac{4}{c} \leq r_j$ over all j yields $n \leq \frac{4}{c}d$.

□

Note that if one of the f_j or $-f_j$ is an interior point of P, then we may use Inequality (2.2) in place of Inequality (2.3) to obtain the bound $n < \frac{1}{c}d$ in Theorem 4.7.

Example 4.8. Consider the 2-dimensional base polytope given in Figure 4.2. The points $f_{1,1} = (-2, 0)$ and $f_{1,2} = (1.5, 0)$ on the 1^{st} coordinate axis and the boundary of P are highlighted in blue. Moreover, the points $f_{2,1} = (0, -1)$ and $f_{2,2} = (0, \frac{2}{3})$ on the 2^{nd} coordinate axis and boundary of P are highlighted in red. In this case, we take $f_1 = (\frac{2}{3}, 0)$ and $f_2 = (0, \frac{2}{3})$.

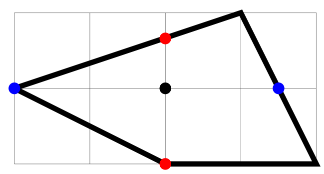


Figure 4.2: Example of the points $f_{j,1}$ and $f_{j,2}$ for a 2-dimensional base polytope

This result has many consequences. Although the bound given in Theorem 4.7 depends on a fixed base polytope, it turns out we need only look at finitely many bases.

Lemma 4.9. *Every lattice polytope P having the origin as an interior point contains a lattice polytope with the origin as its only interior lattice point.*

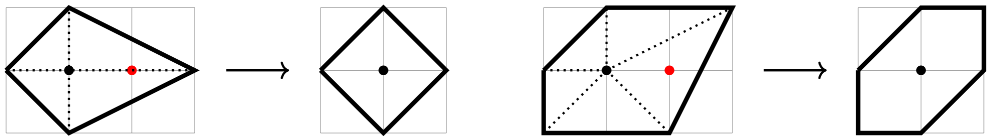
Proof. Let P be as above, and suppose P has $p > 1$ interior lattice points, as the case $p = 1$ is trivial. We proceed by induction on p. Write $P = \text{Conv}\{v_1, \ldots, v_k\}$, where $\{v_1, \ldots, v_k\}$ is the set of lattice points in P. Let $F_1, \ldots, F_l$ be the facets of P. Subdivide P into polytopes P_i sharing only the facet F_i of P and having the origin as a vertex. An example of such a subdivion is given in Figure 4.3.

Suppose one of the facets F of some P_j contains a non-zero interior lattice point of P. In this first case, let $R = \{v_i : v_i \in F_j \cap F\}$. Otherwise some P_j contains an interior lattice point. In this second case, let $R = \{v_i : v_i \in F_j\}$. Take P' to be the convex hull of the points $\{v_1, \ldots, v_k\} - R$. By the choice of R, some interior lattice point of P becomes a vertex of P' and also P' contains the origin as an interior point. As P' contains less interior lattice points than P, by induction, P' contains a lattice polytope with only the origin as an interior lattice point, and hence so does P.

$\square$

Example 4.10. Consider Figure 4.3. Let P be the polytope on the left hand side of Figure 4.3(a). The dotted lines subdivide P into smaller polytopes. Since the point $(1, 0)$ lies on a facet of one of these smaller polytopes, we remove the point $(2, 0)$ from P to obtain P', given on its right.

Let Q be the polytope on the left hand side of Figure 4.3(b); again we have subdivided it into smaller polytopes. Since the point $(1, 0)$ is an interior point of one of these smaller polytopes, we remove the points $(1, -1)$ and $(2, 1)$ from Q to obtain Q', given on its right.



(a) First case in proof (b) Second case in proof

Figure 4.3: Examples of the process in the proof of Lemma 4.9

Theorem 4.11. *In any fixed dimension, there is a uniform upper bound on the number of functions in a Fano CDP.*

Proof. By Lemma 4.9, any base polytope P contains a lattice polytope P' with the origin as its only interior lattice point. The bound given by Theorem 4.7 for P' would also hold for P, because the points f_j picked for P' also lie inside P. Moreover, it is known that there are only finitely many lattice polytopes with exactly one interior lattice point [5]. Taking the maximum bound given by all such P' yields the required result. □

Consider a special case of Theorem 4.7, when we may take $c = 1$ by assuming that, in the proof, the norm $|f_j| = 1$.

Corollary 4.12. *Let $P \subset \mathbb{R}^d$ be the L^1 unit ball. If Ψ is a Fano CDP with base polytope P, and Ψ contains n functions, then $n \leq 4d$. Moreover, this bound is sharp.*

Proof. A construction of Fano CDPs defined over P and consisting of $n = 4d$ functions is suggested by the proof of Theorem 4.7. To reach the upper bound, the values $\psi_i'(-f_j) + \psi_i'(f_j)$ must achieve their upper bound (that is, 2) a maximum number of times, that is, $d-1$ times, for each $i = 1 \ldots n$. One can check that, for fixed $1 \leq j \leq d$, defining

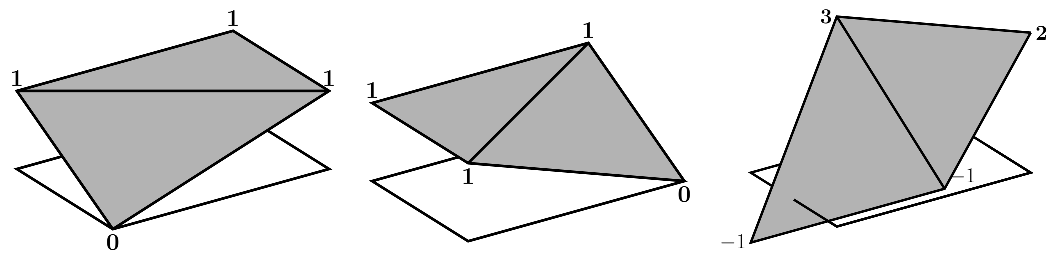
$$\psi_i'(x) = \min\{1, x_j + 1\} \quad \text{for} \quad 4(j-1) < i \leq 4j \text{ and } i \neq 4d$$

and defining

$$\psi_n'(x) = \psi_{4d}'(x) = \min\left\{1 - x_d, 1 - \sum_{k=1}^{d} 2x_k\right\},$$

gives rise to the unique Fano CDP satisfying the requirements. Note that ψ_{4d}' is the same function as ψ_i' for $4(d-1) < i < 4d$ except sheared in a sufficient way so that the functions satisfy the bound given in Inequality (2.3). □

Example 4.13. In Figure 4.4 we depict the 3-dimensional Fano CDP given by this construction. Its base polytope is the convex hull of $\{(\pm 1, 0), (0, \pm 1)\}$. The first four functions $\psi_1', \ldots, \psi_4'$ have vertices at the points $(-1,0,1)$, $(0,-1,1)$, $(1,0,0)$, and $(0,1,1)$. The next three functions $\psi_5', \ldots, \psi_7'$ have vertices at the points $(-1,0,1)$, $(0,-1,1)$, $(1,0,1)$, and $(0,1,0)$. The final function ψ_8' is a copy of ψ_5' except sheared by a factor of 2 in both the x and y direction.



(a) $\psi'_1, \ldots, \psi'_4$ (b) $\psi'_5, \ldots, \psi'_7$ (c) ψ'_8

Figure 4.4: 3-dimensional Fano CDP achieving upper bound given in Corollary 4.12

Next we modify the arguments of Theorem 4.7 slightly. Namely, we bound the integral and non-integral functions separately. Adding this step yields a different bound which also depends on some constant c determined by the base polytope, but without the condition that $c \leq 1$. Unsurprisingly, the bound given also depends on the number of non-integral functions, denoted by m. Recall from Corollary 4.5 that $m \in \{0, 1, 2, 3\}$.

Corollary 4.14. *Let Ψ be a Fano CDP with base polytope $P \subset \mathbb{R}^d$ and n functions, m of which are non-integral. Then*

$$n \leq \Big(m + \frac{4-m}{c}\Big)d$$

for some constant $c > 0$ depending only on P.

Proof. The proof is very similar to that of Theorem 4.7, except we define $f_j = \min\{|f_{j,1}|, |f_{j,2}|\}$ instead of $f_j = \min\{|f_{j,1}|, |f_{j,2}|, 1\}$. Moreover we bound the sum of the function values $\psi'_i(-f_j) + \psi'_i(f_j)$ separately in the cases where ψ'_i is integral and non-integral (these bounds are those given in Lemma 4.6).

□

If $m = 0$, then the bound is $n \leq \frac{4}{c}d$, where $c > 0$, and we can conclude:

Corollary 4.15. *If $P \subset \mathbb{R}^d$ contains the unit ball of radius $r > 4d$, there is no Fano CDP with all integral functions and with P as its base.*

Proof. Let Ψ and P be as above. Then constant c given in Corollary 4.14 satisfies $c > 4d$. Thus we have

$$n \leq \frac{4}{c}d < \Big(\frac{1}{4d}\Big)4d = 1,$$

a contradiction, and so there is no such Ψ.

□

Chapter 5

Finiteness over a Fixed Base

In this chapter we consider how to enumerate Fano CDPs consisting of a fixed base and a fixed number of functions. At some steps of the main proof we appeal to results in areas of algebra for which the background material is not covered in this document. The interested reader can refer to [1] and [6] for background knowledge.

Theorem 5.1. *Let P be a fixed lattice polytope and n a fixed positive integer. Then there are only finitely many Fano CDPs with base polytope P and n functions.*

Proof. The proof given below is similar to the structure of our enumeration strategy in 2-dimensions: By shearing, we can determine certain bounds on the first $n-1$ functions $\psi'_1, \ldots, \psi'_{n-1}$ at each point $u \in P$. These bounds and Inequality (2.3) allow us to bound $\psi'_n(u)$ for each $u \in P$. In turn, we may completely bound the functions $\psi'_1, \ldots, \psi'_{n-1}$ at each point $u \in P$.

Step 1. *Fix regions of linearity.*

Associate to each ψ'_i a set of regions of linearity of the function, which we refer to as R_i. Recall that the regions of linearity R_i correspond to a subdivision of P into lattice polytopes. Since P contains only finitely many lattice points, there are only finitely many ways to define the regions of linearity associated to ψ'_i.

Step 2. *Give upper bounds for $\psi'_1, \ldots, \psi'_{n-1}$.*

Consider the (not necessarily lattice) polytopes given by intersections of the form $R'_1 \cap \cdots \cap R'_{n-1}$, where R'_i is a region of linearity from R_i. Let R be one of the d-dimensional polytopes containing the origin obtained through this process. Note that the restriction of each ψ'_i to R is a linear function for $i = 1 \ldots n-1$. Let C be the cone generated by R. By Theorem 11.1.9 of [1], it contains a unimodular basis $v_1, \ldots, v_d$ (that is, a basis for the lattice). Any element of C is a scalar multiple of a point in R, and hence R contains points $p_1, \ldots, p_d$ where $p_i = \alpha_i v_i$ for some $\alpha_i \in \mathbb{R}$. As the intersection of convex sets is convex [7, p. 3], we can assume that $|p_i| \leq 1$.

The unimodular basis $v_1, \dots, v_d$ corresponds to a basis $v_1^*, \dots, v_d^*$ in the dual space so that $v_i^*(v_j) = \delta_{ij}$ [6, p. 143]. Thus, if we shear ψ_i' over a point p_j, then $\psi_i'(p_k)$ stays fixed for $k \neq j$. Hence we may assume that $0 \leq \psi_i'(p_j) < 1$ for all $i = 1 \dots n-1$ and $j = 1 \dots d$.

As the restriction of ψ_i' to R is a linear function, it is determined by its values at the points in the set $\{0, p_1, \dots, p_d\}$. Let L_i be the extension of this function to all of P. By the concavity of ψ_i', the maximum of L_i bounds ψ_i' by above. Let S be the set of linear functions φ defined on P such that for each point $p \in \{0, p_1, \dots, p_d\}$ either $\varphi(p) = 0$ or $\varphi(p) = 1$. Then each of these functions φ attains a maximum, say M_φ on P. Moreover, since S is finite, $M_i := \max\{M_\varphi : \varphi \in S\}$ exists and provides an upper bound for L_i. Hence M_i is also an upper bound for ψ_i'.

Step 3. *Give lower bound for ψ_n'.*

The upper bounds for $\psi_1', \dots, \psi_{n-1}'$ give a lower bound for ψ_n', through use of Inequality (2.3), that is, $n - 2 \leq \sum \psi_i'$.

Step 4. *Give upper bound for $\psi_n'(u)$ at each $u \in P$.*

Let $u \in P$. Let $\alpha > 0$ be sufficiently small so that $v = -\alpha u$ and the origin are in the same region of planarity of ψ_n'. By the concavity of ψ_n', the line through the points $(0, \psi_n'(0))$ and $(v, \psi_n'(v))$ provides an upper bound for $\psi_n'(u)$. Moreover, this upper bound in maximized by increasing the value for $\psi_n'(0)$ and decreasing the value for $\psi_n'(v)$. Thus the upper bound $\psi_n'(0) \leq 1$ and the lower bound for $\psi_n'(v)$ given in Step 3 provides an upper bound for $\psi_n'(u)$.

Step 5. *Give lower bounds for $\psi_i'(u)$ for each $u \in P$ and $i = 1 \dots n-1$.*

The upper bounds for $\psi_1'(u), \dots, \psi_n'(u)$ yield lower bounds for $\psi_1'(u), \dots, \psi_{n-1}'(u)$, again by Inequality (2.3).

Step 6. *Conclude finiteness by counting possible vertices of graphs.*

As the vertices of the graph of each ψ_i' are integral by definition, and the function values here are bounded above and below, there are only finitely many ways to choose the vertices of $\Gamma(\psi_i')$. Since $\Gamma(\psi_i')$ is determined by its vertices, there are only finitely many ways to define the ψ_i'.

□

Hence fixed P and n yield only finitely many possible Fano CDPs. By Theorem 4.7, we need only consider finitely many possible values of n. Thus we have the following result:

Theorem 5.2. *For any fixed lattice polytope in any dimension, there are only finitely many Fano CDPs consisting of this lattice polytope as its base, up to equivalence.*

Chapter 6

Concluding Remarks

While we have achieved key results that are integral toward proving Conjecture 1.1, further work is still needed. According to Theorem 5.2, we know that we have finiteness of Fano CDPs with more than two functions for a fixed base. Hence, Conjecture 1.1 could be established by showing that the set of isomorphism classes of base polytopes yielding such Fano CDPs is in fact finite.

Furthermore, the proof of Theorem 5.1 suggests a strategy to enumerate Fano CDPs. The strategy in its current state may be sufficient in the enumeration of Fano CDPs over some simple bases, but in general, may require more restrictions and greater precision. Future application of this work could focus on improving and implementing this enumeration procedure. Moreover, improving the enumeration procedure could result in the bounding of permissible base polytopes, just as our enumeration strategies in 2-dimensions were the basis for our proofs of finiteness and restriction of permissible base polytopes.

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Appendix A

Classification: 2-Dimensions

The following tables, through pictorial representation, give a classification of the equivalence classes of 2-dimensional Fano CDPs with more than two functions. Note that in each of the following figures, the left endpoint of the base polytope is -1.

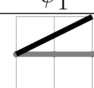
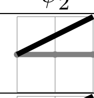
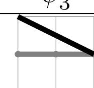
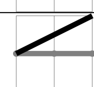
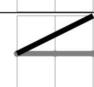
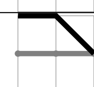
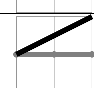
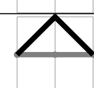
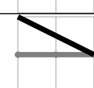
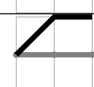
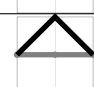
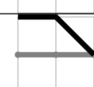
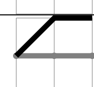
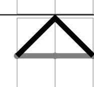
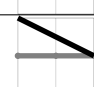
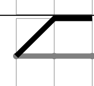
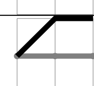
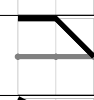
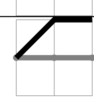
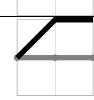
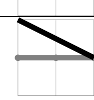
Number	ψ'_1	ψ'_2	ψ'_3
1			
2			
3			
4			
5			
6			
7			

Table A.1: Base polytope length $m = 2$, number of functions $n = 3$

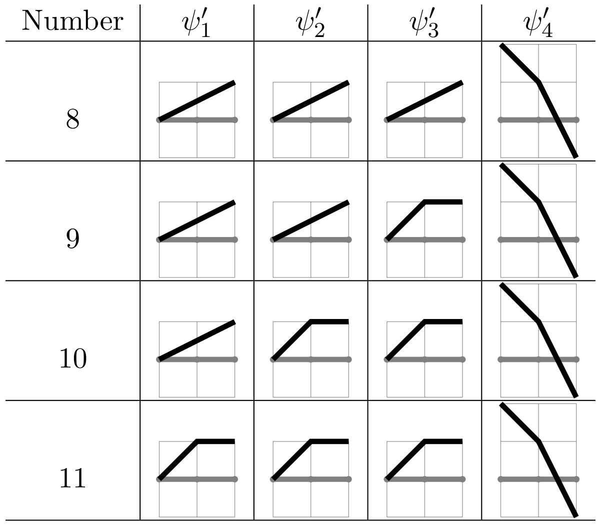


Table A.2: Base polytope length $m = 2$, number of functions $n = 4$

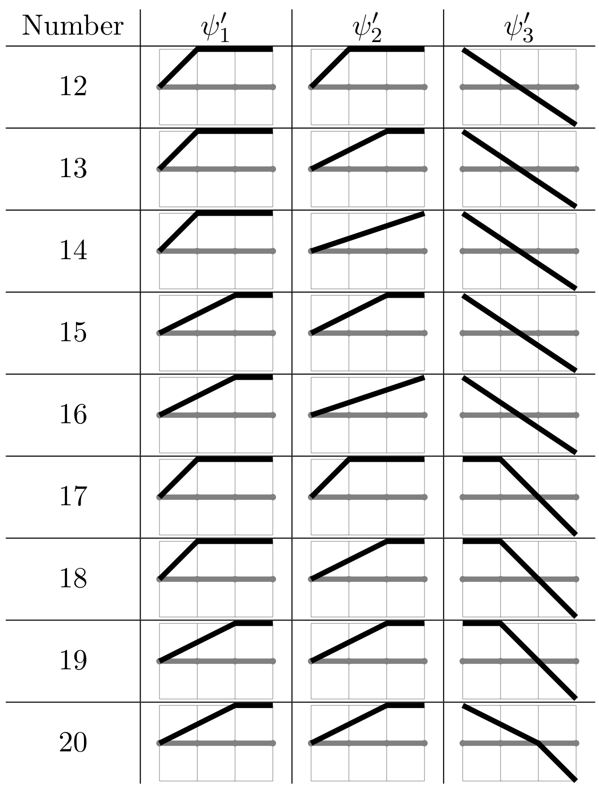


Table A.3: Base polytope length $m = 3$, number of functions $n = 3$

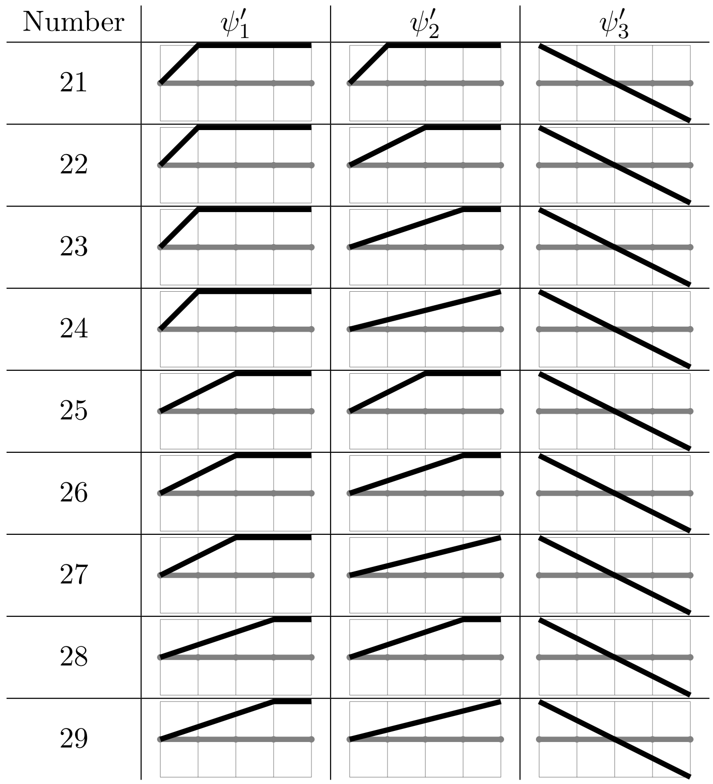


Table A.4: Base polytope length $m = 4$, number of functions $n = 3$

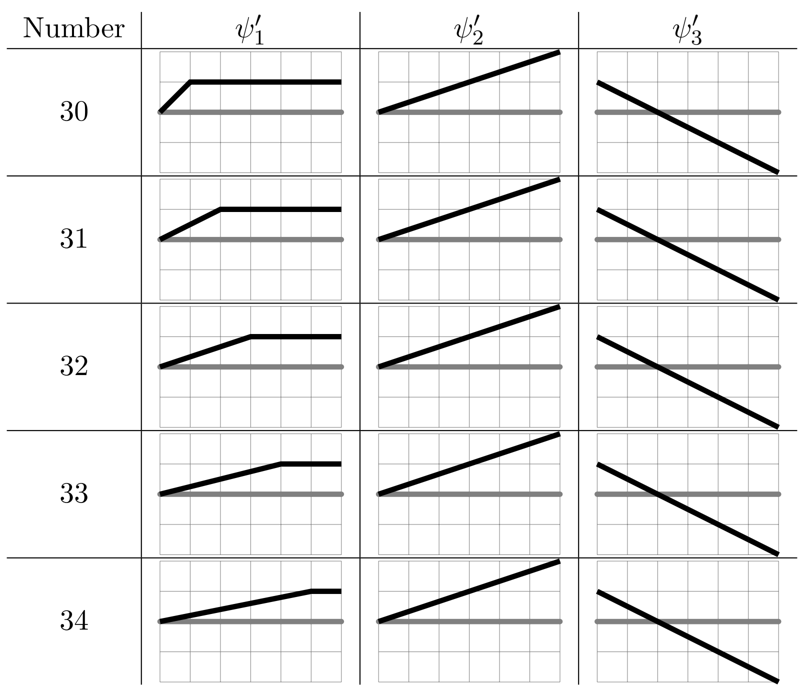


Table A.5: Base polytope length $m = 6$, number of functions $n = 3$