D-finite By Design

Séminaire Combinatoire Philippe Flajolet

Marni Mishna

SFU Department of Mathematics
Simon Fraser University

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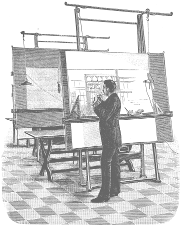

Happy *1 000 000-th* BIRTHDAY to *Philippe*

"Almost anything is non-holonomic unless it is holonomic by design."

-Flajolet Gerhold Salvy 2004/06

On the non-holonomic character of logarithms, powers, and the n^{th} prime function

“Conjecture 4.
Every D-finite globally bounded function is the diagonal of a rational function.”

-Christol, 1990

Globally bounded solutions to differential equations

Collections of evidence

Example

$$\sum_n \binom{2n}{n}^2 x^n = \Delta\left(\frac{1}{1-y-z-xy-xz}\right)$$

Is this always possible? Is there an effective method? What would it mean, combinatorially?

Collections of evidence

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[Bostan, Boukraa, Christol, Maillard 12]

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[Bostan, Boukraa, Christol, Maillard 12]

Today: Evidence from lattice path enumeration; Diagonals arising other other contexts; Some non D-finite classes.

D-finite functions

- A (univariate) series is D-finite if it satisfies a linear differential equation with polynomial coefficients

$$p_0(t)f(t) + p_1(t)f'(t) + p_2(t)f''(t) + \cdots + p_k(t)f^{(k)}(t) = 0$$

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sum, product, derivative, algebraic substitution,
Hadamard product, ...

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- Multivariable extensions

To prove something is D-finite

- Find an explicit differential equation or recurrence
  - e.g. by guess and verify
- Use closure properties and known functions
  - e.g. Show that it is the diagonal of an algebraic function

To prove that something is not D-finite

- Find nested exponentials: e.g. $\exp(\exp(z) - 1)$
- Demonstrate an infinite number of singularities
- Improper asymptotic behaviour
  - e.g. $a_n \sim \log\log(n)$ [Flajolet, Gerhold, Salvy 04/06]
  - e.g. $a_n \sim \gamma\rho^n n^\alpha$ with ρ transcendental or α irrational if $a_n \in \mathbb{Z}$

Globally bounded condition

Definition. Let $f(z) = \sum f_n z^n \in \mathbb{Q}[[z]]$. Then, $f(z)$ is globally bounded if it has a positive radius of convergence and $\exists r, s \in \mathbb{Q}$ such that $rf(sz)$ has integer coefficients.

- Not globally bounded: $\sum n! z^n$, $\sum z^n / n!$
- Globally bounded: $\sum_n \binom{2n}{n}^2 z^n$

Diagonals

$$\Delta\left(\sum_{a_1,\ldots,a_k} F_{a_1 a_2 \ldots a_k} x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}\right) := \sum_n F_{nn\ldots n} x^n$$

- Every *algebraic* power series f in $K[[x_1, \ldots, x_n]]$ arises as the diagonal of a rational power series in $2n$ variables.

  [Furstenberg 67; Denef, Lipshitz 87]

- The diagonal of a rational function is D-finite. [Christol 90]
- The Hadamard product of two diagonals of rational functions is a diagonal of rational functions.

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  [Furstenberg 67; Denef, Lipshitz 87]
- The diagonal of a D-finite function is D-finite. [Lipshitz 90]
- The Hadamard product of two diagonals of rational functions is a diagonal of rational functions.

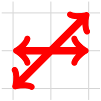
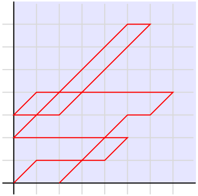
Lattice paths

A laboratory for discovery

Laboratory: Lattice path models

Lattice path models are an excellent case study for functional equations with multiple catalytic variables

1. Fix a lattice, and a set of directions $\mathcal{S}$.
2. How many walks of length n stay in the given region, taking steps from $\mathcal{S}$? (exactly/asymptotically)
3. Is the generating function D-finite?
4. To what extent can the decision be automated?

$\mathcal{S} =$  A walk: 

$$S_n \sim \frac{4\sqrt{3}}{3\Gamma(1/3)} 4^n n^{-2/3}$$

Algebraic OGF [Bostan Kauers 11]

1/4 plane, small steps

- 79 non-isomorphic, non-trivial cases.
- 23 D-finite: All but one succumb to *Orbit sum method*; the leftover is algebraic
- 5 proven non-D-finite (singular models)
- Conjecture: Remaining 51 are not D-finite
- There is a fast way to “distinguish” between the D-finite and non D-finite

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All models are globally bounded and the 23 D-finite models are all are diagonals of rational functions.

D-finite Models: The orbit sums

$W(t) = \sum w(n) t^n \quad w(n) = \text{number of walks of length } n$

- Additional *"catalytic" variables* track other properties.

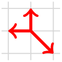
$$Q(x,y) \equiv Q(x,y;t) = \sum \underbrace{q(i,j,n)}_{\#\text{walks of length } n \text{ ending at } (i,j)} x^i y^j t^n$$

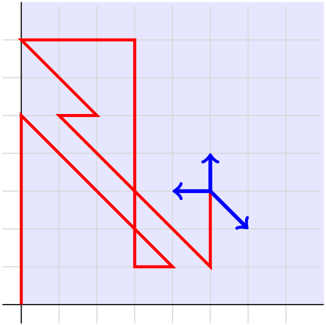
- We want *evaluations*: $\underbrace{Q(1,1)}_{\text{all walks}} \quad \underbrace{Q(1,0)}_{\text{return to axis}} \quad \underbrace{Q(0,0)}_{\text{excursions}}$
- Example: {NE, E, SW, W}

$$W(t) = 1 + 2t + 7t^2 + \dots$$
$$Q(x,y) = 1 + (xy + x)t + (2 + y + x^2y^2 + 2x^2y + x^3)t^2 + \dots$$

How do we get the functional equation?

A walk of length n is a walk of length $n-1$, plus a step.

$\mathcal{S} =$ 



$$Q(x,y) = 1 + t(y + x/y + 1/x)\,Q(x,y) - t/x\,Q(0,y) - tx/y\,Q(x,0)$$

We view this as an equation in 3 unknowns.

What do we *do* with the functional equation?

$$\underbrace{\big(xy - t(xy^2 + x^2 + y)\big)}_{\text{the kernel } K} Q(x,y) = xy - ty\,Q(0,y) - tx^2 Q(x,0)$$

- We define a group G_K of transformations that keep the kernel invariant [Bousquet-Mélou, M. 10]
- e.g. $g : (x,y) \mapsto (y/x, y) \quad g \in G_K$:

$$\begin{aligned} K(y/x, y) &= 1 - t(y + (y/x)/y + 1/(y/x)) = 1 - t(y + 1/x + x/y) \\ &= K(x,y) \end{aligned}$$

- G_K is generated by two involutions which are explicit because K is quadratic.

$$G_K = \left\{ \left(\frac{y}{x}, y\right), \left(x, \frac{x}{y}\right), (x,y), \left(\frac{y}{x}, \frac{1}{x}\right), \left(\frac{1}{y}, \frac{1}{x}\right), \left(\frac{1}{y}, \frac{x}{y}\right) \right\}$$

- *This differs from other strategies where you seek to annihilate the kernel.*

1. Orbit Sums

Apply transformations $g : (x, y) \mapsto (\alpha, \beta)$ from G_K to generate

6 new equations. (12 unknowns!)

(α,β)	$\alpha\beta\, K\, Q(\alpha,\beta) =$	$\alpha\beta$	$-t\alpha^2 Q(\alpha,0)$	$-t\beta Q(0,\beta)$
(x,y)	$xy\, K\, Q(x,y) =$	xy	$-tx^2 Q(x,0)$	$-tyQ(0,y)$
$(\frac{y}{x},y)$	$\frac{y^2}{x}\, K\, Q(\frac{y}{x},y) =$	$\frac{y^2}{x}$	$-t(\frac{y}{x})^2 Q(\frac{y}{x},0)$	$-tyQ(0,y)$
$(\frac{y}{x},\frac{1}{x})$	$\frac{y}{x^2}\, K\, Q(\frac{y}{x},\frac{1}{x}) =$	$\frac{y}{x^2}$	$-t(\frac{y}{x})^2 Q(\frac{y}{x},0)$	$-t\frac{1}{x} Q(0,\frac{1}{x})$
$(\frac{1}{y},\frac{1}{x})$	$\frac{1}{xy}\, K\, Q(\frac{1}{y},\frac{1}{x}) =$	$\frac{1}{xy}$	$-t(\frac{1}{x}) Q(\frac{1}{x},0)$	$-t\frac{1}{x} Q(0,\frac{1}{x})$
$(\frac{1}{y},\frac{x}{y})$	$\frac{x}{y^2}\, K\, Q(\frac{1}{y},\frac{x}{y}) =$	$\frac{x}{y^2}$	$\ldots$	
$(x,\frac{x}{y})$	$\frac{x^2}{y}\, K\, Q(x,\frac{x}{y}) =$	$\frac{x^2}{y}$	$\ldots$	

2. Cancel extra variables by taking an alternating sum

$$
\begin{array}{c|lllll}
+ & xy\,K\,Q(x,y) = & xy & -\,tx^2Q(x,0) & -\,tyQ(0,y) \\
- & \frac{y^2}{x}\,K\,Q(\frac{y}{x},y) = & \frac{y^2}{x} & -\,t(\frac{y}{x})^2Q(\frac{y}{x},0) & -\,tyQ(0,y) \\
+ & \frac{y}{x^2}\,K\,Q(\frac{y}{x},\frac{1}{x}) = & \frac{y}{x^2} & -\,t(\frac{y}{x})^2Q(\frac{y}{x},0) & -\,t\frac{1}{x}Q(0,\frac{1}{x}) \\
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\hline
\end{array}
$$

2. Cancel extra variables by taking an alternating sum

$+$	$xy\,K\,Q(x,y) = xy$
$-$	$\frac{y^2}{x}\,K\,Q(\frac{y}{x},y) = \frac{y^2}{x}$
$+$	$\frac{y}{x^2}\,K\,Q(\frac{y}{x},\frac{1}{x}) = \frac{y}{x^2}$
$-$	$\frac{1}{xy}\,K\,Q(\frac{1}{y},\frac{1}{x}) = \frac{1}{xy}$
$+$	$\frac{x}{y^2}\,K\,Q(\frac{1}{y},\frac{x}{y}) = \frac{x}{y^2}$
$-$	$\frac{x^2}{y}\,K\,Q(x,\frac{x}{y}) = \frac{x^2}{y}$

Poof!

$$\sum_{g\in G_K} (-1)^{\text{ord}}\tilde{g}(xy)Q(g(x,y)) = \underbrace{\frac{\sum_{g\in G_K}(-1)^{\text{ord}}\tilde{g}(xy)}{K}}_{\text{When } G_K \text{ is finite this is rational}}$$

3. Extract subseries of positive powers of x and y to determine $Q(x, y)$

$$\sum_{g \in G_K} (-1)^{\mathrm{ord}} \tilde{g}(xy) Q(g(x, y)) = \frac{\sum_{g \in G_K} (-1)^{\mathrm{ord}} \tilde{g}(xy)}{K}$$

$$LHS = xyQ(x, y) - \frac{y^2}{x} Q\left(\frac{y}{x}, y\right) + \frac{y}{x^2} Q\left(\frac{y}{x}, \frac{1}{x}\right) - \frac{1}{xy} Q\left(\frac{1}{y}, \frac{1}{x}\right) + \frac{x}{y^2} Q\left(\frac{1}{y}, \frac{x}{y}\right) - \frac{x^2}{y} Q\left(x, \frac{x}{y}\right)$$

$$= \sum_{i,j} A_{i,j} x^i y^j + B_{ij} \frac{1}{x}^i y^j + C_{ij} \frac{1}{x}^i \frac{1}{y}^j + D_{ij} x^i \frac{1}{y}^j$$

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$$\implies xyQ(x, y) = [x^{>0} y^{>0}] RHS$$

This is a Hadamard product of two rational functions, hence a diagonal of a rational function.

Summary of steps

1. Write the functional equation for $Q(x, y)$ in terms of combinatorially relevant evaluations
2. Compute the kernel K, and group of transformation fixing the kernel G_K
3. Find the orbit sum: apply the elements of G_K to the equation and take an alternating sum to eliminate the introduced series
4. The RHS will be a rational function $R(x, y)$ *hopefully not equal to 0...*
5. $Q(x, y)$ is obtained from $R(x, y)$ by positive series extraction *This implies D-finiteness.*
6. Evaluate $W(t) = Q(1, 1)$

Degeneracy

Theorem

This method works as described in every finite group case except when $\sum_{g \in G_K} (-1)^{\text{ord}} \tilde{g}(xy) = 0$.

And if it does..?

- In a cases where $|G_K| = 6$, taking sum of half of the group leads to answer with some "aggressive" coefficient extraction.
- One case remains: Gessel walks {N, NE, S, SW}. Two other approaches are able to handle this, and it is algebraic, hence also a diagonal of a rational function.

The case of the infinite group

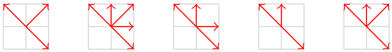
$(x, y) \to (\tau(x, y), \pi(x, y)) \to \dots$

- To prove the group is infinite
  - Test 1: Iterate the map on an actual point e.g. (1, 4). If the group is finite you will always return to this point.
  - Test 2: Apply this to (t, t^2); if iterations increase exponent then infinite
  - Test 3: Prove the transformation is equivalent to a matrix map with eigenvalues not equal to the roots of unity.

Orbit sum method does not apply. They all seem to be non-Dfinite and divide into two types: singular, non-singular.

The 5 Singular models are not D-finite

[M., Rechnitzer 09]; [Melczer, M. 12]



- Kernel has solution $K(x, Y(x)) = 0$ with $Y(x) = xt + O(t^2)$ so symmetric models satisfy

$$S(t) = \frac{1}{1-|\mathcal{S}|t}\left(1 - 2\sum_{n=0}^{\infty}(-1)^n\, Y^{\circ n}(1)\, Y^{\circ n+1}(1)\right).$$

Each $Y^{\circ n}$ contributes $O(n)$ simple poles.
Tough part: Showing that they actually exist.

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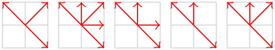

$$S(t) = \frac{1}{1 - |\mathcal{S}|t}\left(1 - 2\sum_{n=0}^{\infty}(-1)^n Y^{\circ n}(1)\, Y^{\circ n+1}(1)\right)$$

- We show that poles from $Y^{\circ n}$ are distinct from $Y^{\circ k}$, $k \neq n$ The right parameterization makes this step trivial. (Repeated composition becomes powers)
- Show that the rest of sum converges at those points.
- Under $t \mapsto q/(1+q^2)$, poles coverge to unit circle.

The 51 non-singular models are *probably* not D-finite

Supporting evidence

- Non singular excursions are not D-finite
  [Bostan Raschel Salvy 12] (asymptotic form)
- $Q(x, y; t)$ is not D-finite as function of 3 variables
  [Kurkova Raschel 11] (infinite set of singularities)

Conjecture: Infinite group $\Longrightarrow$ Non-D-finite.

Recent analyses *may* have found 3D models with finite group but non-D-finite generating function.

Some D-finite classes arising as diagonals

- Some pattern avoiding permutations [Bousquet-Mélou 02]
- 3-non crossing set partitions [Bousquet-Mélou, Xin 05]
- Non crossing set partitions with r coloured arcs [Marberg 12]

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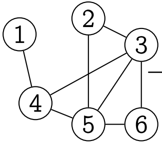
(lattice paths in disguise)

More Diagonals

k-regular objects

k-regular simple labelled graphs

Monomial Encoding



$\rightarrow x_1 x_2^2 x_3^4 x_4^3 x_5^4 x_6^2$

All simple graphs

$G(\mathbf{x}) = \prod_{i \leq j} (1 + x_i x_j)$

k-regular simple labelled graphs

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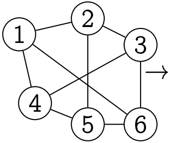

$\rightarrow x_1^3 x_2^3 x_3^3 x_4^3 x_5^3 x_6^3$

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k-regular simple labelled graphs

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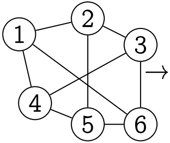

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$G^{(k)}(t)$ is D-finite for every k because it is a scalar product of two “D-finite symmetric functions”. [Gessel 90]

$$G^{(k)}(t) = \langle \underbrace{\exp(h_k t)}_{\text{extractor}}, \underbrace{\exp\left(\sum \frac{1}{2n}(p_n^2 + p_{2n})\right)}_{G(x)=\sum_n e_n[e_2]} \rangle$$

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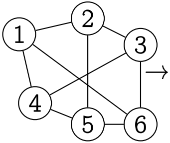

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$G_n^{(k)} \sim \frac{(nk)!}{(nk/2)!} \left(\frac{2^{k/2}}{k!}\right)^n \exp(-\frac{k^2-1}{4} + O(1/n))$ [McKay, Wormald 91]

Not globally bounded.

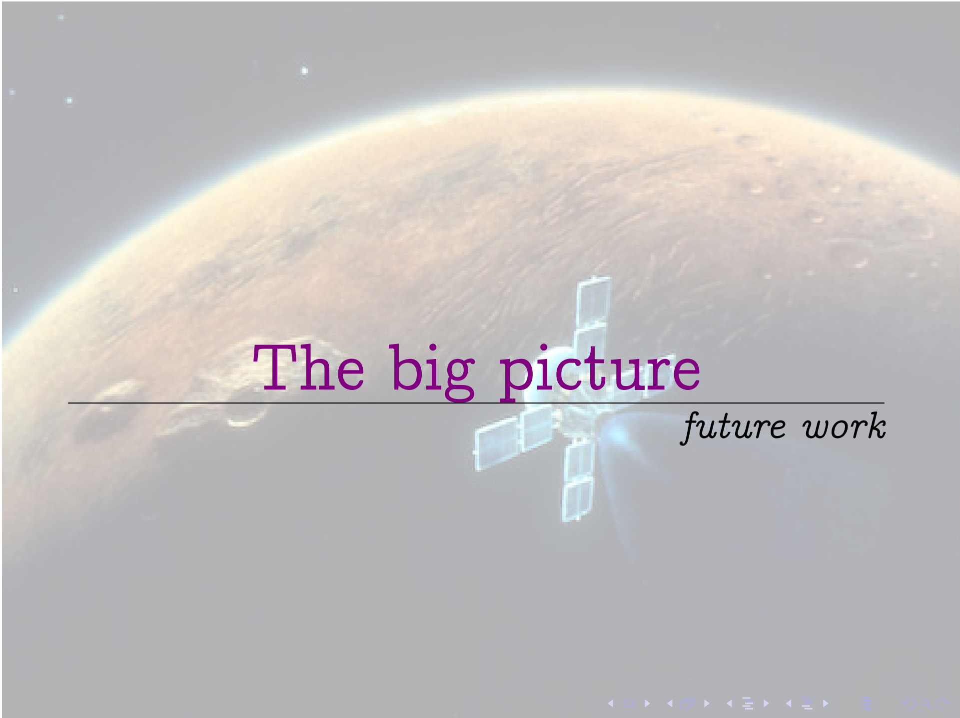
Many regular classes are D-finite

Idea: Encode all objects with symmetric series and use the scalar product of symmetric functions to extract GF of a regular subclass. [Gessel 90; Jackson, Goulden, Reilly 82; Chyzak, M., Salvy 05]
The encodings can be explained using Théorie des Espèces [M. 07],

$$\text{EGF of regular objects} = \langle \exp(h_k t), F \rangle$$

Class	Encoded GF	$=F$
simple graphs	$\prod_{i<j} 1 + x_i x_j$	$E[e_2]$
with loops	$\prod_{i\leq j} 1 + x_i x_j$	$E[h_2]$
multiple edges no loops	$\prod_{i<j} \frac{1}{1-x_i x_j}$	$H[e_2]$
Hypergraphs		$E[e_3]$
Young tableaux multiple entries	$\sum s_\lambda$	$E[e_1 + e_2]$

$E = \sum e_n$ (elementary) $H = \sum h_n$ (complete)



The big picture

future work

Upcoming in lattice path enumeration

- Asymptotic formulas (excursions, singularities)+ combinatorial interpretations of these formulas
- 3D / Repeated steps/ Larger steps (Issue: find the good group)
- Connecting the various approaches
- Simplifying proofs of non-D-finiteness

A possible framework for multiple catalytic variables

$$Q(t) = 1 + tQ(t)^2$$

- Algebraic equation, 1 variable: "easy algebraic"

A possible framework for multiple catalytic variables

$$G(u,t) = 1 + tu^2 G(u,t)^2 + tu\frac{uG(u,t) - G(1,t)}{u-1}$$

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$$k(x,y,t)\,Q(x,y,t) = 1 - t\,q(x,t)\,Q(x,0,t) - t\,r(x,y)\,Q(0,y,t)$$

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- More variables ...

Combinatorial interpretations of D-finite functions

- Translate differential equation [various]
- Using théorie des espèces [M. 07, Labelle, Lamathe 09/10]
- Purely as lattice paths [Kotek, Makowsky 12]
- Matrix approach [Reutenuauer 12]

If Conjecture 4. is true is there a way to formalize it into an interpretation of D-finite functions?

Wild speculation

1. If a lattice model ogf is D-finite, can it be expressed as a diagonal of a rational? (always globally bounded) Gessel walk as diagonal?
2. To what extent does [Bostan, Boukraa, Christol, Maillard 12] apply to lattice paths? (Is it all already known?)
3. Is there are lattice model interpretation for all D-finite series with positive integer coefficients?

Merci Beaucoup!