Topological Graph Theory*
Lecture 4: Circle packing representations

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Summary: A circle packing of a plane graph G is a set of circles $\{C_v \mid v \in V(G)\}$ in $\mathbb{R}^2$ such that for $u \neq v$ C_v and C_u have disjoint interiors, C_v and C_u intersect if an only if $uv \in E(G)$ and such that by putting vertices $v \in V(G)$ in the centers of C_v and joining adjacent vertices u, v with a strait line segment we get a plane representation of G, which is equivalent to G. We show that every 3-connected plane graph has a circle packing representation and show some corollaries.

1 Definitions

In this lecture we assume that all graphs are 2-connected.

Definition 1.1. Let G be a plane graph. A circle packing of G (CP of G) is a set of circles $\{C_v \mid v \in V(G)\}$ such that

- The interiors of C_v are pairwise disjoint.
- C_u and C_v intersect if and only if u and v are adjacent.
- By putting vertices $v \in V(G)$ into the centers of corresponding C_v and embedding every edge uv by a strait line segment joining u and v we get a plane representation of G equivalent to G.

If we consider circle packings in the *extended plane*, the circle packing may contain special circle C_ω which corresponds to a vertex of the graph G we put in infinity.

If we consider $\mathbb{R}^2$ as $\mathbb{C}^*$ then we can define a *Möbius transformation* $w : \mathbb{C}^* \to \mathbb{C}^*$ as

$$w(z) = \frac{az + b}{cz + d}, \qquad ad - bc \neq 0.$$

Möbius transformation maps circles and lines into circles and lines.

Lemma 1.2. *If a graph G has a CP representation and $v \in V(G)$ then G has a CP representation such that the circle corresponding to v is centered at infinity.*

* Lecture Notes for a course given by Bojan Mohar at the Simon Fraser University, Winter 2006.

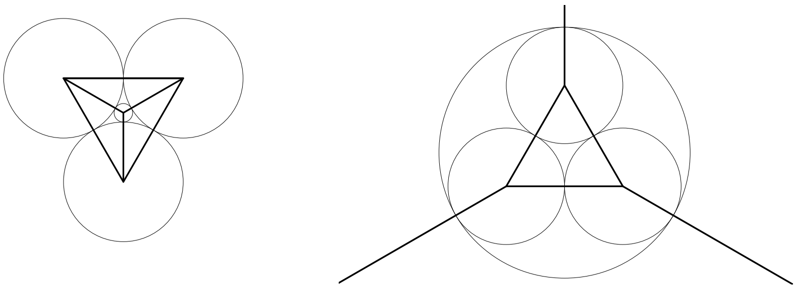

Figure 1: Circle packing and extended circle packing representation of K_4

Let G be a connected plane graph. Construct a new graph G^* by putting a vertex v_f in each face f of G and connecting v_{f_1} and v_{f_2} by an edge e^* if faces f_1 and f_2 share an edge e (e^* is the dual edge to e). The graph G^* is called the *geometric dual* of plane graph G.

Lemma 1.3. *Let G be a plane graph. Then either G or G^* has a vertex of degree at most 3.*

Proof. Assume that the $\deg(v) \geq 4$ for every vertex $v \in V(G)$ and $\deg(f) \geq 4$ for every face in $F(G)$. Counting argument gives that $2|V(G)| \leq |E(G)|$ and $2|F(G)| \leq |E(G)|$, which contradicts the Euler formula. □

A *primal-dual CP* (PDCP) is a pair of simultaneous CP representations of G and G^* such that for any dual edges $e = uv$ and $e^* = u^*v^*$ the circles C_u and C_v touch at the same point as the circles C_{u^*} and C_{v^*} and the lines representing e and e^* intersect perpendicularly.

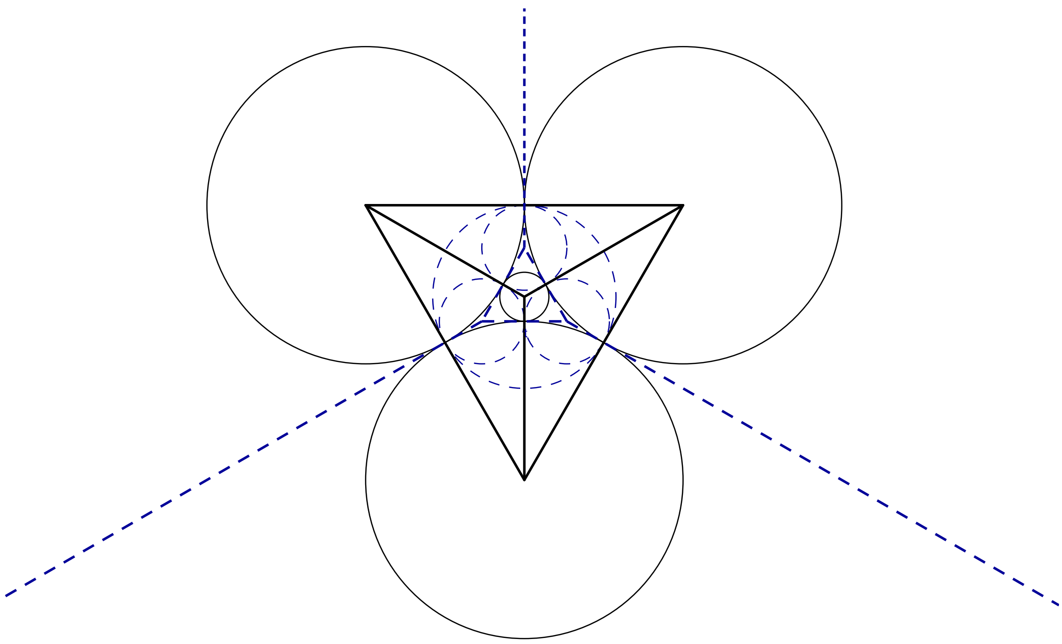

Figure 2: Primal dual circle packing representation of K_4

We will show that every 3-connected plane graph admits a PDCP.

2 Properties of circle packing representations

Let G be a plane graph. Define a graph Γ as the graph whose vertices correspond to vertices and faces of G and vertices ν and τ are connected if ν corresponds to a face and τ to a vertex incident with that face.

Lemma 2.1. *Let G be a 2-connected plane graph with at least 4 vertices and Γ its vertex-face graph. The following are equivalent:*

1. *G is 3-connected.*
2. *Every 4-cycle in Γ is facial*
3. *For ever prober subset $S \subset V(\Gamma)$ that contains at least 5 vertices of Γ we have*

$$2|S| - |E(\Gamma(S))| \geq 5. \tag{1}$$

Proof. If a 4-cycle C in Γ is not facial then the vertices on C corresponding to vertices of G separate G, so 1. implies 2. If G is not 3-connected then the separating vertices $\{x, y\}$ in G are on a non-facial 4-cycle in Γ, so 2 implies 1.

By Euler formula $2|S| - |E(\Gamma(S))| \geq 4$ (Γ is bipartite) and equality holds iff $\Gamma(S)$ is a qaudrangulation. If S is a proper subset with at least 5 vertices, then $\Gamma(S)$ is a quadrangulation, then one of the 4-cycles (boundary of the infinite face) is not facial, so 2. implies 3. If C is a non-facial 4-cycle in Γ then $V(C)$ with the vertices in the interior or exterior will give equality, so 3. implies 2. □

We assume that the vertex of Γ corresponding to the unbounded face of G is at the infinity and denote it by ω. We define $\Gamma' = \Gamma - \omega$.

Lemma 2.2. *Let r_ν, $\nu \in V(\Gamma)$ be the radii of a PDCP of G. If if $\nu \in V(\Gamma')$ and $\nu\omega \notin E(\Gamma)$ then*

$$\sum_{\substack{\tau \\ \nu\tau \in E(\Gamma)}} \arctan \frac{r_\tau}{r_\nu} = \pi. \tag{2}$$

Let $v_1, \dots, v_k$ be the vertices of Γ such that $v_i\omega \in E(\Gamma)$, $i = i, \dots, k$ and let $\alpha_i = \sum_\tau \arctan \frac{r_\tau}{r_{v_i}}$ where the sum is over all neighbors τ of v_i in Γ'. Then

$$0 < \alpha_i < \tau \quad (1 \leq i \leq k) \qquad \text{and} \qquad \sum_{i=1}^{k} \alpha_i = (k-2)\pi. \tag{3}$$

Proof. Let $\nu \in V(\Gamma)$. If $\nu \neq \omega$, then the sum in (2) is half of the sum of angles around ν, which implies the equality. If $\nu = \omega$ then α_i is the angle at v_i in the outer facial cycle of G, which implies (3). □

A *weak PDCP* of G is a simultaneous CP representation of G and $G^* - \omega$ such that for each edge $\nu\tau \in E(\Gamma')$ the circles C_ν and C_τ cross at the right angle. We will show that the existence of positive numbers r_ν satisfying (2) and (3) is sufficient for the existence of a weak PDCP.

3 Existence of circle packing representations

Lemma 3.1. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a covering map (continuous, onto, for each p and $f(p)$ there exist open neighborhoods U and W of p and $f(p)$ such that f restricted to U is a homeomorphism of U to W. Suppose that the set $S = \{q \in \mathbb{R}^2 \mid |f^{-1}(q)| > 1\}$ is bounded. Then f is a homeomorphism.*

Proof. We prove that both S and the complement of S are open. Since S is bounded, it is empty. □

Lemma 3.2. *Let G be a 2-connected plane graph with polygonal edges. Let H be a drawing of G in the plane (possibly with edge crossings) such that all edges of H are polygonal arcs. Suppose further that:*

1. *For each $x \in V(G)$ the edges incident with x in H are pairwise non crossing and leave x in the same clockwise order as in G.*
2. *Each facial cycle in G corresponds to a simple closed curve in H.*
3. *If C is a facial cycle bounding a bounded face in G and e is an edge of G leaving C, then the first segment of e is in the exterior of C in H.*

Then H is a plane representation of G.

Proof. Let $f : G \to H$ be an isomorphism. We extend f to a continuous map of the point set of G onto the point set the point set of H such that is is 1-1 on each edge of G. Let C_0 denote the outer cycle in G. For each facial cycle $C \neq C_0$ we can extend f using Jordan-Schönflies Theorem to $\text{int}(C)$ such that restriction of f onto $\overline{\text{int}}(C)$ is homeomorphism onto of $\text{int}(C)$ onto $f(\overline{\text{int}}(C))$.

For each $p \in \text{int}(C_0)$ the image $f(p)$ is in interior of $f(\overline{\text{int}}(C_0))$. This is clear if p is in $\text{int}(C)$ for some face C of G. Is p is on some edge of G, use condition 3. and if p is a vertex of G use 1. and 3. to get a neighborhood of $f(p)$ in $f(\overline{\text{int}}(C_0)$. So the boundary of $f(\overline{\text{int}}(C_0))$ is a subset of $f(C_0)$. This implies that $f(\overline{\text{int}}(C_0)) = \overline{\text{int}}(f(C_0))$. We can extend f onto $\overline{\text{ext}}(C_0)$ to get a continuous map $f : \mathbb{R}^2 \to \mathbb{R}^2$ which is by Lemma 3.1 homeomorphism. In particular, H is a plane representation of G. □

Lemma 3.3. *Let G be a 3-connected plane graph and Γ its vertex-face graph. If there are positive numbers r_ν, $\nu \in V(\Gamma')$, such that (2) and (3) are satisfied, then there exists a weak PDCP of G and G^* with radii r_ν and with the same local clockwise orientations as in G, G^*.*

Proof. Given radii r_ν all facial quadrangles in Γ are uniquely defined. First choose the position of arbitrary τ_0 and one of its neighbors τ_1 at distance $\sqrt{r_{\tau_0}^2 + r_{\tau_1}^2}$. Using the clockwise order of neighbors of τ_0 and the position of τ_1 all neighbors of τ have uniquely determined positions. Using a path P from τ_0 to $\tau \in V(\Gamma)$ we get positions for all other vertices τ. If we change the path P over a facial quadrangle in Γ, the position of τ does not change, so the position is independent of the choice of P. We have a drawing of G in the plane, which is by Lemma 3.2 a plane representation of G. □

Lemma 3.4. *Let G be a 3-connected plane graph with outer cycle $C = v_1 v_2 \cdots v_k$. Let $\alpha_1, \alpha_2 \ldots, \alpha_k$ be real numbers such that $0 < \alpha_i < \pi$ $(i = 1, \ldots, k)$ and $\alpha_i + \cdots + \alpha_k = (k-2)\pi$. Then there are positive numbers r_ν, $\nu \in V(\Gamma')$ such that (2) holds for $\nu \neq v_1, \ldots, v_k$ and for each $i = 1, \ldots, k$,*

$$2 \sum_{v_i \tau \in E(\Gamma')} \arctan \frac{r_\tau}{r_{v_i}} = \alpha_i, \tag{4}$$

where the summation is taken over all neighbors τ *of* v_i *in* Γ'. *The numbers* r_ν *are unique up to a multiplicative constant.*

Proof. Suppose we have a sequence of numbers $r = (r_\nu | \nu \in V(\Gamma'))$. For each $\nu \in V(\Gamma') \backslash \{v_1, \ldots, v_k\}$ we define

$$\vartheta_\nu(r) = \sum_{\nu\tau \in E(\Gamma)} \arctan \frac{r_\tau}{r_\nu} - \pi$$

and for $i = 1, \ldots, k$

$$\vartheta_{v_i}(r) = \sum_{v_i\tau \in E(\Gamma')} \arctan \frac{r_\tau}{r_{v_i}} - \frac{1}{2}\alpha_i.$$

Then the number

$$\mu(r) = \sum_{\nu \in V(\Gamma')} \vartheta_\nu(r)^2$$

is a measure for how far r is from a solution. To prove the theorem we find a sequence $r = (r_\nu)$ such that $\mu(r) = 0$.

Claim 3.5.

$$\sum_{\nu \in V(\Gamma')} \vartheta_\nu(r) = 0.$$

Proof. By simple computation

$$\begin{aligned} \sum_{v \in V(\Gamma')} \vartheta_\nu(r) &= \sum_{\nu\tau \in E(\Gamma')} \left(\arctan \frac{r_\tau}{r_\nu} + \arctan \frac{r_\nu}{r_\tau} \right) \\ &\quad -\pi(|V(\Gamma')| - k) - \frac{1}{2} \sum_{i=1}^{k} \alpha_i \end{aligned}$$

Now use $\arctan x + \arctan \frac{1}{x} = \frac{\pi}{2}$ and

$$2|V(\Gamma')| = |E(\Gamma)| + 2 = |E(\Gamma')| + k + 2.$$

□

Let S be a proper subset of $V(\Gamma')$. Denote by t the number of vertices among $v_1, \ldots, v_k$ that are contained in S.

Claim 3.6. *If* $|S| \geq 4$ *or if* $|S| \in \{2, 3\}$ *and* $t = 0$ *then*

$$2|S| - |E(\Gamma(S))| \geq t + 3.$$

If $|S| \in \{2, 3\}$ *and* $t > 0$ *then*

$$2|S| - |E(\Gamma(S))| \geq t + 2.$$

Proof. For $|S| \geq 4$ use Lemma 2.1 for the set $S' = S \cup \{\omega\}$. Other cases are checked directly. □

Let Q be the set of all sequences $r = (r_\nu \mid \nu \in V(\Gamma'))$ such that $0 < r_\nu \leq 1$, $r_\nu = 1$ if $\vartheta_\nu(r) > 0$ and $r_\nu = 1$ for at least one $\nu \in V(\Gamma')$. Q is nonempty since the sequence $r_\nu = 1$ is in Q. Let $m = \inf\{\mu(r) \mid r \in Q\}$.

Claim 3.7. *The infimum is attained: there is some sequence r such that $\mu(r) = m$.*

Proof. Let $r^{(i)}$ be a sequence such that $\mu(r^{(i)}) \to m$ and $i \to \infty$. We may assume that the numbers $r_\nu^{(i)}$ converge. Let S be the set of vertices for which $\lim_{i\to\infty} r_\nu^{(i)} \neq 0$. We need to show that $S = V(\Gamma')$.

Suppose S is a proper subset of $V(\Gamma')$. We show that

$$\sum_{\nu \in S} \vartheta_\nu(r^{(i)}) < 0$$

for large i. Let t be the number of vertices $v_i, \ldots, v_k$ in S. We compute

$$\begin{aligned}\sum_{\nu \in S} \vartheta_\nu(r^{(i)}) &= \frac{\pi}{2}|E(\Gamma(S))| - \pi(|S| - t) \\ &\quad -\frac{1}{2}\sum_{v_j \in S} \alpha_j + \sum_{\substack{\nu\tau \in E(\Gamma') \\ \nu \in S, \tau \notin S}} \arctan \frac{r_\tau^{(i)}}{r_\nu^{(i)}}\end{aligned}$$

Since the last sum tends to 0 as $i \to \infty$ we get that $\sum_{\nu \in S} \vartheta_\nu(r^{(i)})$ tends to

$$-\frac{\pi}{2}(2|S| - |E(\Gamma(S))| - t - 2) + \frac{1}{2}\sum_{v_j \in S}(\pi - \alpha_j) - \pi.$$

This is negative (the first term is negative by Claim 3.6 and the second term is negative since we can rewrite the condition $\alpha_i + \cdots + \alpha_k = (k-2)\pi$ as $\sum_{i=1}^k (\pi - \alpha_i) = 2\pi$) which implies that

$$\sum_{\nu \notin S} \vartheta_\nu(r^{(i)}) > 0$$

for large i. This is a contradiction to the definition of S, so $S = V(\Gamma')$. ☐

Let $r = \lim_{i\to\infty} r^{(i)}$. Since ϑ_ν are continuous functions, $r \in Q$.

Claim 3.8. *The minimum is zero: $m = 0$.*

Suppose $m > 0$. Let S' be the set of vertices ν with $\vartheta_\nu(r) < 0$. S' is a proper subset of $V(\Gamma')$. Define r' as $r'_\nu = r_\nu$ for $\nu \notin S$ and $r'_\nu = \alpha r_\nu$ for $0 < \alpha < 1$ such that $r' \in Q$ (choose α close to 1 so that the sign of $\vartheta_\nu(r')$ is the same as the sign of $\vartheta_\nu(r)$ for all $\nu \in V(\Gamma')$. For such α we get $\mu(r') < \mu(r)$, contradiction.

Claim 3.9. *The minimizing r is unique.*

Let r and r' be distinct and $\mu(r) = \mu(r') = 0$. Then $\vartheta(r_\nu) = \vartheta(r'_\nu) = 0$ for all $\nu \in V(\Gamma')$. We can assume that $S = \{\nu \mid r_\nu > r'_\nu\}$ is nonempty. S is a proper subset of $V(\Gamma')$. Then

$$0 = \sum_{\nu \in S} \vartheta_\nu(r) - \sum_{\nu \in S} \vartheta_\nu(r') = \sum_{\nu\tau} \left(\arctan \frac{r_\tau}{r_\nu} - \arctan \frac{r'_\tau}{r'_\nu} \right) < 0$$

where the last sum is over $\nu\tau \in E(\Gamma')$, $\nu \in S$ and $\tau \notin S$.

☐

Theorem 3.10 (Brightwell and Scheinerman)**.** *Let G be a 3-connected plane graph. Then G admits a PDCP representation. The PDCP of G is unique up to factional linear transformations are reflections in the plane.*

Proof. If the outer cycle of G is a 3-cycle let $\alpha_1 = \alpha_2 = \alpha_3 = \frac{\pi}{3}$. By Lemma 3.4 there exist a sequence $r = (r_\nu \mid \nu \in V(\Gamma'))$ which satisfies (2) and (3). By Lemma 3.3 there exists a weak PDCP with these radii. In particular $r_{v_1} = r_{v_2} = r_{v_3}$, which implies that the weak PDCP can be extended to a PDCP. By uniqueness of radii the resulting PDCP is unique once C_1, C_2 and C_3 are prescribed.

If the outer cycle of G has length greater than 3, then either G or G^* has a facial 3-cycle. Using Möbius transformation we redraw G so that this is an outer facial cycle and use previous paragraph.

For any PDCP representation of G there exists a Möbious transformation which takes this PDCP into a PDCP with prescribed C_1, C_2, C_3 which shows the uniqueness of PDCP. □

4 Corollaries

Corollary 4.1 (Koebe-Andreev-Thurston)**.** *Every plane graph admits a circle packing representation.*

Theorem 4.2 (Brightwell and Scheinerman)**.** *If G is a planar 3-connected graph, then G and its dual G^* can be embedded in the plane with strait lines and with the outer vertex of G^* at infinity such that they form a geometric dual pair. Both embeddings are convex and each pair of dual edges is perpendicular.*

Theorem 4.3. *If G is a 3-connected planar graph, then there is a convex polyhedron Q in $\mathbb{R}^3$ whose graph is isomorphic to G such that all edges of Q are tangent to the unit sphere in $\mathbb{R}^3$.*

Proof. Use inverse of the stereographic projection to map circles of the PDCP of G onto circles on the sphere. Let Π_ν be the plane which intersects the sphere in C_ν, $\nu \in G^*$. These spheres define polyhedron Q. □

Theorem 4.4 (Steinitz)**.** *A graph G is the graph of a convex polytope in $\mathbb{R}^3$ if and only if it is planar and 3-connected.*

Theorem 4.5 (Mani)**.** *If G is a 3-connected planar graph, then there is a convex polyhedron Q in $\mathbb{R}^3$ whose graph is isomorphic to G such that every automorphism of G induces a symmetry of Q.*