Topological Graph Theory*
Lecture 9: Planarizing Cycles

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Summary: These notes cover the tenth week of lectures. Topics included are cycle double cover, surface minors, planarizing cycles graph colourings and Four Colour Theorem.

1 Cycle Double Cover

Proposition 1.1. *Let G be a 2-connected graph. An embedding Π of G has face-width at least 2 if and only if all facial walks of G are cycles.*

- Question: When does there exist an embedding in which all faces are cycles?

If Π is an embedding of a 2-connected graph G with $fw(G, \Pi) \geq 2$, then all facial cycles of G have the property that every edge is contained in precisely two of them. Such a collection of cycles (not necessarily associated with an embedding) is called a *cycle double cover (CDC)* of G.

Conjecture 1.2 (Cycle Double Cover Conjecture)**.** *Every 2-connected graph has a CDC.*

Conjecture 1.3. *Every 2-connected graph has an embedding of face-width at least 2 in some (orientable) surface.*

Conjecture 1.2 is a strenthening of Conjecture 1.3. A lot of work has been done in order to find proof of either one, but that hasn't been accomplished just yet. However, there are some improvements, for example it's been shown that Conjecture 1.2 can be reduced to cubic graphs, i.e. if it works for cubic graphs, it works for all graphs.

2 Face-width and Surface Minors

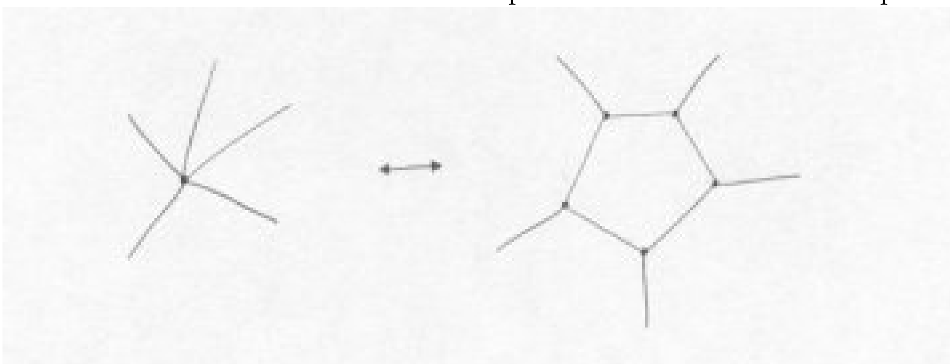
Definition 2.1. Let G be a Π-embedded graph. Then by successively deleting edges and contracting edges that are not loops we can obtain the induced embedding Π^* of some connected minor G^* of G. Then G^* is called a *surface minor* of G.

Theorem 2.2. *For every planar graph H there exists an integer k such that H is a (surface) minor of the $k \times k$ grid.*

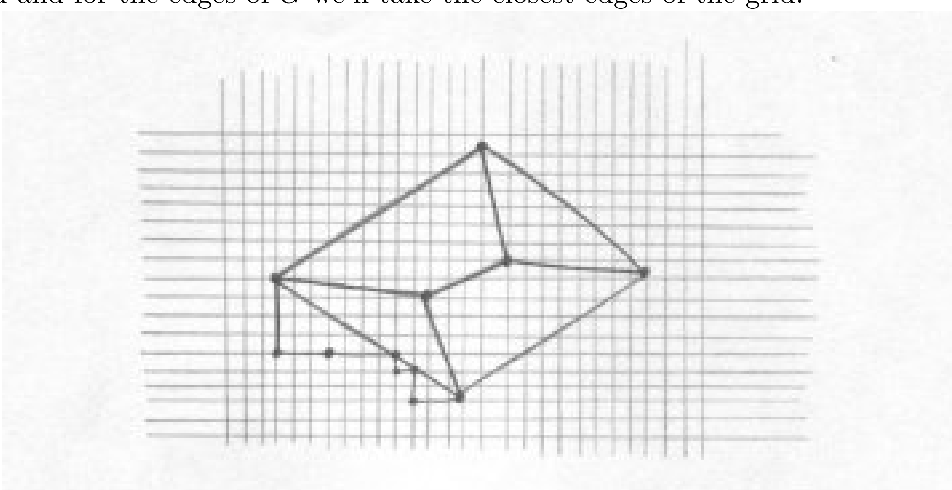
* Lecture Notes for a course given by Bojan Mohar at the Simon Fraser University, Winter 2006.

So grids are in a certain sense universal for planar graphs and the above Theorem clearly won't work for non-planar graphs since $k \times k$ grid cannot be made non-planar by edge-deletion and/or edge-contraction.

Proof. First let's show that H is a minor of a cubic planar graph. By a process illustrated in the picture below we can see that it suffices to prove the Theorem for cubic planar graphs.



For a cubic H take a straight-line representation of H in $\mathbb{R}^2$, take a fine grid and put it over the graph. With a fine enough grid we can make the vertices of G coincide with the vertices of the grid and for the edges of G we'll take the closest edges of the grid.



Now it's easy to see that G is indeed a subgraph of the grid. $\square$

Note that in the proof we could take the grid as fine as needed, but it has been shown that a grid of order $O(|H|)$ sufficies.

Theorem 2.3. *Let H be a graph embedded in a surface of Euler genus at least g, where $g \geq 1$. Then there exists an integer $k = k(H, g)$ such that every graph G embedded in the same surface as H with the face-width at least k contains H as a surface minor.*

So counterpart of a grid for non-planar graphs is a sequence of graphs whose face-width grows infinitely.

Corollary 2.4. *If G is a 3-connected graph embedded in $\mathbb{S}_g$ or $\mathbb{N}_g$ with sufficiently large face-width, then every embedding of G in the same surface is equivalent to it.*

Note that this is a generalization of Whitney's Theorem.

Proof. Let H be a graph in $S = \mathbb{S}_g$ or $S = \mathbb{N}_g$ such that H is 3-connected and uniquely embeddable. Such H exists, just consider an LEW-embedded graph (take any graph and put a dense grid in each of its faces). Suppose H is a cubic graph. Then G as in Theorem 2.3 contains H as a surface minor. Since H is cubic, a subdivision of H is a subgraph of G. It implies that every facial walk of Π is a contractible cycle of Π^*. Therefore the two embeddings are equivalent. $\square$

Corollary 2.5. *If $fw(G, \Pi)$ in $\mathbb{S}_g$ or $\mathbb{N}_g$, where $g \geq 2$, then G contains a non-contractible surface-separating cycle.*

In fact, it was proven that on any surface with face-width at least 11 will do, and this bound was later improved to 6. However, the best face-width bound is not known.

Conjecture 2.6. *Every graph G embedded in $\mathbb{S}_g$ or $\mathbb{N}_g$, where $g \geq 2$, with face-width at least 3 contains a non-contractible surface-separating cycle.*

Similarly for uniqueness of embeddings it has been shown that face-width at least $2g + 3$ or $\frac{clog(g)}{log(log(g))}$ will be enough.

3 Planarazing Cycles

Definition 3.1. Suppose that a graph G is embedded in $\mathbb{S}_g$ with $g \geq 0$. A collection of disjoint cycles $C_1, \ldots C_k$ form a *planarizing set of cycles* if cutting along all of $C_1, \ldots C_k$ gives rise to an embedding of genus 0.

Note that cutting along a cycle creates 2 facial walks, one for each copy of the cycle. Every cut therefore decreases an Euler genus by 2, so we'll need precisely g of those cycles to get Euler genus to be 0. Thus $k = g$.

Theorem 3.2. *Let d and g be positive integers. If G is a triangulation of $\mathbb{S}_g$ of edge-width at least $8(d+1)(2^g-1)$, then G contains a planarizing set of induced cycles such that $dist_G(C_i, C_j) \geq d$.*

Proof. The whole result except for the bound follows from Theorem 2.3, so we will only provide a sketch of the proof for that bound. It uses the induction on g. If $g = 0$ there's nothing to prove, since we'd be cutting along 0 cycles. If $g = 1$, take C_1 to be the shortest non-contractible cycle. By 3-Path-Property it's induced. So we may assume that $g \geq 2$. Let C be the shortest non-contractible cycle. There are 2 cases:

- C is non-surface-separating
- C is surface-separating

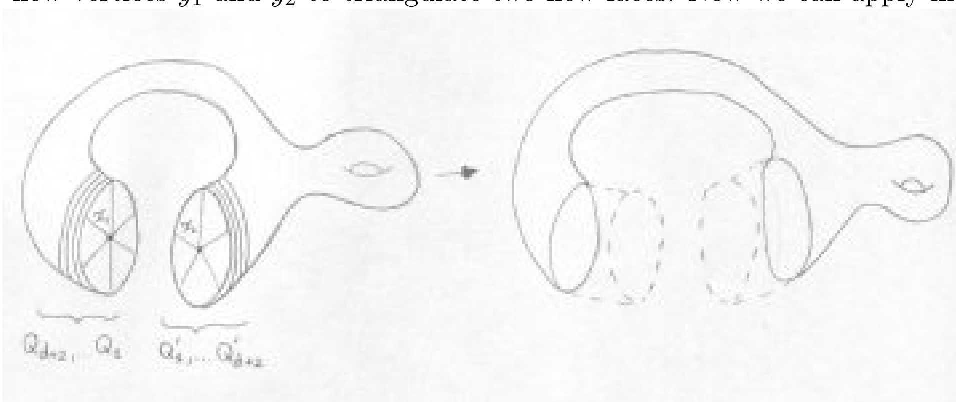
Let G^* be the graph obtained by cutting along C and triangulating new faces by adding vertices x_1 and x_2 inside them. Then,

$$ew(G^*) \geq ew(G) = 4(d+1)(2^g-1) = 8(d+1)(2^g-1) + 4(d+1).$$

So we have a little more than we need to apply the induction hypothesis, 4(d+1) more to be precise.

In a triangulation for every vertex x and integer $q \leq \frac{1}{2}ew(G^*) - \frac{1}{2}$ we define the *q-canonical cycles* $Q_1, \dots Q_q$ recursively so that they are all contractible and $int(Q_i) \supseteq Int(Q_{i-1})\ \forall i$.

Next we'll cut along $Q_1, \dots Q_{d+2}$ and $Q_1^*, \dots Q_{d+2}^*$. This results in a new surface and we'll add two new vertices y_1 and y_2 to triangulate two new faces. Now we can apply induction.



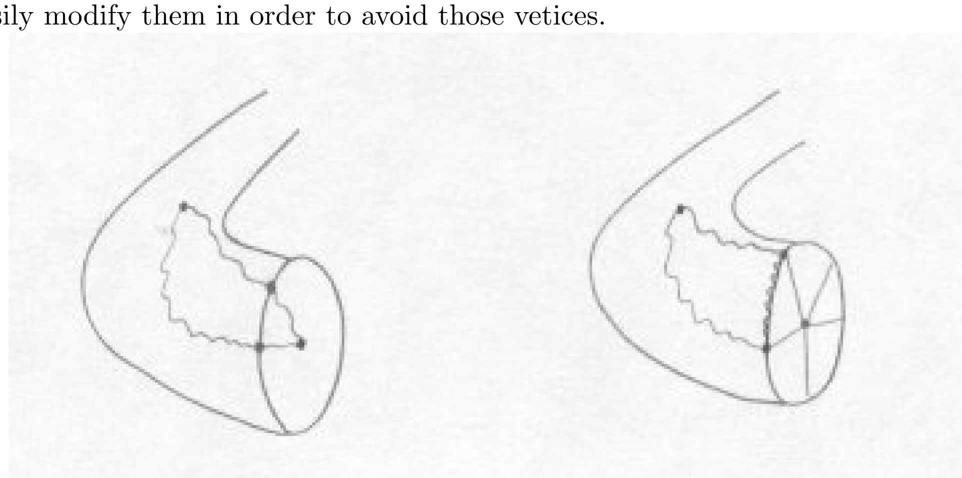
Cutting along each cycle results in decrease of edge-width by at most 2. So

$$ew(G^* *) \geq ew(G^*) - 2(d+1+d+1) = 8(d+1)(2^g - 1),$$

which is precisely what we need.

Therefore by induction hypothesis there exist $C_1, \dots C_{g-1}$ in G^* that are planarizing and at distance at least d from each other.

So the only thing to be taken care of are the cycles that pass through x_1 or x_2, but we can easily modify them in order to avoid those vetices.



Thus the above cycles together with C form a planarizing set for G.

Surface-separating case can be treated similarly.

□

Notice the importance of this Theorem: since after cutting along the specified cycles we get a planar graph, we can apply our knowledge of the planar graphs and then modify the conclusions to work for the original G.

4 Colourings of Graphs and Surfaces

Definition 4.1. Let G be a graph and let $\mathcal{C}$ be a set of colours (usually $\mathcal{C} \subseteq \mathbb{N}$). Then for every $v \in V(G)$ let $L(v) \subseteq \mathcal{C}$ be *a set of admissible colours for* v. An *L-colouring of a graph G* is a function $c : V(G) \to \mathcal{C}$ such that $\forall v \in V(G)$, $c(v) \in L(v)$ and $c(u) \neq c(v)$ if $uv \in E(G)$.

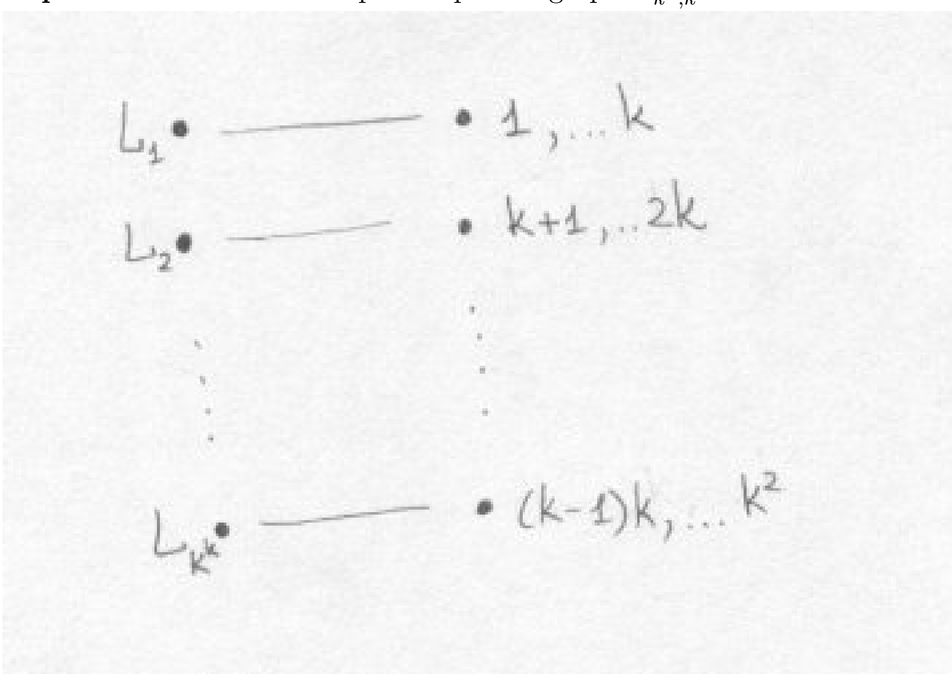
Note that the usual k-colourings are a special case where $L(v) = \{1, 2 \dots k\} \quad \forall v$.

Definition 4.2. The *choice number* or *list chromatic number* of G is the minimum integer k such that for every list assignment $L : V(G) \to \mathcal{C}$, where $|L(v)| \geq k \quad \forall v \in V(G)$ there exists an L-colouring of G. We will denote choice number of G by $ch(G)$.

Clearly $ch(G) \geq \chi(G)$.

Claim 4.3. *There exists a bipartite graph whose choice number can be made infinitely large.*

Example 4.4. Consider a complete bipartite graph $B_{k^k,k}$ as follows:



Here L_i is a k-subset that uses precisely one element of each list on the right. Then $ch(B_{k^k,k}) > k$.

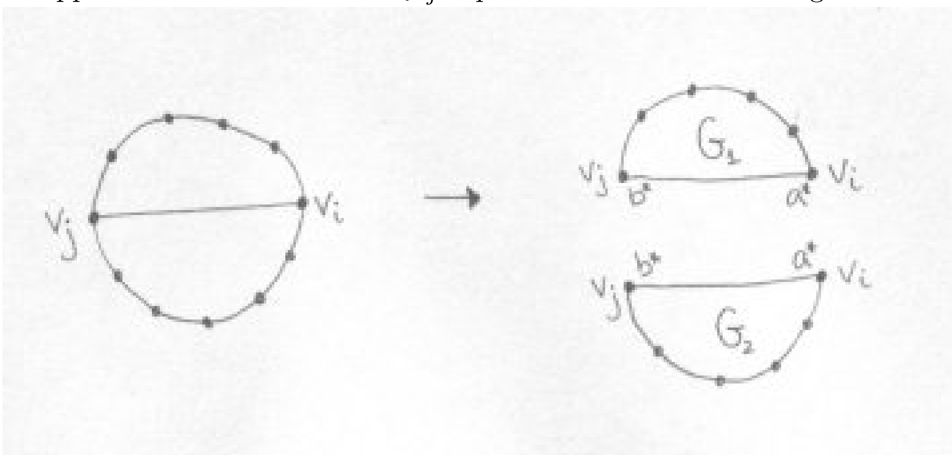
Theorem 4.5. *(Thomassen) Every planar graph has choice number at most 5 (i.e. it's 5-choosable).*

This Theorem is a Corollary of the following Lemma.

Lemma 4.6. *Let L be a list assignment for a plane graph G. Suppose that the outer face of G is bounded by a cycle $C = v_1 v_2 \dots v_r$ and suppose also that $|L(v_1)| \geq 1, |L(v_2)| \geq 1$, $|L(v_1) \bigcap L(v_2)| \geq 2$, $|L(v_i)| \geq 3$ for $i = 3, \dots r$ and for every vertex $v \in V(G) \backslash V(C) \quad |L(v)| \geq 5$. Then G is L-colourable.*

Note that Theorem 4.5 follows if we let $|L(v)| = 5$ for every $v \in V(G)$.

Proof. Proof will use the induction on the number of vertices of G. We may assume that all faces of G (except possibly for C) are 3-cycles and we may also assume that $L(v_1) = \{a\}, L(v_2) = \{b\}$, where $b \neq a$ and $|L(v_i)| = 3, |L(v)| = 5$ for v and all v_i as defined in the statement of the Theorem. Then either C has a chord or C is an induced cycle.
First suppose that C has a chord v_iv_j. Split C into two as following:



Now we will apply induction hypothesis to G_1 to get an L-colouring for it. Let a^* and b^* be colours of v_i and v_j in G_1. After letting $L(v_i) = \{a^*\}$ and $L(v_j) = \{b^*\}$ we can apply induction hypothesis to G_2 to get a valid L-colouring of G.
Suppose now that C is an induced cycle. Let p and q be two distinct colours in $L(v_r)$ not equal to a. By removing p and q from the lists of colours of neighbours of v_r on the interior of C, we can apply induction hypothesis to $G - v_r$. We may assume that v_{r-1} was not coloured with q and then the obtained colouring can be extended to L-colouring of G. $\square$

5 Four Colour Theorem

Theorem 5.1. *Every planar graph can be 4-coloured.*

This theorem was finally proved in 1977 by Appel and Haken and we will provide some ideas for the proof.

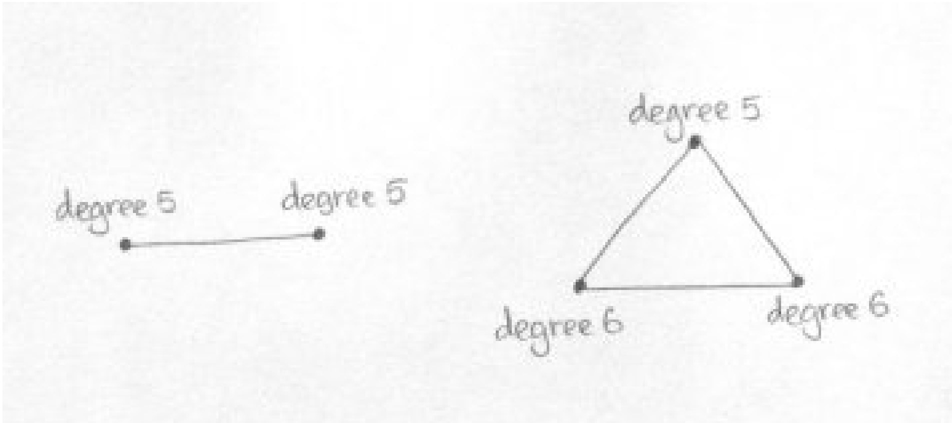
- Minimal counterexample
  Let G be a minimal counterexample. It can be shown that it implies that G is a triangulation with minimum degree 5 and is therefore 5-connected.

- Unavoidability
  Every minimal counterexample contains a configuration in a list $Q_1, Q_2, \dots Q_n$, where $n \approx 1300$.

- Reducibility
  $Q_1, Q_2, \dots Q_n$ cannot occur in a minimal counterexample.

Let's now consider an axample of an unavoidability result.

Proposition 5.2. *Every 5-connected triangulation of genus 0 with minimum degree 5 contains one of the following 2 configurations:*



Proof. We will use a technique known as discharging. We initially assign a charge $c(v) \in (R)$ to every vertex of G such that $\sum_{v \in V(G)} c(v) > 0$. We then specify the rules of discharging based on the degrees of the vertices around v to redistribute the charge, so that the new charges $c^*(v)$ satisfy the following:

$$\sum_{v \in V(G)} c^*(v) = \sum_{v \in V(G)} c(v) > 0.$$

All that's left to prove now is that $c^*(v) \leq 0 \quad \forall v \in V(G)$. For planar triangulations we can use $c(v) = 6 - deg(v)$ for initial charge. Then Euler's formula implies that

$$\sum_{v \in V(G)} c(v) = \sum_{v \in V(G)} (6 - deg(v)) = 12.$$

Define the discharging rule as follows: for every vertex of degree 5 send $\frac{1}{3}$ of the charge to its neighbours of degree at least 7. If v is a vertex of degree 5, then it has at least 3 neighbours of degree at least 7 and therefore $c^*(v) \leq 1 - \frac{1}{3} \times 3 = 0$. If $deg(v) = 6$, then $c^*(v) = c(v) = 0$. Finally if $deg(v) = k \geq 7$, then v has at most $\lfloor \frac{k}{2} \rfloor$ neighbours of degree 5. Then

$$c^*(v) \leq c(v) + \frac{1}{3} \lfloor \frac{k}{2} \rfloor = 6 - k + \frac{1}{3} \lfloor \frac{k}{2} \rfloor.$$

So for $k = 7$, $c^*(v) = 0$ and for $k \geq 8$ we get:

$$c^*(v) \leq 6 - k + \frac{1}{3} \lfloor \frac{k}{2} \rfloor \leq 6 - k + \frac{k}{6} = 6 - \frac{5}{6} k < 0.$$

Therefore the total charge is non-positive and this contradiction proves that one of the configurations in the statement must be present. $\square$