Topological Graph Theory*
Lecture 3-4: Characterizations of Planar Graphs

Notes taken by Dan Benvenuti

Burnaby, 2006

Summary: An indepth look at various characterizations of planar graphs including: Kuratowski subgraphs, bridges, cycle spaces, and 2-bases.

1 Kuratowski's Theorem

Definition 1.1. A *Kuratowski subgraph* is a subgraph homeomorphic to a subdivision of K_5 or $K_{3,3}$, denoted $SK_5/SK_{3,3}$.

Lemma 1.2. *Let G be a graph with cycle C and $X, Y \subseteq V(G)$ then one of the following holds:*

1. $|X| = 1$ *or* $|Y| = 1$
2. $X = Y$
3. $\exists$ *vertices* x_1, y_1, x_2, x_1 *occuring in that order on C with* $x_i \in X$ *and* $y_i \in Y$ *for* $i = 1 \dots 2$
4. $\exists$ *vertices* $u, v \in V(C) \ni$ *there are two paths, P and Q, from u to v with* $C = P \cup Q$ *and* $X \subseteq V(P), Y \subseteq V(Q)$

Proof. Assume X and Y do not satisfy (1) or (2). Furthermore, assume $x_1 \in X \backslash Y$. Walk along C in both directions until vertices $y_1, y_2 \in Y$ are reached. Denote the y_1y_2-path on C not using x_1 as Q. If $x_2 \in X$ is in Q then (4). Else, (3). □

Lemma 1.3. *Let G be a 3-connected graph that does not contain SK_5 or $SK_{3,3}$ as a subgraph, then G can be embedded in $\mathbb{R}^2 \ni$ all faces are convex and the unbounded face is the complement of a convex set. In particular, all edges are straight line segments.*

Proof. Show $G' = G//e$ has a convex embedding by induction on $|G|$. Let z denote the vertex formed by contracting edge $e \in E(G)$. Then, $G' - z$ is 2-connected with a new face, C, bounded by a cycle in G. Consider the two cases, C is the bounded face, and C is the unbounded face. Apply Lemma 1.2 to obtain the desired result from *Case 4*. □

Corollary 1.4. *Every 2-connected planar graph has a convex embedding in $\mathbb{R}^2$.*

* Lecture Notes for a course given by Bojan Mohar at the Simon Fraser University, Winter 2006.

Lemma 1.5. *Suppose G with $|G| \geq 4$ has no $SK_5/SK_{3,3}$, but adding any edge between non-adjacent vertices creates such a subdivision then G is 3-connected.*

Proof. Proceed by induction on $|G|$. Clearly, G is 2-connected. Suppose G has cutset $\{x, y\}$, where $e = xy$. Denote $G = G_1 \cap G_2$ where $G_1 \cup G_2 = e$. Observe that G_1 and G_2 are 3-connected. Let z_i be a vertex in the same face as edge e in $G_i, i = 1 \ldots 2$. Adding the edge $z_1 z_2$ in G yields the desired contradiction. □

Theorem 1.6. Kuratowski's Theorem: *Graph G is planar $\iff$ G does not contain $SK_5/SK_{3,3}$.*

Proof. Apply Lemma 1.3 and Lemma 1.5. □

Theorem 1.7. *If G is a planar graph then G has a planar representation whose edges are all straight lines.*

2 Other Characterizations of Planar Graphs

Definition 2.1. Graph G contains K as a minor if K can be obtained from a subgraph of G by a sequence of edge contractions.

Theorem 2.2. *G is a planar graph $\iff$ it contains neither K_5 nor $K_{3,3}$ as a minor.*

Observation 2.3. *The following are direct consquences of previous results:*

1. *If G contains a subdivision of K then G contains K as a minor.*
2. *If G contains K_5 as a minor then G contains either SK_5 or $SK_{3,3}$.*

Definition 2.4. A *chord* of cycle C in graph G is an edge $e \in E(G)$ with endpoints on C but $e \notin E(C)$.

Definition 2.5. Let C be a cycle of graph G. A *bridge* of C is either a connected component H of $G - V(C)$ togehter with all edges joining H to C, or a chord of C.

Definition 2.6. The *vertices of attachment* of a bridge B on cycle C are the vertices $V(C) \cap V(B)$.

Definition 2.7. Two bridges B_1 and B_2 of cycle C *overlap* if:

1. B_1 and B_2 have three (or more) common vertices of attachment, or
2. C contains distinct vertices $b_{1,1}, b_{2,1}, b_{1,2}, b_{2,2}$ in the given order, where $b_{i,j} \in B_i$ for $i, j = 1 \ldots 2$.

Definition 2.8. Bridges B_1 and B_2 *skew-overlap* if they exhibit *Case 2* of 2.7.

Lemma 2.9. *If C is a cycle of planar graph G with overlapping bridges B_1 and B_2 then B_1 and B_2 lie in distinct faces with respect to C.*

Proof. Suppose both B_1 and B_2 lie in the face face with respect to C (WLOG suppose they lie on the exterior of C). Form graph G' by adding an additional vertex, v, to the opposing face (the interior) and placing edges from v to all vertices of attachment. This contradcts the planarity of G by obtaining $SK_5/SK3,3$ in both cases of Defintion 2.7. □

Theorem 2.10. *A graph G is not planar $\iff$ $\exists$ a cycle C in G that contains three (or more) bridges, B_1, B_2, B_3, that pairwise overlap.*

Proof. As shown in Lemma 2.9 at most one overlapping bridge may reside in a given face of C to retain planarity. By the *Jordan Curve Theorem* cycle C seperates the plane into two components. Hence, C can have at most two overlapping bridges. □

Theorem 2.11. *Let G be a planar graph with $x, y \in V(G)$. $G + xy$ is not planar $\iff$ G contains a cycle $C \ni x$ and y are in distinct overlapping bridges of C.*

Proof. ($\Leftarrow$) Given x and y reside in overlapping bridges, they reside is different faces in G seperated by C. Thus, edge xy cannot be added while conserving planarity. ($\Rightarrow$) Given $G + xy$ is not planar $\exists K = SK_5/SK_{3,3}$ with cycle C in K seperating x from y. Since C is also a cycle of G with x and y seperated, x and y are distinct. □

Definition 2.12. The *overlap graph* of cycle C in graph G has C-bridges as vertices and two such vertices are adjacent if the corresponding bridges overlap.

Definition 2.13. The *skew-overlap graph* is a spanning subgraph of the overlap graph with vertices adjacent $\iff$ the corresponding bridges skew-overlap.

Theorem 2.14. *The following are equivalent:*

1. *graph G is not planar*
2. *the overlap graph of some cycle C is non-bipartite*
3. *the skew-overlap graph of some cycle C is non-bipartite*
4. *the skew-overlap graph of some cycle C contains a 3-cycle*

Proof. Trivally, $4 \to 3 \to 2 \to 1$. We need only show $1 \to 4$. Proceed by induction on the number of edges which need to be removed to obtain $SK_5/SK_{3,3}$. So $\exists e \in E(G) \ni G - e$ is non-planar. By induction $G - e$ has a cycle C whose skew-overlap graph, H, contains 3-cycle T. Let H' be the graph formed from H by identifying two vertices. Use these vertices, cycle C, and Lemma 2.9 to obtain the desired result. □

3 Algebraic Graph Theory

Definition 3.1. The *cycle space* of graph G denoted, $Z(G)$, has $E(G) \supseteq A \in Z(G) \iff A$ generates an Eulerian subgraph of G.

Definition 3.2. The *symmetric difference* of $A, B \subseteq E(G)$ is defined, $A + B = (A \cup B) \setminus (A \cap B)$.

Observation 3.3. *The symmetric difference of two sets is associative ($A + B = B + A$) and if $s \in S = S_1 + \cdots + S_k$ then s belongs to an odd number of S_i.*

Definition 3.4. The *power set of $E(G)$* denoted, $P(E)$, is a vector space over $GF(2)$ with operation "+".

Observation 3.5. $dim(P(E)) = \|G\| = |E(G)|$.

Definition 3.6. Given a graph G and spanning tree T, a *fundamental cycle* of T is the unique cycle, $C(e,T)$, formed by adding any non-tree edge in G to T.

Proposition 3.7. *If G is a connected graph, then:*

$$dim(Z(G) = \|G\| - |G| + 1 = |E(G)| - (n-1)$$

Proof. Construct a basis for $Z(G)$ via a spanning tree, T. Then

$$\{C(e,T) : e \in E(G) \backslash E(T)\}$$

is a basis of $Z(G)$, as the $C(e,T)$ are independent and, as a set, have the desired cardinality. □

Proposition 3.8. *Let G be a 2-connected planar graph then the set of all facial cycles of G generates $Z(G)$. Moreover, if any facial cycle is removed then we obtain a basis for $Z(G)$.*

Proof. It is sufficent to show every cycle is the sum of facial cycles inside the disk bounded by the current identified cycle, C. The result follows from *Euler's Formula*, $\|G\| - |G| + f = 2 \Rightarrow f - 1 = |G| - \|G\| + 1 = dim(Z(G))$. □

Definition 3.9. A basis, $\mathcal{B}$, of a cycle space is a *2-basis* if every edge of G belongs to at most two elements of $\mathcal{B}$.

Theorem 3.10. (MacLane) *Let G e a 2-connected graph. G has a 2-basis of $Z(G) \iff G$ is planar. Moreover, if $\mathcal{B}$ is a 2-basis then $\mathcal{B}$ consists of facial cycles (excluding one) of some planar embedding of G.*

Proof. ($\Rightarrow$) First observe that if G is not planar then it does not have a 2-basis. ($\Leftarrow$) Given G is a 2-connectd planar graph. Let $C_1 \dots C_r$ denote the facial cycles of G, then $r = |G| - \|G\| + 2 = 1 + dim(Z(G))$. But $C_1 \dots C_{r-1}$ generates $Z(G)$.

All that remains is to show that every 2-basis of $\mathcal{B}$ corresponds to an embedding of G. Let $\mathcal{B}'$ be a 2-basis of G', the graph formed by subdiving each edge of G once. Construct G'' by adding a new vertex v_C in the interior of every cycle $C \in \mathcal{B}'$ and connecting it to all vertices on C. Then $\mathcal{B}''$ is a 2-basis of G'', so G'' is planar. Deleting all v_C yield a planar embedding of G. □

Corollary 3.11. *If G is a plane graph whose faces are bounded by an even number of edges then G is bipartite.*

Proof. Suppose G is not bipartite then it contains an odd cycle, C. By Theorem 3.10, C is the sum of facial cycles. Hence, one such cycle must be odd. □

4 3-Connected Planar Graphs

Definition 4.1. A cycle is *induced* if it has no chords.

Definition 4.2. A cycle, C, of graph G is *non-seperating* if $G - V(C)$ is connected.

Theorem 4.3. *Let G be a 3-connected planar graph. Cycle C is a facial cycle in some planar representation of $G \iff C$ is induced and non-seperating.*

Proof. ($\Rightarrow$) Given C is a facial cycle. If $e \in E(G)$ is a chord of of C then $G - e$ is disconnected, hence C contains only a single bridge. ($\Leftarrow$) By the *Jordan Curve Theorem* an induced non-seperating cycle is facial. □

Theorem 4.4. (Whitney's Uniqueness Theorem) *All* 3*-connected planar graphs have unique embeddings in* $\mathbb{R}^2$.

Proof. Observe that Theorem 4.3 implies that the faces of a 3-connected planar graph are determined from the graph, without regard to any planar drawing. Since any two drawings of G have the same faces, the result follows. □

Theorem 4.5. (Tutte's Non-seperating Cycles Theorem) *If graph G is* 3*-connected and $e \in E(G)$ then G contains at least two induced non-seperating cycles Q_1 and $Q_2 \ni Q_1 \cap Q_2 = e$.*

Corollary 4.6. *Given graph G is* 3*-connected. G is nonplanar $\iff \exists e \in E(G) \ni e$ is contained in at least three induced non-seperating cycles.*

Proof. Follows directly from Theorem 4.5. □

Proposition 4.7. *Let G be a* 3*-connected graph (which is not K_5). G is not planar $\iff$ it contains $SK_{3,3}$.*

Proof. ($\Leftarrow$) This trivially follows from *Kuratowski's Theroem.* ($\Rightarrow$) Given G is not planar, but not K_5. It must be that $SK_5 \subseteq G$. Thus, G contains an SK_5-bridge, B, with vertices of attachment on at least 2 paths of SK_5 corresponding to edges in K_5. Any such drawing contains a $K_{3,3}$ subdvision. □