Existence of polyhedral embeddings of graphs*

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Abstract

It is proved that the decision problem about the existence of an embedding of face-width 3 of a given graph is **NP**-complete. A similar result is proved for some related decision problems. This solves a problem raised by Neil Robertson.

1 Introduction

Let C and C' be cycles in a graph G. We say that C and C' *meet properly* if the intersection of C and C' is either empty, a single vertex or an edge.

Let G be a 3-connected graph. A 2-cell embedding of G in some surface is *polyhedral* if every facial walk is a cycle and any two facial cycles meet properly. Equivalently, we require that the graph is 3-connected and that the embedding has face-width at least three [8] (cf. also [1, 5, 6]). Let us recall that the *face-width* (also called the *representativity*) of a (2-cell) embedded graph G is the minimum integer r such that G has r facial walks whose union contains a cycle which is noncontractible on the surface. (In the case when there are no noncontractible cycles, we let the face-width be ∞.)

At the Seventh Vermont Summer Workshop on Combinatorics and Graph Theory in 1995, Neil Robertson asked how difficult it is to see whether a given 3-connected graph admits a polyhedral embedding. In this note we answer his question by proving that the decision problem about the existence of polyhedral embeddings is **NP**-complete. The problem remains

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NP-complete even if we ask about polyhedral embeddings in orientable surfaces and require that the given graph is 6-connected. A similar problem where we ask about embeddings of face-width exactly 3 is also **NP**-complete. However, it is not known if existence of embeddings of face-width 4 or more is still **NP**-complete.

An indication that there is some nontriviality in polyhedral embeddings is that for a complete graph K_n ($n \geq 5$), a polyhedral embedding is necessarily a triangulation, and a significant part of Ringel and Youngs' Map Color Theorem [7] was to determine which complete graphs have such embeddings. Our result is not that much of interest from the computational complexity point of view. Its main message is that any theory on polyhedral embeddings is rich and interesting.

It is worth mentioning that a similar problem concerning embeddings of face-width at least two may be polynomially solvable. This problem is easily reduced to 2-connected graphs (cf. [6, 8]), and there are two long standing conjectures which are closely related to the Cycle Double Cover Conjecture (cf. [4, 10]), and whose affirmative solution would give a trivial answer about existence of embeddings of face-width at least 2.

Conjecture 1.1 (Haggard [3]) *Every 2-connected graph has an embedding of face-width 2 or more.*

Conjecture 1.2 (Jaeger [4]) *Every 2-connected graph has an orientable embedding of face-width 2 or more.*

2 Embeddings and compatible cycles

Our treatment of graph embeddings follows essentially [6]. All graphs are simple, so there are no loops or multiple edges. We only consider 2-cell embeddings into closed surfaces which can be defined combinatorially as follows. An *embedding* of a connected graph G is a pair $\Pi = (\pi, \lambda)$ where $\pi = \{\pi_v \mid v \in V(G)\}$ is a collection of *local clockwise rotations*, i.e., π_v is a cyclic permutation of the edges incident with v ($v \in V(G)$), and $\lambda : E(G) \to \{+1, -1\}$ is a *signature*. The local rotation π_v describes the cyclic clockwise order of edges incident with v on the surface, and the signature $\lambda(uv)$ of the edge uv is positive if and only if the local rotations π_u and π_v both correspond to the clockwise (or both to anticlockwise) rotations when traversing the edge uv on the surface. An embedding of a graph G is *nonorientable* if G contains a cycle whose number of edges with negative signature is odd.

The embedding Π determines a set of Π-*facial walks*. If a Π-facial walk is a cycle, it is also called a Π-*facial cycle*. The underlying surface of the embedding Π is obtained by pasting discs along the Π-facial walks in G.

Let G be a graph. Two subgraphs H_1, H_2 of G are said to be *compatible* if $E(H_1) \cap E(H_2)$ is a matching in G. Equivalently, no two edges of H_1 incident with the same vertex are both contained in H_2.

Let G be a Π-embedded graph and let $v \in V(G)$. Let H be the subgraph of G consisting of all neighbors of v and all edges uw such that vuw is a Π-facial cycle. Then H is called the *link* of v, and is denoted by $\mathrm{link}(v, G, \Pi)$.

Lemma 2.1 *Let G be a Π-embedded graph and let u, v be distinct vertices of G. If no vertex adjacent to v is of degree 4 in G, then* $\mathrm{link}(u, G, \Pi)$ *and* $\mathrm{link}(v, G, \Pi)$ *are compatible subgraphs of G whose maximum degree is at most 2.*

Proof. Each edge vw is contained in at most two facial triangles. Therefore, the maximum degree in the link of v does not exceed 2. If $\mathrm{link}(u, G, \Pi)$ and $\mathrm{link}(v, G, \Pi)$ share two edges aw, bw incident with the same vertex w, then the link of w is the cycle $avbu$ and w is of degree 4. This completes the proof. □

Thomassen [9] proved:

Theorem 2.2 (Thomassen [9]) *The decision problem whether a given cubic bipartite graph contains two compatible Hamilton cycles is* **NP***-complete.*

The cubic bipartite graphs G in Thomassen's proof of Theorem 2.2 in [9] are 2-connected and contain many edges which are contained in any Hamilton cycle of G. Such edges are easily discovered in G. This shows that the same problem is **NP**-complete also when the input graph is 2-connected and has three prescribed edges which are contained in every Hamilton cycle of G.

Let T be a tree of maximum degree d, and suppose that G_0 is a graph and $e_1, \ldots, e_d$ are edges of G_0. Take a distinct copy G_t of G_0 for each vertex $t \in V(T)$. Label each oriented edge tt' of T by a number in $\{1, \ldots, d\}$ so that the edges emanating from the same vertex receive distinct labels. Now, for each edge $tt' \in E(T)$, repeat the following operation. Let a and b be the labels of tt' and $t't$, respectively. Remove the edge $e_a = xy$ from G_t, remove $e_b = x'y'$ from $G_{t'}$, and add the edges xx' and yy'. Let G be any graph resulting from this operation.

Lemma 2.3 *Let G_0 be a graph and let T be a tree of maximum degree d. Let G be a graph constructed as described above, and let $e_1, \ldots, e_d$ be the edges of G_0 used in the construction. Then G contains two compatible Hamilton cycles if and only if G_0 contains two compatible Hamilton cycles each of which contains all edges $e_1, \ldots, e_d$.*

Proof. Suppose first that G contains two compatible Hamilton cycles H_1, H_2. We shall use the notation introduced in the definition of G. Suppose that t is a vertex of degree d in T. The removal of the edges xx' and yy' disconnects the graph G. Let G' be the component of $G - xx' - yy'$ which contains $V(G_t)$. Clearly, xx' and yy' are both contained in H_1 and in H_2. Therefore, $H'_1 = (H_1 \cap G') + xy$ and $H'_2 = (H_2 \cap G') + xy$ are compatible Hamilton cycles of $G' + xy$. By repeating such a reduction for all edges incident with t in T, we obtain two compatible Hamilton cycles of G_t (and hence of G_0) which contain all edges $e_1, \ldots, e_d$.

Suppose now that G_0 contains two compatible Hamilton cycles H_1° and H_2° each of which contains all edges $e_1, \ldots, e_d$. We shall prove by induction on $|V(T)|$ that G admits two compatible Hamilton cycles H_1, H_2 such that all edges $e_1, \ldots, e_d$ in each copy G_t ($t \in V(T)$) which remain in G are contained in H_1 and in H_2. (Here we allow that d is larger than the maximum degree in T.) This is clear if $|V(T)| = 1$. Otherwise, let t be a leaf of T, and let t' be the neighbor of t in T. Let $G' = (G - V(G_t)) + x'y'$. Then G' is obtained from G_0 and $T - t$ in the same way as described before the lemma. By the induction hypothesis, G' has two compatible Hamilton cycles H'_1, H'_2 which contain $e_b = x'y'$ (and all other edges $e_1, \ldots, e_d$ in each copy G_s, $s \in V(T-t)$, which remain in G'). Let $H_j = (H'_j - x'y') \cup (H_j^\circ - xy) + xx' + yy'$, $j = 1, 2$. Then H_1 and H_2 are compatible Hamilton cycles in G with the desired property. □

Lemma 2.4 *Let G_0, T, and G be as in Lemma 2.3. Suppose that T has more than $4k$ leaves where k is a positive integer. If G contains two compatible spanning subgraphs H_1, H_2 such that for $i = 1, 2$, the maximum degree in H_i is at most two and such that the number of connected components of H_i is $\leq k$, then G_0 contains two compatible Hamilton cycles.*

Proof. Note that both H_1 and H_2 are disjoint unions of isolated vertices, paths, and cycles. Let U be the vertex set of G containing all vertices of degree less than 2 in H_1 or in H_2, and containing one vertex of each cycle in H_1 or in H_2. Then $|U| \leq 4k$, and hence there is a leaf t of T such that

$U \cap V(G_t) = \emptyset$. Then $H_1 \cap G_t$ and $H_2 \cap G_t$ give rise to two compatible Hamilton cycles in G_0. □

3 Reduction

Theorem 3.1 *The decision problem "Does a given graph G have a polyhedral embedding" is* **NP***-complete. The problem remains* **NP***-complete also if we ask about polyhedral embeddings in orientable surfaces and require that G is 6-connected.*

Proof. Let G_0 be an arbitrary 2-connected cubic bipartite graph, and let $e_1, e_2, e_3 \in E(G_0)$ be distinct edges of G_0 such that every Hamilton cycle of G_0 contains each of them. By Theorem 2.2 (and the remark following it), it is **NP**-complete to decide if G_0 has two compatible Hamilton cycles. Thus, Theorem 3.1 will follow if we prove that one can construct in polynomial time a 6-connected graph G_1 which has a polyhedral embedding if and only if G_0 has two compatible Hamilton cycles and, moreover, if G_1 has a polyhedral embedding, then it also has an orientable polyhedral embedding.

Let T be a cubic tree (i.e., each vertex of T is of degree 3 or 1) of order 104, so that T has 53 leaves. Construct the graph G as described before Lemma 2.3. Clearly, G is cubic, bipartite and 2-connected. Let $V(G) = V_1 \cup V_2$ be the bipartition of G. Now, define the graph G_1 which is obtained from G as follows. First, replace each vertex $v \in V_2$ by two mutually adjacent vertices v', v'' which are both adjacent to the same three vertices in V_1 as v. Let $V' = \{v' \mid v \in V_2\}$ and $V'' = \{v'' \mid v \in V_2\}$. Finally, add four new vertices a', b', a'', b'' where a' and b' are adjacent to all vertices in $V_1 \cup V'$, and a'', b'' are adjacent to all vertices in $V_1 \cup V''$.

The resulting graph G_1 is 6-connected. To see this, one considers each pair x, y of vertices and shows that there are 6 internally disjoint paths joining x and y. The details are rather straightforward and are left to the reader.

We claim that G_0 contains two compatible Hamilton cycles if and only if G_1 has a polyhedral embedding. First, assume that G_0 admits two compatible Hamilton cycles. Since every Hamilton cycle of G_0 contains e_1, e_2, and e_3, Lemma 2.3 shows that G has two compatible Hamilton cycles, say H_1 and H_2. For $i = 1, 2$, let H_i' (resp. H_i'') be the cycle in G_1 obtained from H_i by replacing each vertex $v \in V_2$ by the vertex $v' \in V'$ (resp. $v'' \in V''$). It is easy to see that G_1 has (a unique) embedding in which all facial cycles are triangles such that the link of a' (resp. b', a'', b'') is H_1' (resp. H_2', H_1'', H_2'').

This embedding is clearly polyhedral. It is also orientable. To see this, orient the facial triangles as follows: $a'v'v_1$ (if $v'v_1 \in E(H_1') \backslash E(H_2')$), $a'v_1v'$ (if $v'v_1 \in E(H_1') \cap E(H_2')$), similarly around b', $v_1v'v''$ (if $v'v_1 \in E(H_1')$), $v_1v''v'$ (if $v'v_1 \in E(H_2')$), where $v_1 \in V_1$, $v' \in V'$, and $v'' \in V''$. Similarly we orient the triangles containing the edges of H_1'' and H_2''. The details are left to the reader.

Conversely, let Π be a polyhedral embedding of G_1. Let us consider the Π-facial cycles containing a'. Each such facial cycle $C = a'v_1v_2 \dots v_k$ is an induced cycle in G_1. (If C had a chord e, then a facial cycle containing e would meet improperly with C.) We say that C is *exceptional* if $k > 2$. It is *strongly exceptional* if $V(C)$ contains at least one of the vertices b', a'', b'', and *weakly exceptional* otherwise. There are at most three strongly exceptional faces containing a' since no two strongly exceptional faces contain the same pair of (nonconsecutive) vertices $\{a', x\}$, $x \in \{b', a'', b''\}$.

An exceptional face $C = a'v_1 \dots v_k$ is induced. Therefore, $v_1, v_k \in V_1 \cup V'$ and $v_2, \dots, v_{k-1} \notin V_1 \cup V'$. Similar conclusions hold for the exceptional faces at the vertices b', a'', and b''. This implies that at most 10 vertices of V'' belong to strongly exceptional faces at the vertices a', b', a'', b''.

Suppose now that C is weakly exceptional. Then $k = 3$ since $v_2 \in V''$, and hence v_3 is a neighbor of a'. Let $v \in V_2$ be the vertex such that $v_2 = v''$. If $v_1 \in V'$, then $v_1 = v'$ and thus $v_1v_3 \in E(G_1)$, a contradiction. Hence $v_1, v_3 \in V_1$. As mentioned above, there are at most 10 vertices in V'' contained in a strongly exceptional face. Therefore, there are at most 10 weakly exceptional facial cycles containing a' such that v_2 is contained in some strongly exceptional facial cycle.

Suppose now that C is not such a face. Consider the Π-clockwise ordering around v''. If the edges $v''a''$ and $v''v'$ are consecutive in that ordering, the facial cycle containing these two edges is not induced (as we just proved above when considering the possibility that $v_1 \in V'$). Similarly, $v''b''$ and $v''v'$ are not consecutive around v''. In particular, $v_2 = v''$ belongs to a strongly exceptional face containing a'' and b'', a contradiction. This implies that there are at most 10 weakly exceptional and at most 3 strongly exceptional faces containing a'. This shows that $\mathrm{link}(a', G_1, \Pi)$ is a subgraph of G_1 of maximum degree at most 2 and with at most 13 connected components. The same holds for $\mathrm{link}(b', G_1, \Pi)$. By Lemma 2.1, these links are compatible subgraphs of G_1. Clearly, they give rise to compatible subgraphs in G. Since T has 53 leaves, Lemma 2.4 implies that G_0 (and hence also G by Lemma 2.3) contains two compatible Hamilton cycles. Additionally, as the previous paragraph shows, G_1 admits an orientable polyhedral embedding determined by two compatible Hamilton cycles of G.

We reduced, in polynomial time, the **NP**-complete problem of Theorem 2.2 to the existence of polyhedral embeddings of 6-connected graphs. Since the embedding of G_1 obtained from two compatible Hamilton cycles in G_0 (and in G) is orientable, this completes the proof. □

The proof of Theorem 3.1 shows that in every embedding Π of G_1 of face-width at least 3, the link of a' determines a Hamilton path in one of the subgraphs G_t of G where t is some leaf of T. This can be used to show that G_1 has no embeddings of face-width 4 or more. Hence, the same proof also shows:

Corollary 3.2 *The decision problem "Does a given 6-connected graph G have an (orientable) embedding of face-width exactly 3" is* **NP***-complete.*

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