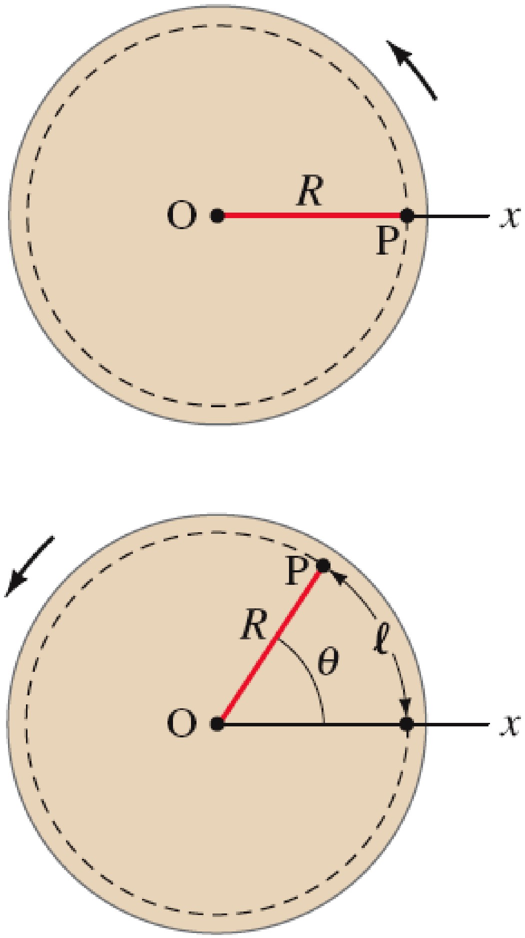
Phys101 Lectures 19, 20
Rotational Motion

Key points:

- **Angular and Linear Quantities**
- **Rotational Dynamics; Torque and Moment of Inertia**
- **Rotational Kinetic Energy**

Ref: 10-1,2,3,4,5,6,8,9.

Angular Quantities

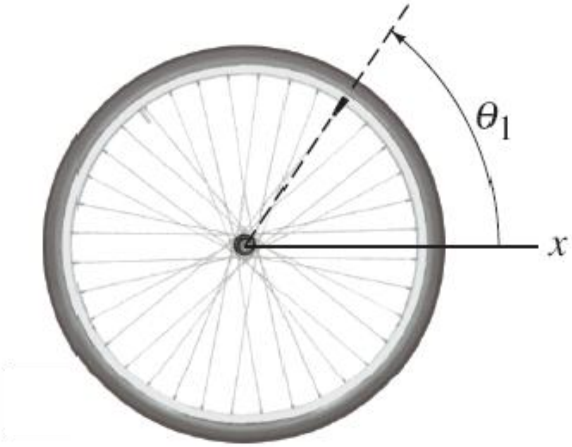


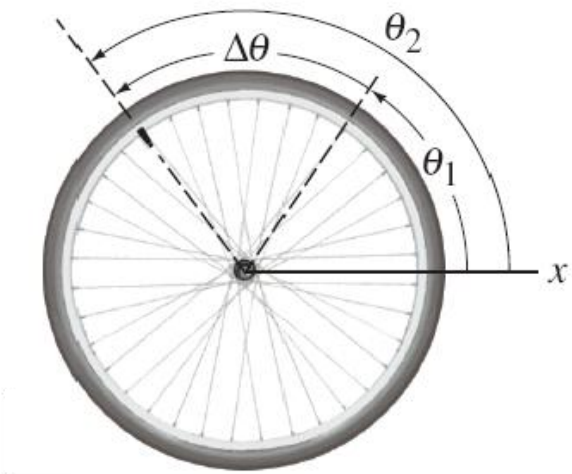
In purely **rotational** motion, all points on the object move in **circles** around the **axis** of rotation (“*O*”). The **radius** of the circle is *R*. All points on a straight line drawn through the axis move through the same **angle** in the same **time**. The angle θ in **radians** is defined:

$$\theta = \frac{l}{R},$$

where *l* is the **arc length**.

10-1 Angular Quantities





Angular displacement:

$$\Delta\theta = \theta_2 - \theta_1 .$$

The average angular velocity is defined as the total angular displacement divided by time:

$$\bar{\omega} = \frac{\Delta\theta}{\Delta t} .$$

The instantaneous angular velocity:

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} .$$

10-1 Angular Quantities

The angular **acceleration** is the rate at which the angular velocity changes with time:

$$\bar{\alpha} = \frac{\omega_2 - \omega_1}{\Delta t} = \frac{\Delta\omega}{\Delta t}.$$

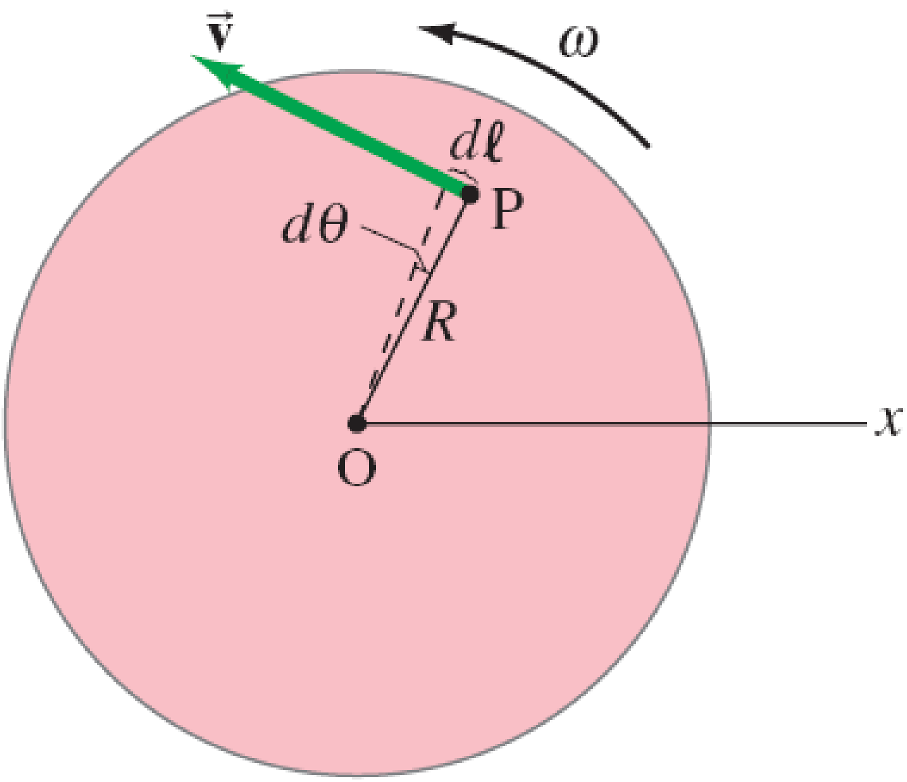
The **instantaneous** acceleration:

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}.$$

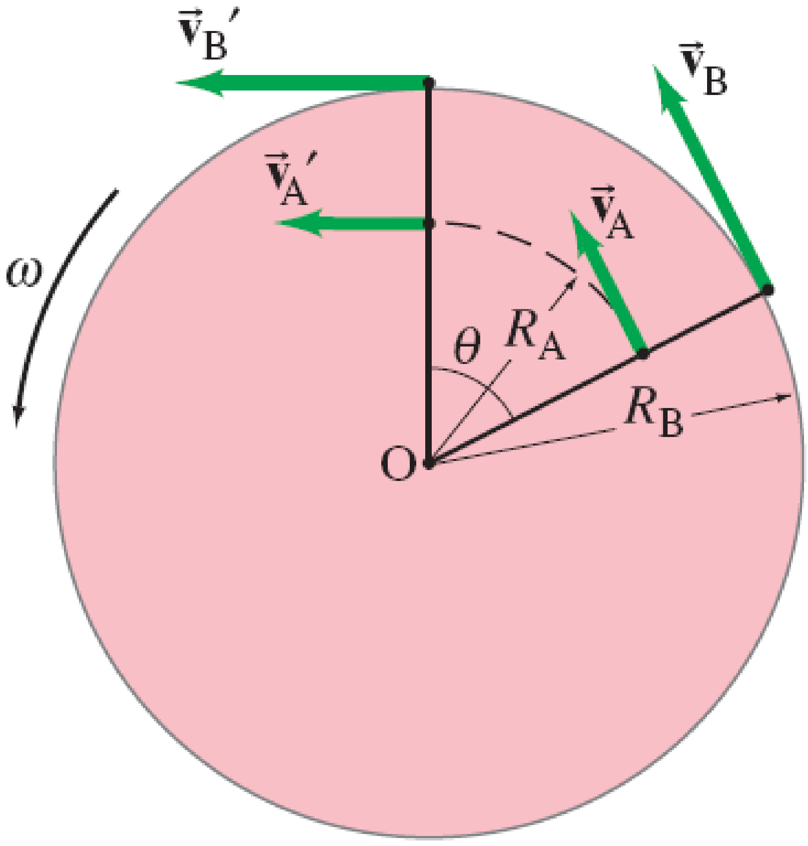
10-1 Angular Quantities

Every point on a rotating body has an **angular velocity** ω and a **linear velocity** v.

They are related: $v = R\omega.$

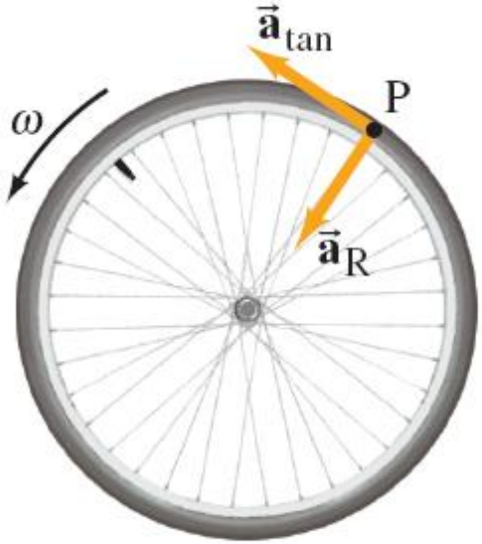


10-1 Angular Quantities



Objects farther from the axis of rotation will move faster.

10-1 Angular Quantities



If the angular velocity of a rotating object **changes**, it has a **tangential** acceleration:

$$a_{\tan} = \frac{dv}{dt} = R\frac{d\omega}{dt} = R\alpha.$$

Even if the angular velocity is constant, each point on the object has a **centripetal** acceleration:

$$a_{R} = \frac{v^2}{R} = \frac{(R\omega)^2}{R} = \omega^2 R.$$

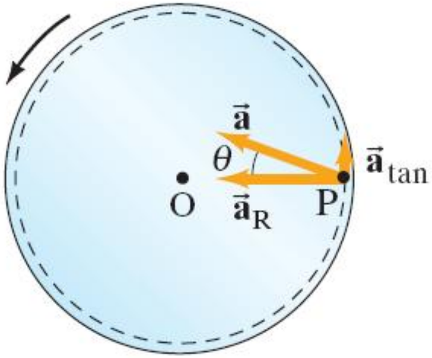
10-1 Angular Quantities

Here is the correspondence between linear and rotational quantities:

TABLE 10–1
Linear and Rotational Quantities

Linear	Type	Rotational	Relation (θ in radians)
x	displacement	θ	$x = R\theta$
v	velocity	ω	$v = R\omega$
$a_{\tan}$	acceleration	α	$a_{\tan} = R\alpha$

(a)



Example 10-3: Angular and linear velocities and accelerations.

A carousel is initially at rest. At $t = 0$ it is given a constant angular acceleration $\alpha = 0.060\ rad/s^2$, which increases its angular velocity for 8.0 s. At $t = 8.0$ s, determine the magnitude of the following quantities: (a) the angular velocity of the carousel; (b) the linear velocity of a child located 2.5 m from the center; (c) the tangential (linear) acceleration of that child; (d) the centripetal acceleration of the child; and (e) the total linear acceleration of the child.

10-1 Angular Quantities

The **frequency** is the number of complete **revolutions** per second:

$$f = \frac{\omega}{2\pi}.$$

Frequencies are measured in **hertz**:

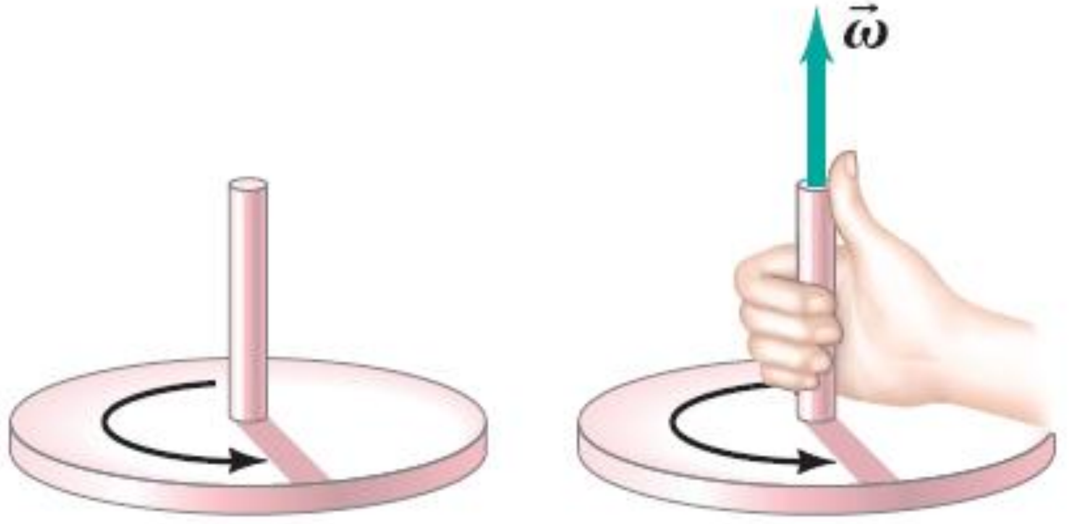
$$1\ \mathrm{Hz} = 1\ \mathrm{s}^{-1}.$$

The **period** is the time one revolution takes:

$$T = \frac{1}{f}.$$

10-2 Vector Nature of Angular Quantities

The **angular velocity** vector points along the **axis** of rotation, with the direction given by the right-hand rule. If the direction of the rotation axis does not change, the **angular acceleration** vector points along it as well.



10-3 Constant Angular Acceleration

The equations of motion for **constant** angular acceleration are the same as those for **linear** motion, with the substitution of the **angular** quantities for the **linear** ones.

Angular	Linear
$\omega = \omega_0 + \alpha t$	$v = v_0 + at$
$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$	$x = v_0 t + \frac{1}{2}at^2$
$\omega^2 = \omega_0^2 + 2\alpha\theta$	$v^2 = v_0^2 + 2ax$
$\overline{\omega} = \dfrac{\omega + \omega_0}{2}$	$\overline{v} = \dfrac{v + v_0}{2}$

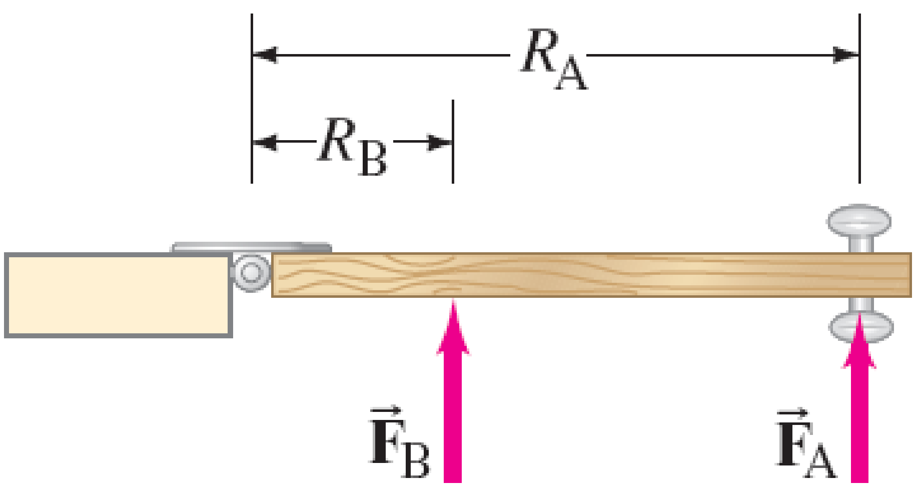
Example 10-6: Centrifuge acceleration.

A centrifuge rotor is accelerated from rest to 20,000 rpm in 30 s. (a) What is its average angular acceleration? (b) Through how many revolutions has the centrifuge rotor turned during its acceleration period, assuming constant angular acceleration?

10-4 Torque

To make an object start rotating, a force is needed; the position and direction of the force matter as well.

The perpendicular distance from the axis of rotation to the line along which the force acts is called the lever arm.



10-4 Torque

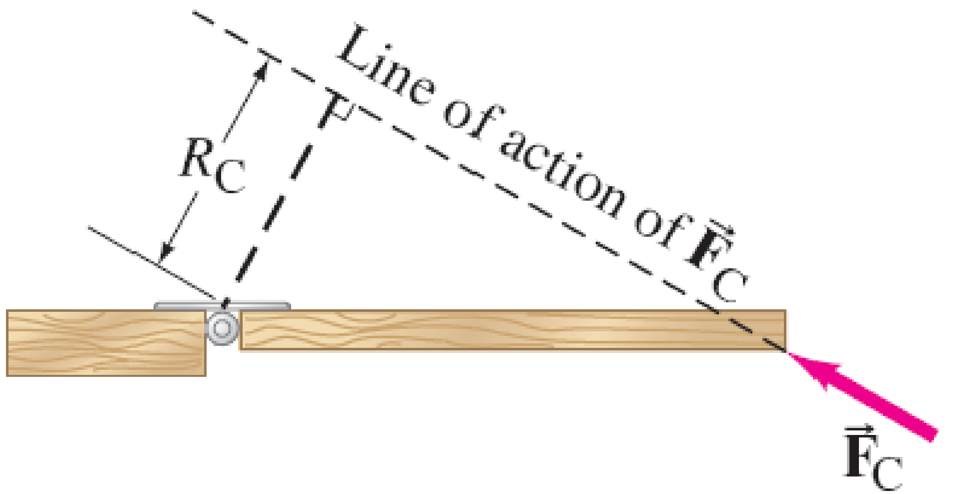
Axis of rotation

Axis of rotation

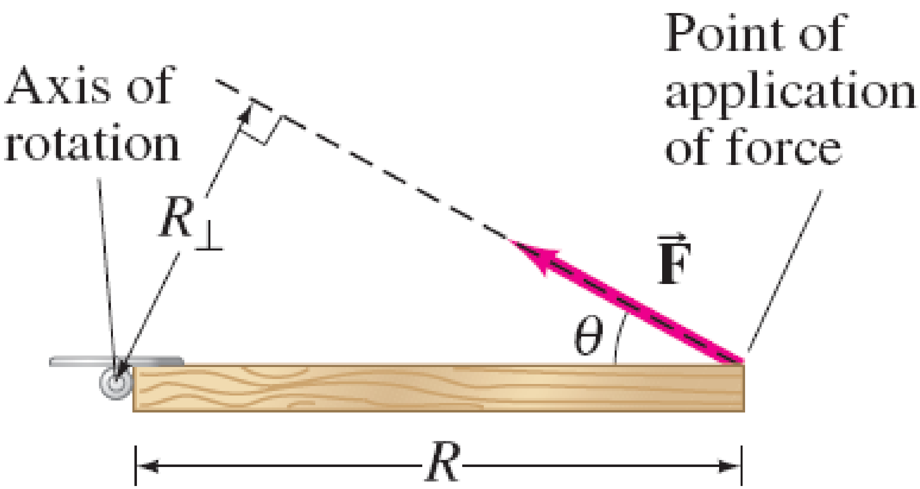
A longer lever arm is very helpful in rotating objects.

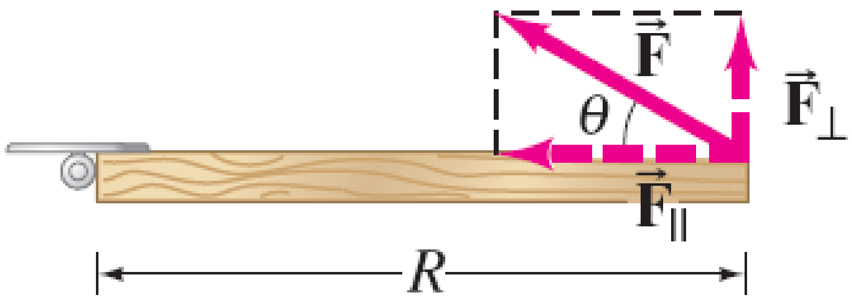
10-4 Torque

Here, the lever arm for F_A is the distance from the **knob** to the **hinge**; the lever arm for F_D is **zero**; and the lever arm for F_C is as shown.



10-4 Torque



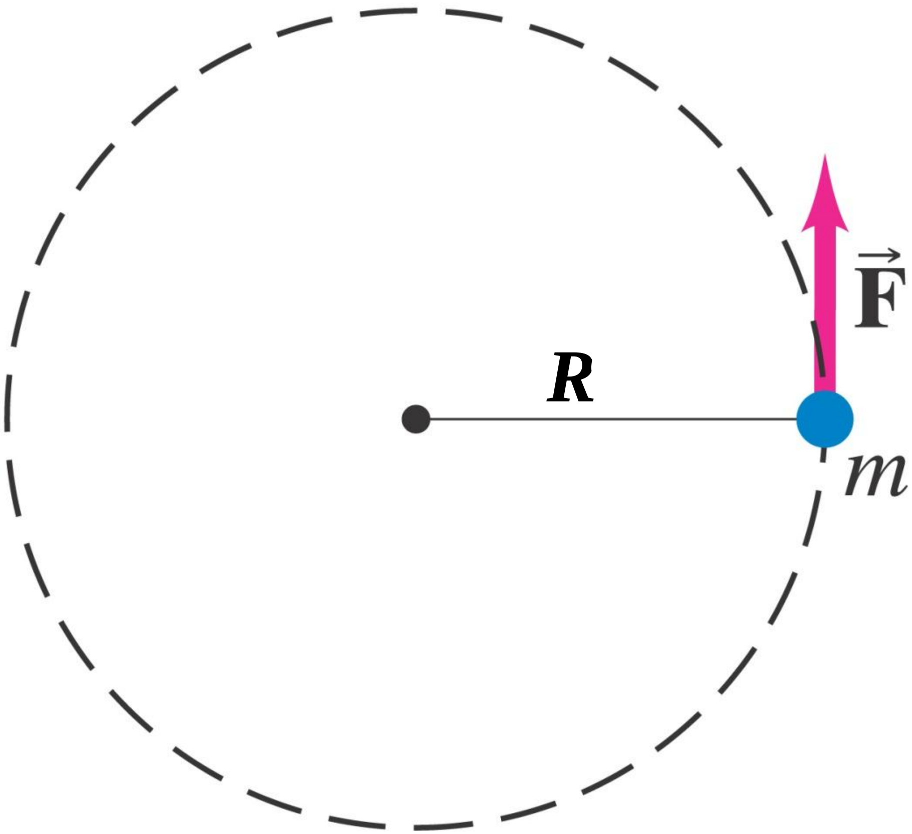


The **torque** is defined as:

$$\tau = R_\perp F.$$

10-5 Rotational Dynamics; Torque and Rotational Inertia

Knowing that $F = ma$, we see that $\tau = mR^2\alpha$.



This is for a single point mass; what about an **extended** object?

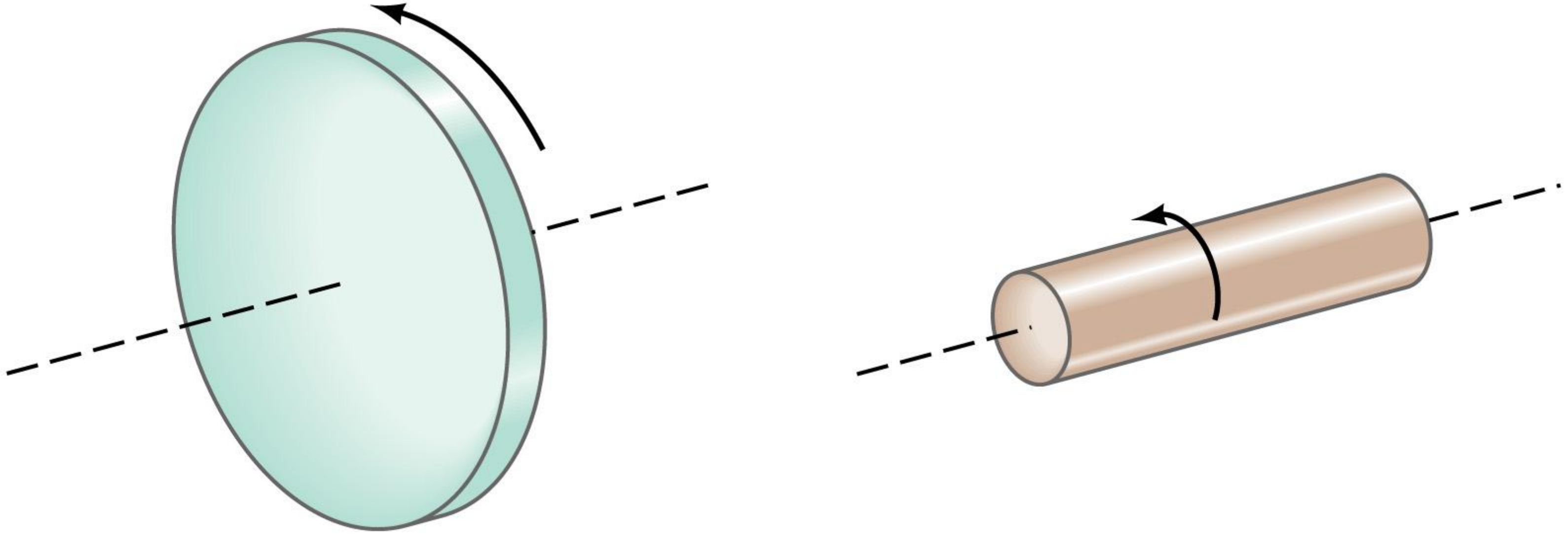
As the angular acceleration is the same for the whole object, we can write:

$$\Sigma\tau = \left(\Sigma mR^2\right)\alpha.$$

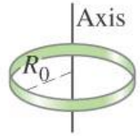
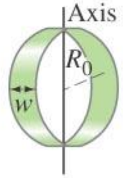
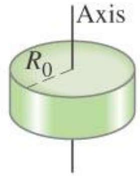
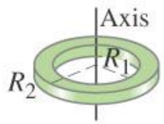
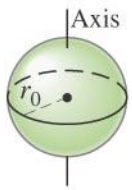
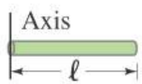
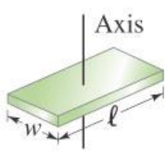
10-5 Rotational Dynamics; Torque and Rotational Inertia

The quantity $I = \Sigma m_i R_i^2$ is called the **rotational inertia** of an object.

The **distribution** of mass matters here—these two objects have the same mass, but the one on the left has a greater **rotational inertia**, as so much of its mass is far from the axis of rotation.



10-5 Rotational Dynamics; Torque and Rotational Inertia

	Object	Location of axis		Moment of inertia
(a)	**Thin hoop,** radius R_0	Through center		MR_0^2
(b)	**Thin hoop,** radius R_0 width w	Through central diameter		$\frac{1}{2}MR_0^2 + \frac{1}{12}Mw^2$
(c)	**Solid cylinder,** radius R_0	Through center		$\frac{1}{2}MR_0^2$
(d)	**Hollow cylinder,** inner radius R_1 outer radius R_2	Through center		$\frac{1}{2}M(R_1^2 + R_2^2)$
(e)	**Uniform sphere,** radius r_0	Through center		$\frac{2}{5}Mr_0^2$
(f)	**Long uniform rod,** length ℓ	Through center		$\frac{1}{12}M\ell^2$
(g)	**Long uniform rod,** length ℓ	Through end		$\frac{1}{3}M\ell^2$
(h)	**Rectangular thin plate,** length ℓ, width w	Through center		$\frac{1}{12}M(\ell^2 + w^2)$

The **rotational inertia** of an object depends not only on its **mass distribution** but also the location of the **axis** of rotation—compare (f) and (g), for example.

10-6 Solving Problems in Rotational Dynamics

1. **Draw** a diagram.
2. Decide what the **system** comprises.
3. Draw a **free-body diagram** for each object under consideration, including all the **forces** acting on it and where they act.
4. Find the **axis of rotation**; calculate the **torques** around it.

10-6 Solving Problems in Rotational Dynamics

5. Apply Newton's second law for **rotation**. If the **rotational inertia** is not provided, you need to find it **before** proceeding with this step.

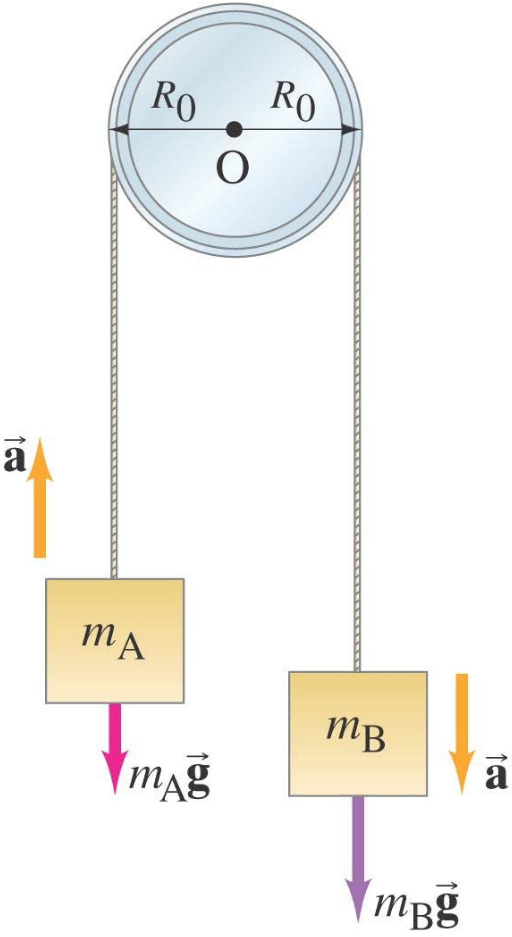
6. Apply Newton's second law for **translation** and other laws and principles as needed.

7. **Solve**.

8. **Check** your answer for units and correct order of magnitude.

Example: Atwood's machine.

An Atwood machine consists of two masses, m_A and m_B, which are connected by a cord of negligible mass that passes over a pulley. If the pulley has radius R_0 and moment of inertia I about its axle, determine the acceleration of the masses m_A and m_B.



10-8 Rotational Kinetic Energy

The **kinetic energy** of a rotating object is given by

$$K = \Sigma\left(\tfrac{1}{2}mv^2\right).$$

By substituting the rotational quantities, we find that the rotational kinetic energy can be written:

$$\text{rotational } K = \tfrac{1}{2}I\omega^2.$$

A object that both **translational** and **rotational** motion also has both translational and rotational **kinetic energy**:

$$K = \tfrac{1}{2}Mv_{\mathrm{CM}}^2 + \tfrac{1}{2}I_{\mathrm{CM}}\omega^2.$$

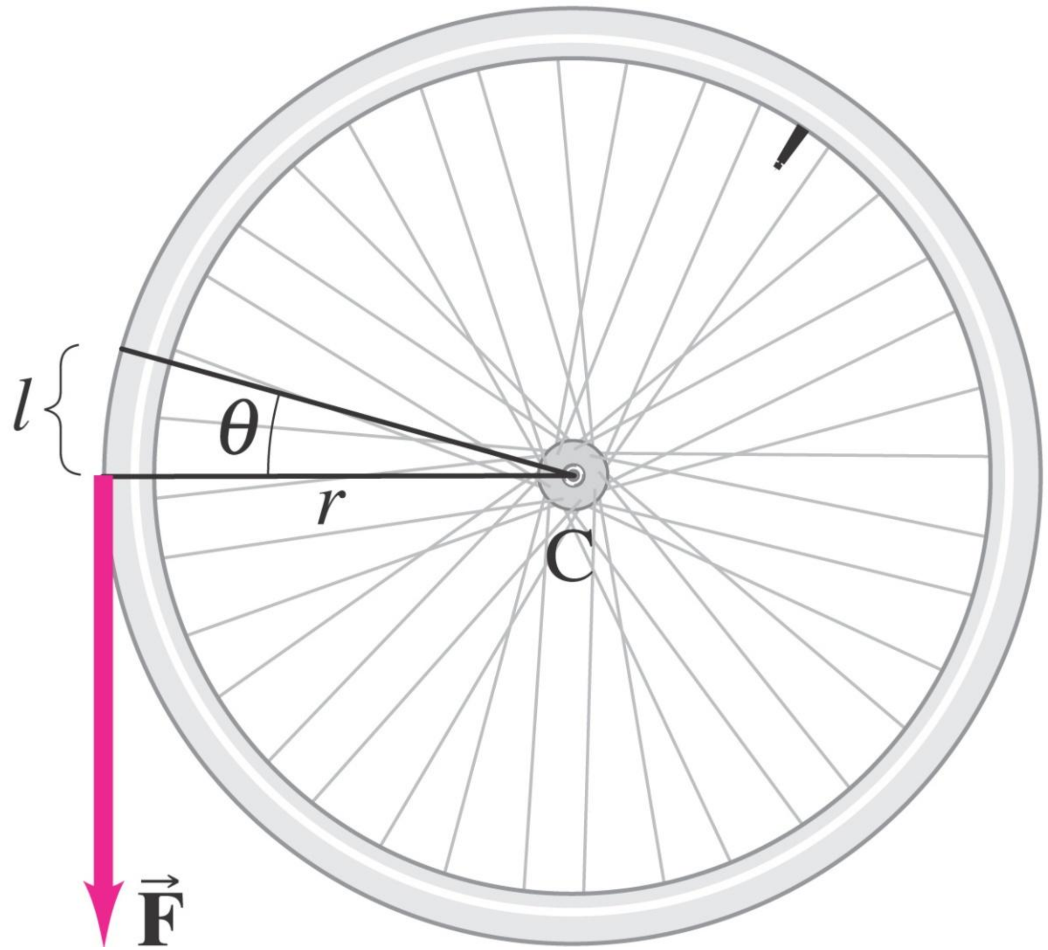
Example: old final #19

A pencil, 16 cm long, is released from a vertical position with the eraser end resting on a table. The eraser does not slip. Treat the pencil like a uniform rod.
(a) What is the angular acceleration of the pencil when it makes a 30° angle with the vertical?
(b) What is the angular speed of the pencil when it makes a 30° angle with the vertical?

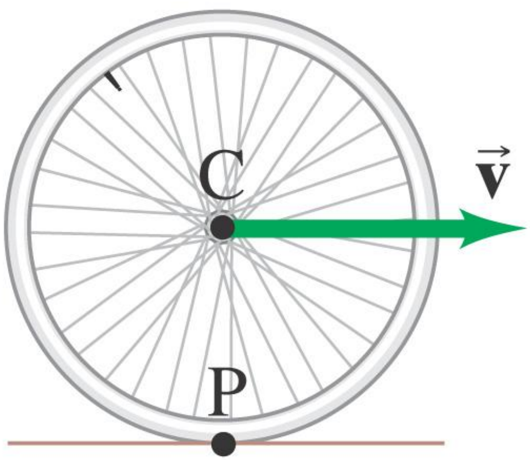
10-8 Rotational Kinetic Energy

The torque does work as it moves the wheel through an angle θ:

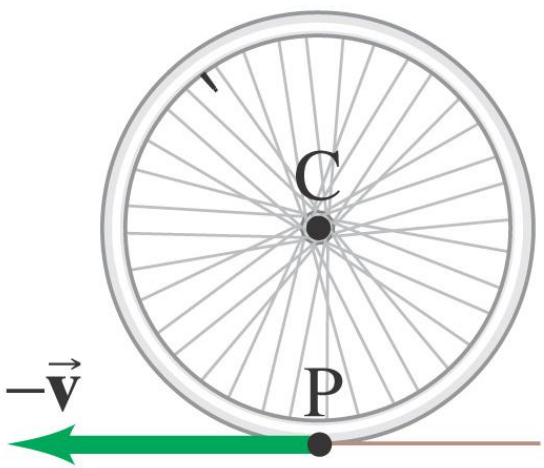
$$W = \tau \Delta \theta.$$



Rotational Plus Translational Motion; Rolling



In (a), a wheel is **rolling** without slipping. The point P, touching the ground, is instantaneously at **rest**, and the center moves with velocity $\vec{\mathbf{v}}$.



In (b) the same wheel is seen from a **reference frame** where C is at rest. Now point P is moving with velocity $-\vec{\mathbf{v}}$.

The **linear speed** of the wheel is related to its **angular speed**:

$$v = R\omega.$$

Demo: Which one reaches the bottom first? Why?