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Market demand curves and Dupuit–Marshall consumers' surpluses: a general equilibrium analysis

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Abstract

This paper investigates a claim of Hicks that the area to the left of a *market demand curve*—calculated by assuming that other markets are in equilibrium—is equal to the aggregate overall Dupuit–Marshall consumers' surplus together with producers' surplus in other markets, and extends it to general equilibrium in an economy with a competitive private sector. The usefulness of the result is questioned by considering path independence and the possibility of consistency with a Bergson–Samuelson social-welfare function. In the only interesting possibility, there must be *no* income effects on non-numéraire goods for each person. © 1999 Elsevier Science B.V. All rights reserved.

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[*P*]*olitical economy, being concerned only with wealth, can take account of the intensity of a wish only through its monetary expression. Political economy only bakes bread for those who can buy it, and leaves to social economy the care of supplying it to those with nothing of value to give in exchange.* Dupuit (1969) *p.* 262

This involves the consideration that a pound's worth of satisfaction to an ordinary poor man is a much greater thing than a pound's worth of satisfaction to an ordinary rich man ... In earlier generations, many statesmen, and even some economists, neglected to make adequate allowance for considerations of this class, especially when constructing schemes of taxation; and their words or deeds seemed to imply a want of sympathy with the sufferings of the poor, though more often they were simply due to lack of thought. Marshall (1920) *p.* 108

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1. Introduction

In consumer's surplus analysis, an interesting argument, originally due to Hicks (1946a), (1946b), is used to justify the use of a kind of partial-equilibrium analysis. It is most clearly stated in Sugden and Williams (1978). Suppose that an economy consists of an undistorted competitive private sector and a government sector that produces marketed goods. If the government increases its production of one good, price is changed in that market, but price and quantity changes in other markets are also induced. The argument claims that the area to the left of the *market demand curve* in the original market is equal to the sum of Dupuit–Marshall consumers' surpluses in all markets plus producers' surpluses in the other markets. The market demand curve is the one that obtains when all other markets clear on the transition path (see Section 2 for a discussion). Adding this surplus to the change in government-sector profit provides a cost-benefit test.

We show that this argument is true in general equilibrium. As in Hicks and Sugden/Williams, price changes are generated by changing a public-sector input-output vector. In principle, all prices in the economy could change. The area to the left of the market demand curve(s) (translated into a general-equilibrium setting) is equal to the sum across all consumers of the sum of Dupuit–Marshall surpluses and producers' surpluses in all markets. In general equilibrium, the Dupuit–Marshall surpluses include changes in consumers' (lump-sum or full) incomes. In addition, we show that project profitability at the simple average of before-project and after-project prices approximates the cost-benefit test.

Having established this result, we turn to its interpretation and usefulness. It is well known that, when more than one price changes, Dupuit–Marshall surpluses suffer from a path-dependency problem (Silberberg, 1972; Chipman and Moore, 1976). That is, the value of the surplus can be different depending on the path of integration. Accordingly, we investigate the possibilities for path independence. In addition, we ask whether the aggregate surplus is consistent with a Bergson–Samuelson social-welfare function. If there is only one consumer, this requires the surplus to accord with his or her well-being.

We study four cases which differ in the way prices and incomes are normalized. First, we allow prices and incomes to be unrestricted, and show that it is not possible to have path independence. As a consequence, it is impossible for the surplus to be consistent with a single consumer's preferences or, in the many-consumer case, with a social-welfare function.

The second case uses normalized prices—prices divided by aggregate income. In the single-consumer case, path independence and consistency with individual preferences requires homotheticity (Silberberg, 1972; Chipman and Moore, 1976). In the many-consumer case, we show that there must be an aggregate consumer with homothetic preferences. This requires individuals to have parallel quasi-homothetic preferences with a restriction across consumers that ensures a homothetic aggregate.

The third normalization requires prices to sum to one. As in the first case, this leads to an impossibility.

The fourth and last normalization sets the price of a numéraire good to one. This implies that there must be no income effects for non-numéraire goods. Therefore, the requirement for individual preferences is different for different choices of numéraire.

Thus, there are two circumstances in which Hicks's result has normative significance. But these two possibilities are not encouraging. We live in many-consumer economies and we cannot reasonably expect to find preferences that support an aggregate consumer, nor can we bargain on the absence of income effects on all but the numéraire good for all consumers. In addition, these two possibilities force the evaluator to be indifferent to income inequality, a standard but ethically unattractive feature of most cost-benefit analysis.

2. Partial equilibrium

Before presenting our main general equilibrium result, we discuss the problem in partial equilibrium. Following Sugden and Williams (1978), suppose that the government operates a rail service between a city and one of its suburbs, as illustrated in Fig. 1. A project is planned which will reduce the price of rail trips from p_r^b to p_r^a (b stands for 'before', a stands for 'after'). Cheaper transportation results in increased demand in the (competitive) suburban rental housing market, shifting the demand curve to the right—from $D_h(p_r^b, p_h)$ to $D_h(p_r^a, p_h)$. This results in an increase in the price of housing from p_h^b to p_h^a which, in turn, shifts the demand curve for rail trips to the left—from $D_r(p_r, p_h^b)$ to $D_r(p_r, p_h^a)$. The initial equilibrium is at C and K, and the equilibrium after the change is at E and I.

Dupuit–Marshall consumers' surpluses can be measured in many different ways. If consumers' surplus in the rail market is calculated first, it is $\hat{s}_r^c = \text{ACFD}$. Once p_r has changed, the relevant housing demand curve is $D_h(p_r^a, p_h)$ and consumers' surplus in the housing market is $\hat{s}_h^c = -\text{GILJ}$. Producers' surplus (the increase in landlords' profit) in the housing market is $s_h^p = \text{GIKJ}$, and consumers' surplus plus producers' surplus in housing is

$$\hat{s}^c + s_h^p := \hat{s}_r^c + \hat{s}_h^c + s_h^p = \text{ACFD} - \text{GILJ} + \text{GIKJ} = \text{ACFD} - \text{ILK}. \tag{1}$$

Alternatively, the consumers' surplus in housing can be computed first, and, in this case, it is $\tilde{s}_h^c = -\text{GHKJ}$, the area to the left of the original demand curve $D_h(p_r^b, p_h)$. The relevant demand curve for the rail-market surplus becomes $D_r(p_r, p_h^a)$, and consumers' surplus is $\tilde{s}_r^c = \text{ABED}$. Using this method, consumers' surplus plus producers' surplus in housing is

$$\tilde{s}^c + s_h^p := \tilde{s}_r^c + \tilde{s}_h^c + s_h^c = \text{ABED} - \text{GHKJ} + \text{GIKJ} = \text{ABED} + \text{HIK}. \tag{2}$$

A third method for calculating the surpluses can be obtained by considering a price path which reduces the price of rail trips, keeping the housing market in equilibrium. When this is done, price and quantity in the rail market move along the market demand curve $D_r^M(p_r)$. The claim made by Hicks (1946a), (1946b) and by Sugden and Williams (1978) is that the area ACED to the left of the market demand curve is equal to the overall consumers' surplus plus producers' surplus in the housing market. This can be seen intuitively by thinking of the change in p_r as a sequence of very small changes. This makes the 'corrections' −ILK and HIK in (1) and (2) very small, and the sum of consumers' surplus and producers' surplus approaches ACED, the shaded area in Fig. 1.

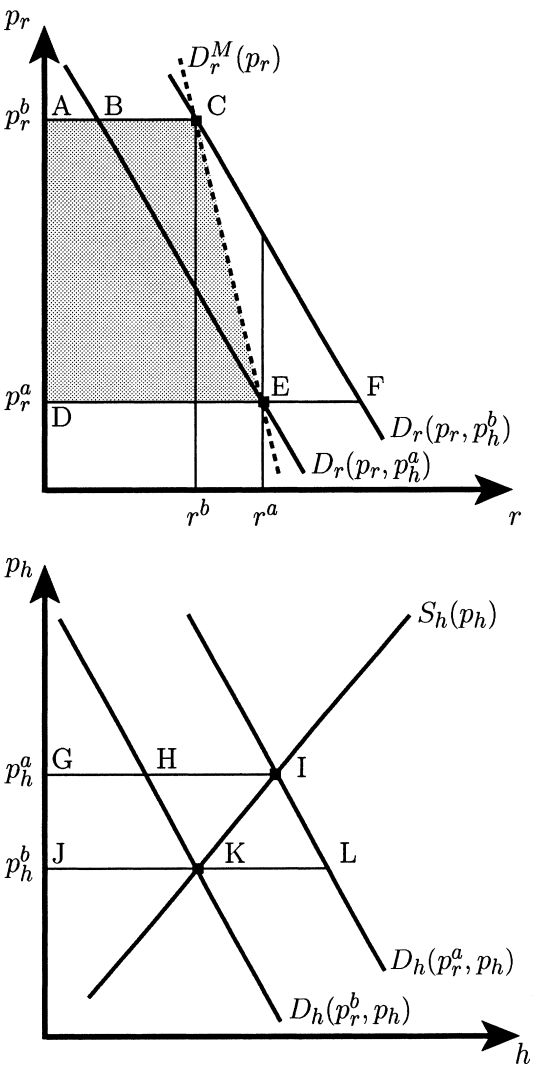


Fig. 1. Market demand curves and Dupuit–Marshall consumers' surpluses.

In general, these three numbers are not the same; in order for them to coincide, the consumers' surpluses must be *path independent*, a requirement that is not often satisfied.[1] If path independence is satisfied, the consumers' surplus is well defined for all price paths.

To prove the result mathematically, we define the Dupuit–Marshall consumers' surplus as a line integral along a continuous price path from (p_r^b, p_h^b) to (p_r^a, p_h^a) given by

[1] See Silberberg (1972), Chipman and Moore (1976) and Section 4.

the functions $P_r: [0,1] \mapsto \mathscr{R}$ and $P_h: [0,1] \mapsto \mathscr{R}$ with $(p_r^b, p_h^b) = (P_r(0), P_h(0))$ and $(p_r^a, p_h^a) = (P_r(1), P_h(1))$. Then

$$s^c = -\int_{(p_r^b, p_h^b)}^{(p_r^a, p_h^a)} [D_r(p_r, p_h)\mathrm{d}p_r + D_h(p_r, p_h)\mathrm{d}p_h]$$

$$= -\int_0^1 [D_r(P_r(t), P_h(t))\mathrm{d}P_r(t) + D_h(P_r(t), P_h(t))\mathrm{d}P_h(t)]. \tag{3}$$

If the functions P_r and P_h are differentiable, then (3) can be written as

$$s^c = -\int_0^1 [D_r(P_r(t), P_h(t))P_r'(t) + D_h(P_r(t), P_h(t))P_h'(t)]\mathrm{d}t. \tag{4}$$

Producers' surplus in the housing market is

$$s_h^p = \int_{p_h^b}^{p_h^a} S_h(p_h)\mathrm{d}p_h = \int_0^1 S_h(P_h(t))\mathrm{d}P_h(t), \tag{5}$$

where S_h is the housing supply function. It follows that the sum of overall consumers' surplus and producers' surplus in the housing market is given by

$$s^c + s_h^p = -\int_0^1 D_r(P_r(t), P_h(t))\mathrm{d}P_r(t) + \int_0^1 [-D_h(P_r(t), P_h(t)) + S_h(P_h(t))]\mathrm{d}P_h(t). \tag{6}$$

Along the market equilibrium path, $D_h(P_r^M(t), P_h^M(t)) = S_h(P_h^M(t))$, where $P_r^M(t)$ and $P_h^M(t)$ are equilibrium or market prices and, defining s_m^c as the overall consumers' surplus along the market-equilibrium path,

$$s_m^c + s_h^p = -\int_0^1 D_r(P_r^M(t), P_h^M(t))\mathrm{d}P_r^M(t)$$

$$=: -\int_{p_r^b}^{p_r^a} D_r^M(p_r)\mathrm{d}p_r = \mathrm{ACED}. \tag{7}$$

A cost-benefit test can be performed by adding the consumers' surplus ACED to the change in government profit that results from the project.

A path-independent example is provided by the demand functions[2]

[2]These demand functions can be rationalized by the utility function $U(r,h) = \min\{2r, h\}$ with income equal to 300.

$$D_r(p_r,p_h)=\frac{300}{p_r+2p_h},\quad D_h(p_r,p_h)=\frac{600}{p_r+2p_h}, \tag{8}$$

and the supply function

$$S_h(p_h)=20, \tag{9}$$

with $p_r^b=10$ and $p_r^a=5$. The decrease in p_r induces an increase in p_h from $p_h^b=10$ to $p_h^a=12.5$.

Then, computing consumers' surplus in the rail market first,

$$\begin{aligned}\hat{s}^c+s_h^p&=-\int_{10}^{5}\frac{300}{p_r+20}\mathrm{d}p_r-\int_{10}^{12.5}\frac{600}{5+2p_h}\mathrm{d}p_h+\int_{10}^{12.5}20\,\mathrm{d}p_h\\&=[300\ln(p_r+20)]_5^{10}-[300\ln(5+2p_h)]_{10}^{12.5}+[20p_h]_{10}^{12.5}\\&=300\ln\frac{30}{25}-300\ln\frac{30}{25}+50=50.\end{aligned} \tag{10}$$

When consumers' surplus in housing is calculated first,

$$\begin{aligned}\tilde{s}^c+s_h^p&=-\int_{10}^{12.5}\frac{600}{10+2p_h}\mathrm{d}p_h-\int_{10}^{5}\frac{300}{p_r+25}\mathrm{d}p_r+\int_{10}^{12.5}20\,\mathrm{d}p_h\\&=-[300\ln(10+2p_h)]_{10}^{12.5}+[300\ln(p_r+25)]_5^{10}+[20p_h]_{10}^{12.5}\\&=-300\ln\frac{35}{30}+300\ln\frac{35}{30}+50=50.\end{aligned} \tag{11}$$

For any value of p_r, equilibrium in the housing market requires

$$\frac{600}{p_r+2p_h}=20, \tag{12}$$

so

$$p_h=15-\frac{p_r}{2}, \tag{13}$$

and the market demand curve is given by

$$D_r^M(p_r)=\frac{300}{p_r+30-p_r}=10. \tag{14}$$

It follows that

$$s_m^c+s_h^p=-\int_{10}^{5}10\,\mathrm{d}p_r=[10p_r]_5^{10}=50, \tag{15}$$

which is the area to the left of the market demand curve.

A path-dependent example is provided by the demand functions[3]

$$D_r(p_r,p_h)=\frac{5p_h}{p_r}+5,\quad D_h(p_r,p_h)=\frac{300-5p_r}{p_h}-5, \tag{16}$$

and

$$S(p_h)=20, \tag{17}$$

with $p_r^b=10$ and $p_r^a=5$. p_h increases from $p_h^b=10$ to $p_h^a=11$.

Calculating the rail-market surpluses first,

$$\begin{aligned}\hat{s}^c+s_h^p&=-\int_{10}^{5}\left(\frac{5p_h^b}{p_r}+5\right)dp_r-\int_{10}^{11}\left(\frac{300-5p_r^a}{p_h}-5\right)dp_h+\int_{10}^{11}20\,dp_h\\&=-\int_{10}^{5}\left(\frac{50}{p_r}+5\right)dp_r-\int_{10}^{11}\left(\frac{275}{p_h}-5\right)dp_h+\int_{10}^{11}20\,dp_h\\&=59.66-21.21+20=58.45.\end{aligned} \tag{18}$$

If the housing-market surplus is computed first,

$$\begin{aligned}\tilde{s}^c+s_h^p&=-\int_{10}^{11}\left(\frac{300-5p_r^b}{p_h}-5\right)dp_h-\int_{10}^{5}\left(\frac{5p_h^a}{p_r}+5\right)dp_r+\int_{10}^{11}20\,dp_h\\&=-\int_{10}^{11}\left(\frac{250}{p_h}-5\right)dp_h-\int_{10}^{5}\left(\frac{55}{p_r}+5\right)dp_r+\int_{10}^{11}20\,dp_h\\&=-18.83+63.12+20=64.29.\end{aligned} \tag{19}$$

Equilibrium in the housing market requires, for any value of p_r,

$$\frac{300-5p_r}{p_h}-5=20, \tag{20}$$

so

$$p_h=12-\frac{p_r}{5}, \tag{21}$$

and the market demand curve is given by

$$D_r^M(p_r)=\frac{60}{p_r}+4. \tag{22}$$

The area to the left of the market demand curve is

[3]These demand functions can be rationalized by the utility function $U(r,h)=5\ln(r-5)+h$ with income equal to 300.

$$s_{\mathrm{m}}^{\mathrm{c}} + s_{\mathrm{h}}^{\mathrm{p}} = -\int_{10}^{5}\left(\frac{60}{p_{\mathrm{r}}} + 4\right)\mathrm{d}p_{\mathrm{r}} = [60 \ln p_{\mathrm{r}} + 4p_{\mathrm{r}}]_{5}^{10} = 61.59. \tag{23}$$

Note that all three answers are different because of path dependence.

There is an additional important difficulty with the above result. It is that, in general, the demand curves will shift as prices change because producers' surplus $s_{\mathrm{h}}^{\mathrm{p}}$ and the change in net revenue from the project are changes in income. Consequently, the above argument *requires* there to be no income effects in both markets. If there are only two markets, this cannot be true because budget constraints must be satisfied. Consequently, if all prices and incomes change, the consumers' surplus calculated above cannot be correct.

3. General equilibrium: the basic result

Suppose that an economy consists of two sectors—a competitive private sector and a public sector. Only private goods are produced, $m \geq 2$ of them; $\boldsymbol{p} \in \mathscr{R}_{++}^{m}$ is the price vector.

The private-sector technology is described by the profit function Π: $\mathscr{R}_{++}^{m} \mapsto \mathscr{R}_{+}$. Π is homogeneous of degree one, and we assume that it is differentially strongly convex (Diewert et al., 1981). We employ the usual convention that inputs are measured negatively.[4] Using Hotelling's theorem, $\Pi_j(\boldsymbol{p})$, if positive, is the private-sector supply of good j. If $\Pi_j(\boldsymbol{p})$ is negative, it is equal to minus the demand for an input. There are $H \geq 1$ individuals, and V^h: $\mathscr{R}_{++}^{m+1} \mapsto \mathscr{R}$ is person h's indirect utility function. Each V^h is decreasing in p, increasing in y, homogeneous of degree zero, and we assume that each is differentially strongly quasi-convex in p for each y.[5] D_j^h is person h's demand function for good j, $h = 1, \ldots, H$, $j = 1, \ldots, m$, and, using Roy's theorem, is given by

$$D_j^h(\boldsymbol{p}, y^h) = -\frac{V_j^h(\boldsymbol{p}, y^h)}{V_y^h(\boldsymbol{p}, y^h)}. \tag{24}$$

The government sector's input–output vector is $\boldsymbol{g} \in \mathscr{R}^m$. It may be constrained to lie in a production set but, because we use consumers' surplus analysis to search for social improvements, we do not assume that it is chosen optimally. Profit from government operations is $\boldsymbol{p} \cdot \boldsymbol{g}$ and it is paid out to consumers as lump-sum transfers (taxed from them if negative). Endowments are $\omega^h \in \mathscr{R}_{+}^{m}$, $h = 1, \ldots, H$, and $\omega = \Sigma_{h=1}^{H} \omega^h$ is the aggregate endowment. ρ^h is person h's profit share, and $\Sigma_{h=1}^{H} \rho^h = 1$. Similarly, σ^h is person h's share of government profit with $\Sigma_{h=1}^{H} \sigma^h = 1$. The (lump-sum or full) income of person h is y^h and it is equal to

$$y^h = \rho^h \Pi(\boldsymbol{p}) + \boldsymbol{p} \cdot \omega^h + \sigma^h \boldsymbol{p} \cdot \boldsymbol{g}. \tag{25}$$

[4] Given that the economy is assumed to be competitive the assumption of one firm entails no loss of generality.
[5] For details and the entailed restrictions on the expenditure function see Blackorby et al. (1978).

$\boldsymbol{y} = (y^1, \ldots, y^H)$ is the economy's income vector and $y = \Sigma_{h=1}^{H} y^h$ is aggregate income, with

$$y = \sum_{h=1}^{H} y^h = \Pi(\boldsymbol{p}) + \boldsymbol{p} \cdot \omega + \boldsymbol{p} \cdot \boldsymbol{g}. \tag{26}$$

Equilibrium requires, for each $j = 1, \ldots, m$,

$$D_j(\boldsymbol{p},\boldsymbol{y}) := \sum_{h=1}^{H} D_j^h(\boldsymbol{p},y^h) = \Pi_j(\boldsymbol{p}) + \omega_j + g_j. \tag{27}$$

We consider a public project that moves the public input–output vector $\boldsymbol{g}$ from $\boldsymbol{g}^{\mathrm{b}}$ (before) to $\boldsymbol{g}^{\mathrm{a}}$ (after). A path of integration is described by the continuous functions $\boldsymbol{G}: [0,1] \mapsto \mathscr{R}^m$, with $\boldsymbol{g}^{\mathrm{b}} = \boldsymbol{G}(0)$ and $\boldsymbol{g}^{\mathrm{a}} = \boldsymbol{G}(1)$, and $\boldsymbol{P}: [0,1] \mapsto \mathscr{R}^m$, with $\boldsymbol{p}^{\mathrm{b}} = \boldsymbol{P}(0)$ and $\boldsymbol{p}^{\mathrm{a}} = \boldsymbol{P}(1)$. Equilibrium prices corresponding to $\boldsymbol{g}^{\mathrm{b}}$ and $\boldsymbol{g}^{\mathrm{a}}$ are $\boldsymbol{p}^{\mathrm{b}}$ and $\boldsymbol{p}^{\mathrm{a}}$. The income vector along the paths $\boldsymbol{G}$ and $\boldsymbol{P}$ is $\boldsymbol{Y}(t) = (Y^1(t), \ldots, Y^H(t))$, $t \in [0,1]$, and is given by (25) evaluated at $\boldsymbol{g} = \boldsymbol{G}(t)$ and $\boldsymbol{p} = \boldsymbol{P}(t)$ with $\boldsymbol{y}^{\mathrm{b}} = \boldsymbol{Y}(0)$ and $\boldsymbol{y}^{\mathrm{a}} = \boldsymbol{Y}(1)$. Aggregate income along the paths is $Y(t) = \Sigma_{h=1}^{H} Y^h(t)$ with $Y(0) = y^{\mathrm{b}}$ and $Y(1) = y^{\mathrm{a}}$, and is given by (26) evaluated at $\boldsymbol{g} = \boldsymbol{G}(t)$ and $\boldsymbol{p} = \boldsymbol{P}(t)$.

The Hicksian partial-equilibrium claim discussed in Section 2 above can be demonstrated in this general equilibrium model. The aggregate Dupuit–Marshall consumers' surplus, allowing for income change, is

$$s_{\mathrm{ba}} = \int_{(\boldsymbol{p}^{\mathrm{b}},\boldsymbol{y}^{\mathrm{b}})}^{(\boldsymbol{p}^{\mathrm{a}},\boldsymbol{y}^{\mathrm{a}})} \left[-\sum_{j=1}^{m} D_j(\boldsymbol{p},\boldsymbol{y})\mathrm{d}p_j + \mathrm{d}y \right]$$

$$= \int_0^1 \left[-\sum_{j=1}^{m} D_j(\boldsymbol{P}(t),\boldsymbol{Y}(t))\mathrm{d}P_j(t) + \mathrm{d}Y(t) \right]. \tag{28}$$

It is equal to the sum of the individual consumer's surpluses $s_{\mathrm{ba}}^1, \ldots, s_{\mathrm{ba}}^H$, and

$$s_{\mathrm{ba}} = \sum_{h=1}^{H} s_{\mathrm{ba}}^h = \sum_{h=1}^{H} \int_{(\boldsymbol{p}^{\mathrm{b}},\boldsymbol{y}^{\mathrm{b}})}^{(\boldsymbol{p}^{\mathrm{a}},\boldsymbol{y}^{\mathrm{a}})} \left[-\sum_{j=1}^{m} D_j^h(\boldsymbol{p},y^h)\mathrm{d}p_j + \mathrm{d}y^h \right]$$

$$= \sum_{h=1}^{H} \int_0^1 \left[-\sum_{j=1}^{m} D_j^h(\boldsymbol{P}(t),Y^h(t))\mathrm{d}P_j(t) + \mathrm{d}Y^h(t) \right]. \tag{29}$$

Theorem 1 shows that this consumers' surplus can be computed in another way in an undistorted competitive economy: one in which all agents are price takers, producer and consumer prices are equal, and all taxes and/or transfers are lump-sum. The general equilibrium analogue to the market-demand-curve surplus of (7) is computed along a particular path where all markets are in equilibrium. Along a market-equilibrium path, $\boldsymbol{P}(t) = \boldsymbol{P}^{\mathrm{M}}(t)$ and $\boldsymbol{Y}(t) = \boldsymbol{Y}^{\mathrm{M}}(t) = (Y^{1\mathrm{M}}(t), \ldots, Y^{H\mathrm{M}}(t))$, where

$$D_j(\boldsymbol{P}^{\mathrm{M}}(t),\boldsymbol{Y}^{\mathrm{M}}(t)) = \sum_{h=1}^{H} D_j^h(\boldsymbol{P}^{\mathrm{M}}(t),Y^{hM}(t))$$

$$= \Pi_j(\boldsymbol{P}^{\mathrm{M}}(t)) + \omega_j + G_j(t), \tag{30}$$

$j = 1, \ldots, m$, for all $\boldsymbol{G}(t)$, $t \in [0,1]$. It is possible for these paths to be different even if the end points $\boldsymbol{g}^{\mathrm{b}}$ and $\boldsymbol{g}^{\mathrm{a}}$ are the same because the paths $\{\boldsymbol{G}(t) | t \in [0,1]\}$ can be different, $t \in (0,1)$.

Theorem 1. *For any market-equilibrium price path, the aggregate Dupuit–Marshall consumers' surplus (28) is equal to*

$$s_{\mathrm{ba}} = \sum_{h=1}^{H} s_{\mathrm{ba}}^h = -\int_0^1 \sum_{j=1}^{m} G_j(t) dP_j^M(t) + [\boldsymbol{p}^a \cdot \boldsymbol{g}^a - \boldsymbol{p}^b \cdot \boldsymbol{g}^b] \tag{31}$$

or, equivalently,

$$s_{\mathrm{ba}} = \sum_{h=1}^{H} s_{\mathrm{ba}}^h = \int_0^1 \sum_{j=1}^{m} P_j^M(t) dG_j(t). \tag{32}$$

Proof. Along the market-equilibrium path, using (29) and (30) and noting that $Y(t) = \Sigma_{h=1}^{H} Y^h(t)$, $Y(0) = y^{\mathrm{b}}$, and $Y(1) = y^{\mathrm{a}}$,

$$s_{\mathrm{ba}} = \sum_{h=1}^{H} s_{\mathrm{ba}}^h = \int_0^1 \left[-\sum_{j=1}^{m} D_j(\boldsymbol{P}^{\mathrm{M}}(t),\boldsymbol{Y}^{\mathrm{M}}(t)) \mathrm{d}P_j^{\mathrm{M}}(t) + \mathrm{d}Y^{\mathrm{M}}(t) \right]$$

$$= -\int_0^1 \sum_{j=1}^{m} \Pi_j(\boldsymbol{P}^{\mathrm{M}}(t)) \mathrm{d}P_j^{\mathrm{M}}(t) - \int_0^1 \sum_{j=1}^{m} \omega_j \, \mathrm{d}P_j^{\mathrm{M}}(t) - \int_0^1 \sum_{j=1}^{m} G_j(t) \mathrm{d}P_j^{\mathrm{M}}(t) + [y^{\mathrm{a}} - y^{\mathrm{b}}]$$

$$= -[\Pi(\boldsymbol{p}^{\mathrm{a}}) - \Pi(\boldsymbol{p}^{\mathrm{b}})] - [\boldsymbol{p}^{\mathrm{a}} \cdot \omega - \boldsymbol{p}^{\mathrm{b}} \cdot \omega] - \int_0^1 \sum_{j=1}^{m} G_j(t) \mathrm{d}P_j^{\mathrm{M}}(t) + [y^{\mathrm{a}} - y^{\mathrm{b}}]. \tag{33}$$

Using (26), the change in income is

$$y^{\mathrm{a}} - y^{\mathrm{b}} = [\Pi(\boldsymbol{p}^{\mathrm{a}}) - \Pi(\boldsymbol{p}^{\mathrm{b}})] + [\boldsymbol{p}^{\mathrm{a}} \cdot \omega - \boldsymbol{p}^{\mathrm{b}} \cdot \omega] + [\boldsymbol{p}^{\mathrm{a}} \cdot \boldsymbol{g}^{\mathrm{a}} - \boldsymbol{p}^{\mathrm{b}} \cdot \boldsymbol{g}^{\mathrm{b}}], \tag{34}$$

and (33) becomes

$$s_{\mathrm{ba}} = -\int_0^1 \sum_{j=1}^{m} G_j(t) \mathrm{d}P_j^{\mathrm{M}}(t) + [\boldsymbol{p}^{\mathrm{a}} \cdot \boldsymbol{g}^{\mathrm{a}} - \boldsymbol{p}^{\mathrm{b}} \cdot \boldsymbol{g}^{\mathrm{b}}], \tag{35}$$

which is (31).

Integrating (35) by parts yields

$$s_{ba} = [\boldsymbol{p}^a \cdot \boldsymbol{g}^a - \boldsymbol{p}^b \cdot \boldsymbol{g}^b] + \int_0^1 \sum_{j=1}^m P_j^M(t)\mathrm{d}G_j(t) - [\boldsymbol{p}^a \cdot \boldsymbol{g}^a - \boldsymbol{p}^b \cdot \boldsymbol{g}^b], \tag{36}$$

which yields (32). ■

It is straightforward to interpret the results of Theorem 1. Eq. (31) is the general equilibrium analogue of (7), the partial-equilibrium result. Because $\boldsymbol{p} \cdot \boldsymbol{g}$ is government-sector profit, $[\boldsymbol{p}^a \cdot \boldsymbol{g}^a - \boldsymbol{p}^b \cdot \boldsymbol{g}^b]$ is the change in profit. It results in a change in consumers' incomes. The first term is easiest to interpret if the government sector produces a single good ($j=1$) with a change in its price but no change in the prices of government inputs (other prices may change). In that case, the surplus is simply

$$-\int_0^1 G_1(t)\mathrm{d}P_1^M(t) + [\boldsymbol{p}^a \cdot \boldsymbol{g}^a - \boldsymbol{p}^b \cdot \boldsymbol{g}^b]. \tag{37}$$

The first term corresponds to the area to the left of the market demand curve—ACED in Fig. 1—and it should be added to the change in government-sector profit. The summation sign in (31) indicates that, in each market in which the government participates, consumers' surpluses should be computed for each output or input whose price changes. It is true, of course, that there may be private-sector involvement in some or all of the goods (including inputs) that correspond to the non-zero elements of $\boldsymbol{g}$. Eq. (31) indicates that this can be accounted for completely through the market equilibrium path $\{\boldsymbol{P}^M(t) | t \in [0,1]\}$.

Eq. (32) is an area *under* market demand curves. Suppose that some prices ($j \in \mathcal{F} \subseteq \{1, \ldots, m\}$) are unaffected by the change in g, then $p_j^b = p_j^a = \bar{p}_j$ for those values of j, and the corresponding terms in (32) are $\Sigma_{j \in \mathcal{F}} \bar{p}_j [g_j^a - g_j^b]$, the change in government-sector profit. If $\mathcal{F} = \{1, \ldots, m\}$, this is the usual cost-benefit test in an undistorted competitive economy. In the case where all prices can change, it is possible to approximate the path $\{\boldsymbol{P}(t), t \in [0,1]\}$ linearly with

$$P_j(t) = a_j G_j(t) + b_j, \tag{38}$$

$j = 1, \ldots, m$, $\boldsymbol{P}(0) = \boldsymbol{p}^b$, $\boldsymbol{P}(1) = \boldsymbol{p}^a$. In this case, (32) becomes

$$s_{ba} = \left[\frac{\boldsymbol{p}^b + \boldsymbol{p}^a}{2}\right] \cdot [\boldsymbol{g}^a - \boldsymbol{g}^b]. \tag{39}$$

This is the change in government-sector profit computed at the average of $\boldsymbol{p}^b$ and $\boldsymbol{p}^a$, before and after prices. Such averaging, based on partial-equilibrium arguments, is common in cost-benefit analysis and (39) provides a general-equilibrium justification.

4. Interpretation

In this section, we interpret the result of Theorem 1. Although it is interesting and potentially useful, it is important to ask whether and under what conditions the

market-demand-curve surplus is an indicator of welfare change. To do that, we investigate it for all possible paths $\{\boldsymbol{G}(t)|t\in[0,1]\}$ and the corresponding paths $\{\boldsymbol{p}=\boldsymbol{P}(t)|t\in[0,1]\}$. We do not restrict the end points $\boldsymbol{g}^{\mathrm{b}}=\boldsymbol{G}(0)$ and $\boldsymbol{g}^{\mathrm{a}}=\boldsymbol{G}(1)$ and we assume, throughout the section, that individual incomes can move independently along market-equilibrium paths.[6] It is not true that all possible price vectors $\boldsymbol{p}\in\mathscr{R}^{m}_{++}$ are in some path. The results presented below do not depend on this, however. They are local results and hold for feasible prices and incomes.

Because the demand functions are homogeneous of degree zero and because the market-clearing equations only determine relative prices, many different normalizations are possible: dividing by aggregate income, holding income equal to a constant, dividing by one price, and letting the sum of prices be constant. Of course, these normalizations are mutually exclusive and, as we shall see, each leads to a different result. Each subsection below is devoted to one of them.

We use the following result from Courant (1936) on path independence of line integrals of the form

$$\int_{\boldsymbol{x}^{\mathrm{b}}}^{\boldsymbol{x}^{\mathrm{a}}}\sum_{k=1}^{K}f^{k}(\boldsymbol{x})\mathrm{d}x_{k}=\int_{0}^{1}\sum_{k=1}^{K}f^{k}(\boldsymbol{X}(t))\mathrm{d}X_{k}(t), \tag{40}$$

$\boldsymbol{x}=(x_1,\ldots,x_K)=(X_1(t),\ldots,X_K(t))=\boldsymbol{X}(t)\in\mathscr{R}^{K}$ and $\boldsymbol{X}$: $[0,1]\mapsto\mathscr{R}^{K}$, with $\boldsymbol{X}(0)=\boldsymbol{x}^{\mathrm{b}}$ and $\boldsymbol{X}(1)=\boldsymbol{x}^{\mathrm{a}}$. If the integral in (40) is path independent, then there must be a potential function F such that it is equal to $F(\boldsymbol{x}^{\mathrm{a}})-F(\boldsymbol{x}^{\mathrm{b}})$, and it must, therefore, be true that

$$\mathrm{d}F(\boldsymbol{x})=\sum_{k=1}^{K}\frac{\partial F(\boldsymbol{x})}{\partial x_{k}}\mathrm{d}x_{k}=\sum_{k=1}^{K}f^{k}(\boldsymbol{x})\mathrm{d}x_{k} \tag{41}$$

for all $(\mathrm{d}x_1,\ldots,\mathrm{d}x_K)$. This requires

$$\frac{\partial F(\boldsymbol{x})}{\partial x_{k}}=f^{k}(\boldsymbol{x}), \tag{42}$$

$k=1,\ldots,K$, and, because

$$\frac{\partial^{2}F(\boldsymbol{x})}{\partial x_{k}\,\partial x_{l}}=\frac{\partial^{2}F(\boldsymbol{x})}{\partial x_{l}\,\partial x_{k}}, \tag{43}$$

$$\frac{\partial f^{k}(\boldsymbol{x})}{\partial x_{l}}=\frac{\partial f^{l}(\boldsymbol{x})}{\partial x_{k}} \tag{44}$$

for all x.

[6] A sufficient condition for this is that their endowment vectors are not co-linear.

4.1. No normalization

In this subsection, we normalize neither prices nor income and show that it is impossible for the consumers' surplus s_{ba}, described by (31) or (32), to be path independent for either a one- or a many-consumer economy. Theorem 2 was proved by Chipman and Moore (1976) for one consumer.

Theorem 2. *For any $H \geq 1$, there are no well-behaved preferences such that s_{ba}, given by (28) or (29), is path independent.*

Proof. Using (27), (28) and (29), the surplus can be rewritten as

$$s_{\mathrm{ba}} = \int_{(\boldsymbol{p}^{\mathrm{b}},y^{\mathrm{b}})}^{(\boldsymbol{p}^{\mathrm{a}},y^{\mathrm{a}})} \left[-\sum_{j=1}^{m}\sum_{h=1}^{H} D_j^h(\boldsymbol{p},y^h)\mathrm{d}p_j + \sum_{h=1}^{H} \mathrm{d}y^h \right]. \tag{45}$$

Path independence requires

$$\frac{\partial}{\partial y^h}\sum_{h=1}^{H} D_j^h(\boldsymbol{p},y^h) = \frac{\partial(1)}{\partial p_j} = 0, \tag{46}$$

or

$$\frac{\partial}{\partial y^h} D_j^h(\boldsymbol{p},y^h) = 0 \tag{47}$$

for all h and all j which violates the regularity conditions on preferences. ■

Path independence does not necessarily imply, however, that the aggregate consumers' surplus s_{ba} is consistent with a social-welfare function. We say that, for any $H \geq 1$, s_{ba} is *consistent with a Bergson–Samuelson social-welfare function* if and only if there exists an increasing function W: $\mathscr{R}^H \mapsto \mathscr{R}$ such that

$$s_{\mathrm{ba}} \geq 0 \leftrightarrow W(\boldsymbol{u}^{\mathrm{a}}) \geq W(\boldsymbol{u}^{\mathrm{b}}) \leftrightarrow W(u^{1\mathrm{a}},\ldots,u^{H\mathrm{a}}) \geq W(u^{1\mathrm{b}},\ldots,u^{H\mathrm{b}}) \tag{48}$$

for all alternatives B and A and all paths between them. $\boldsymbol{u}^{\mathrm{b}} = (u^{1\mathrm{b}},\ldots,u^{H\mathrm{b}})$ and $\boldsymbol{u}^{\mathrm{a}} = (u^{1\mathrm{a}},\ldots,u^{H\mathrm{a}})$ are the utility vectors in B and A.[7] If there is a single consumer, (48) requires the surplus to be consistent with preferences. If s_{ba} is consistent with a social-welfare function, it must be path independent (see Lemma 1 in Appendix A). This observation together with Theorem 2 is sufficient for Corollary 2.1.

Corollary 2.1. *For any $H \geq 1$, there are no well-behaved preferences such that s_{ba} is consistent with a Bergson–Samuelson social-welfare function.*

[7] We use a social-welfare function for clarity of exposition. The existence of a social-welfare ordering of utility vectors is sufficient.

This result suggests that, without normalizations of price(s) or income, the result of Theorem 1 is nothing more than an intellectual curiosity.

4.2. Normalized prices

In general equilibrium, only relative prices are determined and, if $\boldsymbol{p}^{\mathscr{E}}$ is an equilibrium price vector, so is $\lambda\boldsymbol{p}^{\mathscr{E}}$ for all $\lambda > 0$. We consider first a one-consumer economy in which prices along the equilibrium path can be divided by income without disturbing equilibrium to create the *normalized price path* $\{\boldsymbol{P}^{\mathrm{N}}(t)|t \in [0,1]\}$, where[8]

$$\boldsymbol{p}^{\mathrm{N}} = \boldsymbol{P}^{\mathrm{N}}(t) = (P_1^{\mathrm{N}}(t), \ldots, P_m^{\mathrm{N}}(t)) := \left(\frac{P_1(t)}{Y(t)}, \ldots, \frac{P_m(t)}{Y(t)}\right). \tag{49}$$

In this case the consumers' surplus becomes

$$s_{\mathrm{ba}}^{\mathrm{N}} = -\int_{\boldsymbol{p}^{\mathrm{Nb}}}^{\boldsymbol{p}^{\mathrm{Na}}} \sum_{j=1}^{m} D_j(\boldsymbol{p}^{\mathrm{N}},1)\mathrm{d}p_j^{\mathrm{N}}. \tag{50}$$

Theorem 3. *In a one-consumer economy, the consumer's surplus in normalized prices* s_{ba}^{N}*, given by (50), is path independent if and only if preferences are homothetic; that is,*

$$u = V(\boldsymbol{p},y) = \overset{*}{V}(\alpha(\boldsymbol{p})y) = \overset{*}{V}(\alpha(\boldsymbol{p}^{N})) \tag{51}$$

for all $(\boldsymbol{p},y) \in \mathscr{R}_{++}^{m+1}$*, where* $\alpha: \mathscr{R}_{++}^{m} \mapsto \mathscr{R}$ *is homogeneous of degree minus one and* $\overset{*}{V}$ *is increasing. In this case,*

$$\begin{aligned} s_{ba}^{N} &= ln\ \alpha(\boldsymbol{p}^{Na}) - ln\ \alpha(\boldsymbol{p}^{Nb}) \\ &= ln(\alpha(\boldsymbol{p}^{a})y^{a}) - ln(\alpha(\boldsymbol{p}^{b})y^{b}) \\ &= ln\ \overset{*}{V}^{-1}(u^{a}) - ln\ \overset{*}{V}^{-1}(u^{b}). \end{aligned} \tag{52}$$

Proof. Path independence and (50) imply

$$\frac{\partial D_j(\boldsymbol{p}^{\mathrm{N}},1)}{\partial p_k^{\mathrm{N}}} = \frac{\partial D_k(\boldsymbol{p}^{\mathrm{N}},1)}{\partial p_j^{\mathrm{N}}} \tag{53}$$

for all $j,k \in \{1, \ldots, m\}$ and all $\boldsymbol{p}^{\mathrm{N}} \in \mathscr{R}_{++}^{m}$. For any $y > 0$,

$$\frac{\partial D_j(\boldsymbol{p},y)}{\partial p_k} = \frac{\partial D_j(\boldsymbol{p}^{\mathrm{N}},1)}{\partial p_k^{\mathrm{N}}}\frac{1}{y} \tag{54}$$

for all $j,k = 1, \ldots, m$, and (53) implies

[8] Mathematically, this is equivalent to requiring that $Y(t) = 1$ for all $t \in [0,1]$. This result was proved by Silberberg (1972) and Chipman and Moore (1976).

$$\frac{\partial D_j(\boldsymbol{p},y)}{\partial p_k} = \frac{\partial D_k(\boldsymbol{p},y)}{\partial p_j} \tag{55}$$

for all $j,k \in \{1,\ldots,m\}$ and all $(\boldsymbol{p},y) \in \mathcal{R}^{m+1}_{++}$. Therefore, the cross-partial derivatives of the *uncompensated* demand functions are symmetric, which implies homotheticity and (51) (Silberberg, 1972; Chipman and Moore, 1976). α is homogeneous of degree minus one because each V^h is homogeneous of degree zero.

Given (51),

$$D_j(\boldsymbol{p}^{\mathrm{N}},1) = -\frac{\alpha_j(\boldsymbol{p}^{\mathrm{N}})}{\alpha(\boldsymbol{p}^{\mathrm{N}})}, \tag{56}$$

$j = 1,\ldots,m$. In this case, $s^{\mathrm{N}}_{\mathrm{ba}}$ becomes

$$\begin{aligned} s^{\mathrm{N}}_{\mathrm{ba}} &= \int_{\boldsymbol{p}^{\mathrm{Nb}}}^{\boldsymbol{p}^{\mathrm{Na}}} \sum_{j=1}^{m} \frac{\alpha_j(\boldsymbol{p}^{\mathrm{N}})}{\alpha(\boldsymbol{p}^{\mathrm{N}})} \mathrm{d}p^{\mathrm{N}}_j \\ &= \int_{\boldsymbol{p}^{\mathrm{Nb}}}^{\boldsymbol{p}^{\mathrm{Na}}} \mathrm{d}\ln \alpha(\boldsymbol{p}^{\mathrm{N}}) \\ &= \ln(\alpha(\boldsymbol{p}^{\mathrm{Na}})) - \ln(\alpha(\boldsymbol{p}^{\mathrm{Nb}})) \\ &= \ln(\alpha(\boldsymbol{p}^{\mathrm{a}})y^{a}) - \ln(\alpha(\boldsymbol{p}^{\mathrm{b}})y^{\mathrm{b}}) \\ &= \ln \overset{*}{V}{}^{-1}(u^{\mathrm{a}}) - \ln \overset{*}{V}{}^{-1}(u^{\mathrm{b}}). \quad \blacksquare \end{aligned} \tag{57}$$

$s^{\mathrm{N}}_{\mathrm{ba}}$ is an exact index of welfare change if and only if (51) is satisfied with normalized prices which implies path independence. Accordingly, we have

Corollary 3.1. *In a one-consumer economy, the consumer's surplus in normalized prices s^{N}_{ba}, given by (50), is an exact index of welfare change if and only if preferences are homothetic and (51) is satisfied.*

Although Theorem 3 and its corollary provide a possibility, it is a very restrictive one. Path independence requires symmetry of the cross-derivatives of the *uncompensated* demand functions, which implies homotheticity. Consequently, the utility function is given by (51). Normalizing prices is equivalent to setting income equal to one, $-\sum_{j=1}^{m} D_j(\boldsymbol{p},1)\mathrm{d}p_j = \sum_{j=1}^{m} \alpha_j(\boldsymbol{p})/\alpha(\boldsymbol{p})\mathrm{d}p_j = \mathrm{d}\ln \alpha(\boldsymbol{p})$, and the logarithmic structure of (52) results.

There is an analogue to Theorem 3 for many consumers. Equilibrium is consistent with many normalizations and one of them allows us to choose aggregate income to be equal to one. In that case, prices are normalized by dividing by aggregate income and (49) applies. We show that this implies the existence of an aggregate consumer with homothetic preferences. In addition, individual preferences must be either identical and

homothetic or quasi-homothetic and parallel. In the latter case, preferences must aggregate to overall homotheticity.

The income vector $\tilde{\boldsymbol{y}}$ is defined by

$$\tilde{\boldsymbol{y}} = \left(\frac{y^1}{Y}, \ldots, \frac{y^H}{Y}\right) \tag{58}$$

and, because $\Sigma_{h=1}^{H} \tilde{y}^h = 1$,

$$\tilde{\boldsymbol{y}} = \left(1 - \sum_{h \neq 1} \tilde{y}^h, \tilde{y}^2, \ldots, \tilde{y}^H\right). \tag{59}$$

The aggregate consumers' surplus is written as

$$s_{\mathrm{ba}}^{y} = \int_{(\boldsymbol{p}^{\mathrm{Nb}}, \tilde{\boldsymbol{y}}^{\mathrm{Nb}})}^{(\boldsymbol{p}^{\mathrm{Na}}, \tilde{\boldsymbol{y}}^{\mathrm{Na}})} \left[-\sum_{j=1}^{m} \sum_{h=1}^{H} D_j^h(\boldsymbol{p}^{\mathrm{N}}, \tilde{y}^h) \mathrm{d}p_j^{\mathrm{N}} + \sum_{h=1}^{H} \mathrm{d}\tilde{y}^h \right]$$

$$= \int_{(\boldsymbol{p}^{\mathrm{Nb}}, \tilde{\boldsymbol{y}}^{\mathrm{Nb}})}^{(\boldsymbol{p}^{\mathrm{Na}}, \tilde{\boldsymbol{y}}^{\mathrm{Na}})} \left[-\sum_{j=1}^{m} \sum_{h=1}^{H} D_j^h(\boldsymbol{p}^{\mathrm{N}}, \tilde{y}^h) \mathrm{d}p_j^{\mathrm{N}} \right]. \tag{60}$$

Theorem 4. *For $H \geq 2$, the consumers' surplus in normalized prices s_{ba}^{y}, given by (60), is path independent if and only if*

$$V^h(\boldsymbol{p}, y^h) = \overset{*}{V}{}^h(\alpha(\boldsymbol{p}) y^h + \beta^h(\boldsymbol{p})), \tag{61}$$

where α: $\mathscr{R}_{++}^{m} \mapsto \mathscr{R}$ is homogeneous of degree minus one, β^h: $\mathscr{R}_{++}^{m} \mapsto \mathscr{R}$ is homogeneous of degree zero, and $\overset{*}{V}{}^h$ is increasing, $h = 1, \ldots, H$.[9] *In addition,*

$$\sum_h \beta^h(\boldsymbol{p}) = \textit{constant}. \tag{62}$$

The surplus s_{ba}^{y} can be written as

$$s_{\mathrm{ba}}^{y} = ln\ \alpha(\boldsymbol{p}^{\mathrm{Na}}) - ln\ \alpha(\boldsymbol{p}^{\mathrm{Nb}})$$

$$= ln(\alpha(\boldsymbol{p}^{\mathrm{a}}) Y^{\mathrm{a}}) - ln(\alpha(\boldsymbol{p}^{\mathrm{b}}) Y^{\mathrm{b}}).$$

Proof. Because $\Sigma_{h=1}^{H} \mathrm{d}\tilde{y}^h = 0$, $\mathrm{d}\tilde{y}^1 = -\Sigma_{h \neq 1} \mathrm{d}\tilde{y}^h$, and (60) can be rewritten as

[9] Note that the function α is the same for each person.

$$\int_{(\boldsymbol{p}^{\mathrm{Nb}},\tilde{\boldsymbol{y}}^{\mathrm{b}})}^{(\boldsymbol{p}^{\mathrm{Na}},\tilde{\boldsymbol{y}}^{\mathrm{a}})} \left[-\sum_{j=1}^{m} \left[\sum_{h\neq 1} D_j^h(\boldsymbol{p}^{\mathrm{N}},\tilde{y}^h) + D_j^1\left(\boldsymbol{p}^{\mathrm{N}},1-\sum_{h\neq 1}\tilde{y}^h\right)\right] \mathrm{d}p_j^{\mathrm{N}} + \sum_{h\neq 1}\mathrm{d}\tilde{y}^h - \sum_{h\neq 1}\mathrm{d}\tilde{y}^h \right]. \tag{64}$$

Path independence requires, for $g \neq 1$,

$$\frac{\partial}{\partial \tilde{y}^g}\left[\sum_{h\neq 1} D_j^h(\boldsymbol{p}^{\mathrm{N}},\tilde{y}^h) + D_j^1\left(\boldsymbol{p}^{\mathrm{N}},1-\sum_{h\neq 1}\tilde{y}^h\right)\right] = 0 \tag{65}$$

or that

$$\frac{\partial}{\partial \tilde{y}^g}\left[D_j^g(\boldsymbol{p}^{\mathrm{N}},\tilde{y}^g) + D_j^1\left(\boldsymbol{p}^{\mathrm{N}},1-\sum_{h\neq 1}\tilde{y}^h\right)\right] = 0. \tag{66}$$

This in turn implies that

$$\frac{\partial}{\partial \tilde{y}^g}[D_j^g(\boldsymbol{p}^{\mathrm{N}},\tilde{y}^g)] = \frac{\partial}{\partial \tilde{y}^h}[D_j^h(\boldsymbol{p}^{\mathrm{N}},\tilde{y}^h)] \tag{67}$$

for all g and h. Consequently,

$$D_{jy}^g(\boldsymbol{p},y^g)Y = D_{jy}^h(\boldsymbol{p},y^h)Y \tag{68}$$

or

$$D_{jy}^g(\boldsymbol{p},y^g) = D_{jy}^h(\boldsymbol{p},y^h), \tag{69}$$

so that there must exist an aggregate consumer. It follows that the individual indirect utility functions can be written as in (61) (Gorman, 1953, 1961) with homogeneity properties of α and $\beta^1,\ldots,\beta^H$ implied by homogeneity of degree zero of $V^1,\ldots,V^H$. The aggregate consumer's utility function is $\mathring{V}$, with

$$\mathring{V}(\boldsymbol{p},Y) = \overset{*}{V}\left(\alpha(\boldsymbol{p})Y + \sum_{h=1}^{H}\beta^h(\boldsymbol{p})\right). \tag{70}$$

The consumers' surplus s_{ba}^y is the surplus for the aggregate consumer (in normalized prices). Path independence requires homotheticity by Theorem 3, and (62) results. The surplus is calculated as in Theorem 3. ■

If s_{ba}^y is consistent with a social-welfare function (48), path independence is implied. Corollary 4.1 follows.

Corollary 4.1. *The consumers' surplus in normalized prices s_{ba}^y is consistent with a Bergson–Samuelson social-welfare function if and only if it is path-independent and (61) and (62) are satisfied, with*

$$W(\boldsymbol{u}) = \sum_{h=1}^{H} \overset{*}{V}{}^{h^{-1}}(u^h) = \alpha(\boldsymbol{p})\left(\sum_{h=1}^{H} y^h\right) + \textit{constant}. \tag{71}$$

Corollary 4.1 indicates that the evaluator must be indifferent to inequality in the distribution of income. The result of Theorem 4 and its corollary is connected to results on social evaluation using the sum of compensating or equivalent variations in Blackorby and Donaldson (1985). Consistent aggregation is possible only when an aggregate consumer exists. As in the result above, evaluators must be indifferent to inequality in the distribution of income.

Eq. (61) requires each consumer to have quasi-homothetic preferences. Although there is a representation of the preferences of household h ($\bar{V}^h(\boldsymbol{p},y^h) = \alpha(\boldsymbol{p})y^h + \beta^h(\boldsymbol{p})$) in which the marginal utility of income is independent of the level of income, it is not a constant: because α is homogeneous of degree minus one, it must depend on at least one price. In addition, this property of the marginal utility of income is not shared by all of the (ordinally equivalent) representations of preferences.

4.3. Prices normalized to add to one

In this subsection, we normalize prices to add to one. This means that income and each price must be divided by the sum of unadjusted prices along the path of integration. Writing the resulting prices and income as $\check{\boldsymbol{p}}$ and $\check{y}^h$, with

$$\check{\boldsymbol{p}} = \frac{\boldsymbol{p}}{\sum_{i=1}^m p_i} \text{ and } \check{y}^h = \frac{y^h}{\sum_{i=1}^m p_i}, \tag{72}$$

the consumers' surplus is equal to s_{ba}^{Σ}, which is given by

$$s_{\mathrm{ba}}^{\Sigma} = \int_{(\check{\boldsymbol{p}}^{\mathrm{b}},\check{\boldsymbol{y}}^{\mathrm{b}})}^{(\check{\boldsymbol{p}}^{\mathrm{a}},\check{\boldsymbol{y}}^{\mathrm{a}})} \left[-\sum_{j=1}^m D_j(\check{\boldsymbol{p}},\check{\boldsymbol{y}})\mathrm{d}\check{p}_j + \sum_{h=1}^H \mathrm{d}\check{y}^h \right]. \tag{73}$$

Theorem 5. *For any $H \geq 1$, with prices and incomes normalized so that prices sum to one, there are no well-behaved preferences such that s_{ba}^{Σ}, given by (73), is path independent.*

Proof. Because $\Sigma_{i=1}^m \mathrm{d}\check{p}_i = 0$, writing $\tilde{\boldsymbol{p}} = (\check{p}_2, \ldots, \check{p}_m)$,

$$s_{\mathrm{ba}}^{\Sigma} = \int_{(\tilde{\boldsymbol{p}}^{\mathrm{b}},\check{\boldsymbol{y}}^{\mathrm{b}})}^{(\tilde{\boldsymbol{p}}^{\mathrm{a}},\check{\boldsymbol{y}}^{\mathrm{a}})} \left[\sum_{j=2}^m [D_1(\check{\boldsymbol{p}},\check{\boldsymbol{y}}) - D_j(\check{\boldsymbol{p}},\check{\boldsymbol{y}})]\mathrm{d}\check{p}_j + \sum_{h=1}^H \mathrm{d}\check{y}^h \right]. \tag{74}$$

Path independence requires

$$\frac{\partial}{\partial \check{y}^h} D_1(\check{\boldsymbol{p}},\check{\boldsymbol{y}}) - \frac{\partial}{\partial \check{y}^h} D_j(\check{\boldsymbol{p}},\check{\boldsymbol{y}}) = 0, \tag{75}$$

$j \neq 1$, $h = 1, \ldots, H$, and hence, using homogeneity of degree zero of the demand functions, that

$$\frac{\partial}{\partial y^h}D_1^h(\boldsymbol{p},y^h)=\frac{\partial}{\partial y^h}D_j^h(\boldsymbol{p},y^h) \tag{76}$$

for all h and all $j\neq 1$. The individual budget constraints imply that

$$\sum_{j=1}^{m}p_j\frac{\partial}{\partial y^h}D_j^h(\boldsymbol{p},y^h)=1, \tag{77}$$

and, therefore, using (76),

$$\frac{\partial}{\partial y^h}D_j^h(\boldsymbol{p},y^h)=\frac{1}{\sum_{i=1}^{m}p_i}. \tag{78}$$

Integrating,

$$D_j^h(\boldsymbol{p},y^h)=\frac{y^h}{\sum_{i=1}^{m}p_i}+\phi^{hj}(\boldsymbol{p}) \tag{79}$$

for some function ϕ^{hj}. (79) and Roy's theorem imply that

$$u^h=V^h(\boldsymbol{p},y^h)\overset{\circ}{=}\frac{y^h}{\sum_{i=1}^{m}p_i}+\psi^h(\boldsymbol{p}), \tag{80}$$

where ψ^h is homogeneous of degree zero and $\overset{\circ}{=}$ means 'is ordinally equivalent to'. Consequently, there is an aggregate consumer. The demand functions can be written as

$$D_j^h(\boldsymbol{p},y^h)=\frac{y^h}{\sum_{i=1}^{m}p_i}-\left(\sum_{i=1}^{m}p_i\right)\psi_j^h(\boldsymbol{p})=\check{y}^h-\psi_j^h(\check{\boldsymbol{p}}), \tag{81}$$

where ψ_j^h is the jth partial derivative of ψ^h. Defining $\psi(\boldsymbol{p})=\Sigma_{h=1}^{H}\psi^h(\boldsymbol{p})$,

$$D_j(\check{\boldsymbol{p}},\check{\boldsymbol{y}})=\sum_{h=1}^{H}\check{y}^h-\psi_j(\check{\boldsymbol{p}}). \tag{82}$$

It follows that

$$s_{\mathrm{ba}}^{\Sigma}=-\int_{(\check{\boldsymbol{p}}^{\mathrm{b}},\check{\boldsymbol{y}}^{\mathrm{b}})}^{(\check{\boldsymbol{p}}^{\mathrm{a}},\check{\boldsymbol{y}}^{\mathrm{a}})}\sum_{j=2}^{m}[\psi_1(\check{\boldsymbol{p}})-\psi_j(\check{\boldsymbol{p}})]\mathrm{d}\check{p}_j+\sum_{h=1}^{H}\mathrm{d}\check{y}^h, \tag{83}$$

and path independence requires, for all $j,k\neq 1$,

$$\psi_{1j}(\check{\boldsymbol{p}})-\psi_{kj}(\check{\boldsymbol{p}})=\psi_{1k}(\check{\boldsymbol{p}})-\psi_{jk}(\check{\boldsymbol{p}}), \tag{84}$$

and, therefore,

$$\psi_{1j}(\check{\boldsymbol{p}})=\psi_{1k}(\check{\boldsymbol{p}}). \tag{85}$$

Because (85) holds for all $\check{\boldsymbol{p}}$ and all $j,k\neq 1$, there exist functions δ and ϵ such that

$$\psi_1(\check{\boldsymbol{p}}) = \delta(\check{p}_1)\left(\sum_{i=2}^{m}\check{p}_i\right) + \epsilon(\check{p}_1). \tag{86}$$

Because $\Sigma_{i=1}^{m}\check{p}_i = 1$,

$$\psi_1(\check{\boldsymbol{p}}) = \delta(\check{p}_1)(1-\check{p}_1) + \epsilon(\check{p}_1), \tag{87}$$

which is a function of $\check{p}_1$ alone. Because ψ is homogeneous of degree zero, ψ_1 is homogeneous of degree minus one, and it follows that

$$\psi_1(\check{\boldsymbol{p}}) = \frac{\lambda_1}{\check{p}_1}, \tag{88}$$

where λ_1 is an arbitrary constant.

Although good one was chosen in the argument above to be the residual commodity, any good can play that role and, therefore, (88) generalizes to all goods with

$$\psi_j(\check{\boldsymbol{p}}) = \frac{\lambda_j}{\check{p}_j}, \tag{89}$$

$j = 1, \ldots, m$. Because ψ_j is homogeneous of degree minus one,

$$\psi_j(\boldsymbol{p}) = \psi_j\left(\left(\sum_{i=1}^{m}p_i\right)\check{\boldsymbol{p}}\right) = \frac{\psi_j(\check{\boldsymbol{p}})}{\sum_{i=1}^{m}p_i} = \frac{\lambda_j}{\left(\sum_{i=1}^{m}p_i\right)\check{p}_j} = \frac{\lambda_j}{p_j} \tag{90}$$

for all j and, integrating,

$$\psi(\boldsymbol{p}) = \sum_{j=1}^{m}\lambda_j \ln p_j + \gamma, \tag{91}$$

where γ is a constant. Because ψ is homogeneous of degree zero, $\Sigma_{j=1}^{m}\lambda_j = 0$.

Now consider the case $m = 2$. There is an aggregate consumer with preferences represented by

$$V(\boldsymbol{p},Y) \overset{\text{e}}{=} \frac{Y}{p_1+p_2} + \lambda \ln p_1 - \lambda \ln p_2. \tag{92}$$

If $\lambda = 0$, our curvature assumption is violated. Without loss of generality, let $\lambda > 0$. The expenditure function corresponding to V is E, with

$$E(u,\boldsymbol{p}) = (p_1+p_2)(u - \lambda \ln p_1 + \lambda \ln p_2). \tag{93}$$

Given this,

$$E_{11}(u,\boldsymbol{p}) = -\frac{\lambda(p_1-p_2)}{(p_1)^2}, \tag{94}$$

and our assumptions require it to be negative for all $(u,\boldsymbol{p})$. But, because it is positive when $p_2 > p_1$, an impossibility results. ■

It is also impossible for the consumers' surplus s_{ba}^{Σ} to be consistent with a Bergson–

Samuelson social-welfare function because that requires the integral in (73) to be path independent. This observation proves

Corollary 5.1. *For any $H \geq 1$, there are no well-behaved preferences such that s_{ba}^{Σ} is consistent with a Bergson–Samuelson social-welfare function.*

4.4. A numéraire good

It is also possible to choose some good as a numéraire, normalizing its price to one. If good k is chosen, prices and income become $\bar{p}_j = p_j/p_k$, $j = 1, \ldots, m$, and $\bar{y}^h = y^h/p_k$. Letting $\bar{\boldsymbol{y}} = (\bar{y}^1, \ldots, \bar{y}^H) = \boldsymbol{y}/p_k$, consumers' surplus is

$$\bar{s}^k_{\mathrm{ba}} = \int_{(\bar{\boldsymbol{p}}^{\mathrm{b}}_{-k}, \bar{y}^{\mathrm{b}})}^{(\bar{\boldsymbol{p}}^{\mathrm{a}}_{-k}, \bar{y}^{\mathrm{a}})} \left[-\sum_{j \neq k} D_j(\bar{\boldsymbol{p}}, \bar{\boldsymbol{y}}) \mathrm{d}\bar{p}_j + \sum_{h=1}^{H} \mathrm{d}\bar{y}^h \right], \tag{95}$$

where $\bar{\boldsymbol{p}}_{-k}$ is $\bar{\boldsymbol{p}}$ without its kth element. Theorem 6 characterizes preferences that allow $\bar{s}^k_{\mathrm{ba}}$ to be path independent.

Theorem 6. *For any $H \geq 1$, if good $k \in \{1, \ldots, m\}$ is chosen as numéraire, the consumers' surplus $\bar{s}^k_{\mathrm{ba}}$, given by (95), is path independent if and only if the individual indirect utility functions can be written as*

$$V^h(\boldsymbol{p}, y^h) = \overset{*}{V}{}^h\left(\frac{y^h + \psi^h(\boldsymbol{p})}{p_k} \right), \tag{96}$$

where ψ^h is homogeneous of degree one and $\overset{*}{V}{}^h$ is increasing, $h = 1, \ldots, H$. The surplus is given by

$$\begin{aligned} \bar{s}^k_{\mathrm{ba}} &= \sum_{h=1}^{H} \bar{s}^{kh}_{\mathrm{ba}} = \sum_{h=1}^{H} (\bar{y}^{h\mathrm{a}} + \psi^h(\bar{\boldsymbol{p}}^{\mathrm{a}})) - \sum_{h=1}^{H} (\bar{y}^{h\mathrm{b}} + \psi^h(\bar{\boldsymbol{p}}^{\mathrm{b}})) \\ &= \sum_{h=1}^{H} \left(\frac{y^{ha} + \psi^h(\boldsymbol{p}^{\boldsymbol{a}})}{p^{\mathrm{a}}_k} \right) - \sum_{h=1}^{H} \left(\frac{y^{hb} + \psi^h(\boldsymbol{p}^{\boldsymbol{b}})}{p^{\mathrm{b}}_k} \right). \end{aligned} \tag{97}$$

Proof. The surplus can be written as

$$\bar{s}^k_{\mathrm{ba}} = \int_{(\bar{\boldsymbol{p}}^{\mathrm{b}}_{-k}, \bar{y}^{\mathrm{b}})}^{(\bar{\boldsymbol{p}}^{\mathrm{a}}_{-k}, \bar{y}^{\mathrm{a}})} \left[-\sum_{j \neq k} \sum_{h=1}^{H} D^h_j(\bar{\boldsymbol{p}}, \bar{\boldsymbol{y}}) \mathrm{d}\bar{p}_j + \sum_{h=1}^{H} \mathrm{d}\bar{y}^h \right], \tag{98}$$

and path independence requires

$$\frac{\partial}{\partial \bar{y}^g} \sum_{h=1}^{H} D^h_j(\bar{\boldsymbol{p}}, \bar{\boldsymbol{y}}) = \frac{\partial}{\partial \bar{y}^g} D^g_j(\bar{\boldsymbol{p}}, \bar{y}^h) = \frac{\partial(1)}{\partial \bar{p}_j} = 0 \tag{99}$$

for all $g, j \neq k$, $\bar{\boldsymbol{p}}$, $\bar{\boldsymbol{y}}$. Therefore, for each h and each $j \neq k$,

$$\frac{\partial}{\partial y^h} D_j^h(\boldsymbol{p}, y^h) = 0, \tag{100}$$

because D_j^h is homogeneous of degree zero. By Hotelling's theorem,

$$D_j^h(\boldsymbol{p}, y^h) = \frac{\partial}{\partial p_j} E^h(V^h(\boldsymbol{p}, y^h), \boldsymbol{p}), \tag{101}$$

where E^h is the expenditure function corresponding to V^h. Eqs. (100) and (101) imply

$$E_{ju}^h(u^h, \boldsymbol{p}) = 0 \tag{102}$$

or, equivalently,

$$E_{uj}^h(u^h, \boldsymbol{p}) = 0 \tag{103}$$

for all $(u^h, \boldsymbol{p})$ and all $j \neq k$. Integrating, and taking account of the fact that (103) holds for all $j \neq k$ and that the expenditure function and its utility derivative are homogeneous of degree one in prices,

$$\begin{aligned} E_u^h(u^h, \bar{\boldsymbol{p}}) &= \bar{f}^h(u^h, p_k) \\ &= p_k \bar{f}^h(u^h, 1) \\ &=: p_k f^h(u^h) \end{aligned} \tag{104}$$

for some function f^h. Integrating again yields

$$E^h(u^h, \boldsymbol{p}) = p_k \phi^h(u^h) - \psi^h(\boldsymbol{p}) \tag{105}$$

for functions ϕ^h and ψ^h. Setting $E^h(u^h, \boldsymbol{p}) = y^h$ and inverting yields

$$V^h(\boldsymbol{p}, y^h) = \overset{*}{V}{}^h\left(\frac{y^h + \psi^h(\boldsymbol{p})}{p_k}\right), \tag{106}$$

which is (96). For each h, ψ^h must be homogeneous of degree one because V^h is homogeneous of degree zero.

Given (106),

$$D_j^h(\bar{\boldsymbol{p}}, \bar{y}^h) = -\psi_j^h(\bar{\boldsymbol{p}}), \tag{107}$$

$j \neq k$. Then, the surplus of consumer h is given by

$$\bar{s}_{\mathrm{ba}}^{kh} = \int_{(\bar{\boldsymbol{p}}_{-k}^{\mathrm{b}}, \bar{y}^{h\mathrm{b}})}^{(\bar{\boldsymbol{p}}_{-k}^{\mathrm{a}}, \bar{y}^{h\mathrm{a}})} \left[\sum_{j \neq k} \psi_j^h(\bar{\boldsymbol{p}}) \mathrm{d}\bar{p}_j + \mathrm{d}\bar{y}^h\right]$$

$$= \int_{(\bar{\boldsymbol{p}}^{\mathrm{b}}_{-k}, \bar{y}^{h\mathrm{b}})}^{(\bar{\boldsymbol{p}}^{\mathrm{a}}_{-k}, \bar{y}^{h\mathrm{a}})} [\mathrm{d}\psi^h(\bar{\boldsymbol{p}}) + \mathrm{d}\bar{y}^h]$$

$$= [\bar{y}^{h\mathrm{a}} + \psi^h(\bar{\boldsymbol{p}}^{\mathrm{a}})] - [\bar{y}^{h\mathrm{b}} + \psi^h(\bar{\boldsymbol{p}}^{\mathrm{b}})]. \tag{108}$$

Because $\bar{s}^k_{\mathrm{ba}} = \sum_{h=1}^{H} \bar{s}^{kh}_{\mathrm{ba}}$, the aggregate surplus is given by (97). ■

It follows immediately from Theorem 6 that

Corollary 6.1. *For any $H \geq 1$, the consumers' surplus $\bar{s}^k_{ba}$ is consistent with a Bergson–Samuelson social-welfare function if and only if (96) holds and the social-welfare function W is given by*

$$W(\boldsymbol{u}) \doteq \sum_{h=1}^{H} \overset{*}{V}{}^{h\,-1}(u^h). \tag{109}$$

The indirect utility functions in (96) have no income effects for non-numéraire goods. Because of this, the aggregate Dupuit–Marshall consumers' surplus is the same as the aggregate of the Hicksian compensating and equivalent variations. In addition, as in Section 4.2, there is a representation of each household's preferences $(\tilde{V}^h(\boldsymbol{p}, y^h) = (y^h + \psi^h(\boldsymbol{p}))/p_k)$ such that the marginal utility of income is independent of the level of income. But it is not constant because it depends on p_k. This property is *not* shared by all of the representations of households' preferences. It is true, of course, that the Dupuit–Marshall consumers' surpluses depend on household demand functions and, because of this, are unaffected by applications of increasing transformations to utility functions. Consequently, any attempt to characterize necessary or sufficient conditions for consistent aggregation in terms of marginal utilities of income is both misleading and meaningless.

Each choice of numéraire good results in *different* preferences and a different social-welfare function. Because consumers must have different preferences for different choices of numéraire, this result is unsatisfactory. In each case, however, the evaluator must be indifferent to inequality in the distribution of income.

5. Conclusion

We have extended the claim of Hicks that the aggregate Dupuit–Marshall consumers' surplus associated with the market demand curve(s) is equal to the consumers' surplus in all markets together with producers' surpluses in other markets in partial and general equilibrium in an undistorted economy. This attractive result can be used to justify a cost-benefit test which simply computes project profitability at the average of before- and after-project prices.

There are difficulties with this result in cost-benefit analysis, however. Without normalization, the surplus can be path dependent and, as a consequence, cannot be

consistent with a Bergson–Samuelson social-welfare function. Possibilities arise only when normalized prices (prices divided by aggregate income) or a numéraire are employed. In the first case, an aggregate consumer with homothetic preferences is required. In the second, there is an aggregate consumer, income effects for each person are confined to the numéraire, and preferences must be different for different choices of numéraire. Because of this, a project may pass the cost-benefit test using one numéraire and fail it using another. But preferences are supposed to be primitives in economic models; choosing them to accommodate an arbitrary price normalization is not in the spirit of the cost-benefit enterprise. All of these results suggest that the Dupuit–Marshall consumers' surplus methodology that pervades cost-benefit analysis is theoretically unsatisfactory.

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Appendix A

Lemma 1. *The line integral*

$$\int_{\boldsymbol{x}^{\boldsymbol{b}}}^{\boldsymbol{x}^{\boldsymbol{a}}} \sum_{k=1}^{m} f^k(\boldsymbol{x}) dx_k \tag{A.1}$$

satisfies

$$\int_{\boldsymbol{x}^{\boldsymbol{b}}}^{\boldsymbol{x}^{\boldsymbol{a}}} \sum_{k=1}^{m} f^k(\boldsymbol{x}) dx_k \geq 0 \leftrightarrow F(\boldsymbol{x}^{\boldsymbol{a}}) \geq F(\boldsymbol{x}^{\boldsymbol{b}}) \tag{A.2}$$

for all continuous paths in $\mathcal{S} \subseteq \mathcal{R}^m$ *and some function* $F: \mathcal{S} \mapsto \mathcal{R}$ *if and only if it is path-independent.*

Proof. Consider, for any $\boldsymbol{x}^{\mathrm{b}}, \boldsymbol{x}^{\mathrm{a}} \in \mathcal{S}$, any two paths $\bar{C}$ and $\hat{C}$. Define $\hat{C}'$ as the reverse of $\hat{C}$ (from $\boldsymbol{x}^{\mathrm{a}}$ to $\boldsymbol{x}^{\mathrm{b}}$) and $\tilde{C}$ as the path from $\boldsymbol{x}^{\mathrm{b}}$ to $\boldsymbol{x}^{\mathrm{a}}$ along $\bar{C}$ and back to $\boldsymbol{x}^{\mathrm{b}}$ along $\hat{C}'$. Eq. (A.2) implies that the line integral must be zero on any closed path (from any x to itself). Therefore,

$$\int_{\tilde{C}} \sum_{k=1}^{m} f^k(\boldsymbol{x}) \mathrm{d}x_k = \int_{\bar{C}} \sum_{k=1}^{m} f^k(\boldsymbol{x}) \mathrm{d}x_k + \int_{\hat{C}'} \sum_{k=1}^{m} f^k(\boldsymbol{x}) \mathrm{d}x_k$$

$$= \int_{\bar{C}} \sum_{k=1}^{m} f^k(\boldsymbol{x}) \mathrm{d}x_k - \int_{\hat{C}} \sum_{k=1}^{m} f^k(\boldsymbol{x}) \mathrm{d}x_k = 0. \tag{A.3}$$

Consequently,

$$\int_{\bar{C}} \sum_{k=1}^{m} f^k(\boldsymbol{x}) \mathrm{d}x_k = \int_{\hat{C}} \sum_{k=1}^{m} f^k(\boldsymbol{x}) \mathrm{d}x_k, \tag{A.4}$$

and the integral is path-independent.

If the integral is path-independent, for all paths between $\boldsymbol{x}^{\mathrm{b}}$ and $\boldsymbol{x}^{\mathrm{a}}$,

$$\int_{\boldsymbol{x}^{\mathrm{b}}}^{\boldsymbol{x}^{\mathrm{a}}} \sum_{k=1}^{m} f^k(\boldsymbol{x}) \mathrm{d}x_k = G(\boldsymbol{x}^{\mathrm{a}}) - G(\boldsymbol{x}^{\mathrm{b}}) \tag{A.5}$$

for some function G: $\mathscr{S} \mapsto \mathscr{R}$ and (A.2) is satisfied. ■

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