

Equivalent-Income Functions and Income-Dependent Equivalence Scales*

by

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Abstract

This paper presents and investigates two classes of equivalent-income functions that are generalizations of those that correspond to exact (independent-of-base) absolute and relative equivalence scales. They provide less restrictive household demands, especially for children's goods, and have associated absolute and relative equivalence scales that may depend on income. We show that, under certain conditions, equivalent-income functions and the associated income-dependent equivalence scales can be uniquely estimated from demand data. We estimate them using Canadian data and find that the resulting scales are both plausible and income dependent. In addition, the estimated scales are used to measure inequality and we find that they make a significant difference to the level of and trend in measured inequality.

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1. Introduction

Although economic data report the expenditures made by households, individual levels of well-being (utility) must be known for social evaluation, inequality measurement and poverty-line determination. When making comparisons of economic well-being across households, it is important to take account of ‘economies of scale’ in the consumption of commodities such as housing and special needs of some people such as those with disabilities.

Equivalent-income functions provide comparisons of levels of individual well-being across households. These functions require the specification of a reference household type which is taken to be a single childless adult in this paper. The equivalent income of a household is the income necessary to give a single adult the same utility level that members of the household enjoy. For example, the equivalent income of an adult couple with an income of \$30,000 is likely to be more than \$15,000 (income per person) and less than \$30,000 due to economies of scale in household consumption. If equivalent income is \$20,000, a single adult with \$20,000 has the same utility level as each of the two adults in the couple.

An equivalent-income function determines equivalent income for every household type at every level of household income and every price vector. To perform a social evaluation, an investigator assigns the household equivalent income to each household member. This converts the income distribution into one in which each household is an identical single adult and is equivalent—in terms of utilities—to the actual one.

Absolute and relative equivalence scales are closely related to equivalent-income functions. An absolute equivalence scale is given by the difference between income and equivalent income and a relative equivalence scale is given by the ratio of income to equivalent income. An equivalence scale is said to be *exact* if it does not depend on income. Exact scales are the ones commonly used.

In this paper, we develop two new types of equivalent-income functions: those that are affine in income and those that are ‘log-affine’ in income (the logarithm of the function is

affine in the logarithm of income).¹ They are generalizations of the equivalent-income functions associated with exact absolute and relative equivalence scales. These generalizations allow for equivalence scales that are not exact, that is, scales that depend on income.

If an equivalent-income function is affine in income, we say that it satisfies *Generalized Absolute Equivalence-Scale Exactness* (GAESE) and, if an equivalent-income function is log-affine in income we say that it satisfies *Generalized Relative Equivalence-Scale Exactness* (GRESE). We show that, under certain conditions, these equivalent-income functions are uniquely identifiable from observable demand data. This means that, although there are an infinite number of equivalent-income functions that can rationalize any set of demand data, only one is affine (or log-affine) in income.

The new types of equivalent-income functions share two important features that distinguish them from the equivalent-income functions in common use. First, affine and log-affine equivalent-income functions have associated equivalence scales that depend on income. The affine equivalent-income function has an associated absolute equivalence scale that is affine in income and the log-affine equivalent-income function has an associated relative equivalence scale that is log-affine in income. This means that economies of household formation may vary with income. For example, a two-adult household needs less than twice the income of a single adult in order to achieve the same level of well-being because of shared consumption such as housing and shared tasks such as food preparation. Because housing and food expenditures are a decreasing share of income as income rises, we may expect the relative scale to be increasing in income.

Second, the restrictions imposed on demand behaviour by affine and log-affine equivalent-income functions are weaker than those imposed on demand behaviour by exact equivalence scales. In particular, if an absolute equivalence scale is exact, the income elasticity of demand for children's goods is zero and, if a relative equivalence scale is exact, the income elasticity of demand for children's goods is one. In contrast, the more general equivalent-income functions that we investigate in this paper allow for more general demand functions, which are especially important in the case of children's goods.

Previous research on equivalence scales offers two types of equivalent-income functions. The first type is associated with exact absolute equivalence scales, which are developed in Blackorby and Donaldson [1994]. If utility functions support such scales, we say that *Absolute Equivalence-Scale Exactness* (AESE) is satisfied. The second type of equivalent-income function is associated with exact relative equivalence scales (Lewbel [1989], Blackorby and

¹ A function $f: \mathcal{R} \rightarrow \mathcal{R}$ is affine in x if and only if $f(x) = ax + b$ where $a, b \in \mathcal{R}$. We say that $f: \mathcal{R}_{++} \rightarrow \mathcal{R}_{++}$ is log-affine in x if and only if $\ln f(x) = a \ln x + b$.

Donaldson [1993]).² If utility functions support such scales, we say that *Relative Equivalence-Scale Exactness* (RESE) is satisfied. We develop GAESE and GRESE by generalizing the structures of inter-household comparisons allowed by AESE and RESE. Blackorby and Donaldson [1993, 1994] show that, if either AESE³ or RESE⁴ is a maintained hypothesis, the equivalent-income function can be identified from demand behaviour alone. We provide similar identification theorems for affine and log-affine equivalent-income functions.

Our theorems show that, given GAESE, equivalent-income functions are uniquely identifiable from observable demand data if the reference expenditure function is neither affine nor log-affine⁵ and, given GRESE, equivalent-income functions are uniquely identifiable from demand data if the reference expenditure function is not log-affine. Both AESE and RESE are special cases of GAESE with testable restrictions on preferences. RESE is a special case of GRESE with testable restrictions on preferences.

Econometric research on equivalent-income functions has focussed on those that support exact relative equivalence scales. Researchers have used consumer-demand techniques to assess the scales' sizes, price sensitivity and exactness. A plethora of papers (for a survey, see Browning [1992]; see also Blundell and Lewbel [1991], Dickens, Fry and Pashardes [1993] and Pashardes [1995]) use parametric methods to estimate exact relative equivalence scales and two recent papers (Gozalo [1997]; Pendakur [1999]) use semiparametric methods to estimate exact relative equivalence scales. Most of these papers allow for price-dependence and find that equivalence scales do depend on prices. Indeed, Blundell and Lewbel [1991] argue that price responses are the only observable feature of equivalence scales. Several papers test the exactness of relative equivalence scales. Parametric papers (Blundell and Lewbel [1991], Dickens, Fry and Pashardes [1993], Pashardes [1995]) and semiparametric papers (Gozalo [1997], Pendakur [1999]) alike reject the hypothesis of Relative Equivalence-Scale Exactness (RESE). Pendakur [1999] suggests that exactness may hold for some comparisons but not others and finds that relative equivalence scales are not exact for comparisons of childless households to households with children, but may be exact for comparisons within these

² In addition to the articles referenced in the text, see Blackorby and Donaldson [1991], Blundell and Lewbel [1991], Browning [1988, 1992] Deaton and Muellbauer [1980], Donaldson [1992], Grunau [1988], Lewbel [1989], Muellbauer [1974], and Pollak and Wales [1979].

³ This result holds only if household expenditure functions are not affine (see footnote 5 or Section 2), which means that preferences are not quasi-homothetic. There is a body of research (mostly focussed on aggregation conditions) that finds that commodity demands are not consistent with quasi-homothetic preferences. See Sections 2 and 5 for discussions.

⁴ This result holds only if household expenditure functions are not log-affine (see footnote 5 or Section 2), that is, if preferences are not PIGLOG, which means that expenditure-share equations are not affine in the logarithm of income. There is body of recent research which suggests that expenditure-share equations do not have this form (see, for example, Banks, Blundell and Lewbel [1993]).

⁵ We say that the reference expenditure function E^r is affine if and only if it can be written as $E^r(u, p) = B(p)g(u) + F(p)$ and log-affine if and only if it can be written as $\ln E^r(u, p) = C(p)f(u) + \ln D(p)$. See Section 2 for a discussion.

groups. He further notes that this may be due to the tight restrictions that exactness puts on the demand for children's goods. Our results are consistent with this hypothesis.

In the empirical section of this paper, we use nonlinear parametric estimation techniques and Canadian expenditure data to estimate equivalent-income functions and equivalence scales given both GAESE and GRESE. First, for GAESE, we assume that demand equations are quadratic in income and show how GAESE restricts the demand equations and how equivalent-income functions can be calculated if the restrictions hold. We test GAESE against an unrestricted quadratic alternative, estimate GAESE-restricted equivalent-income functions and test down to AESE and RESE from GAESE. Relative equivalence scales given GAESE are shown to be quite different from those given RESE. For households with more than one person, the former are larger than the latter and, for many household types, increase with income.

Second, for GRESE, we assume that expenditure-share equations are quadratic in the logarithm of income and show how GRESE restricts the demand equations and how equivalent-income functions can be calculated if the restrictions hold. We test GRESE against an unrestricted quadratic alternative, estimate GRESE-restricted equivalent-income functions and test down to RESE. Relative equivalence scales given GRESE are quite different from those given RESE, with the former larger than the latter and increasing in income.

Our estimates given GAESE and GRESE suggest that equivalence scales increase significantly over income for some household types, but are almost constant for others. For example, the GRESE-restricted relative equivalence scale for dual parents with a teenaged child aged 16 to 17 is 2.06 at very low income and 2.88 at very high income. On the other hand, the GRESE-restricted relative equivalence scale for a single parent with one child aged 5 to 15 is 1.52 at very low income and 1.53 at very high income. The dependence of equivalence scales on income permits greater flexibility in the demand for child goods, allowing the demand for (GAESE) or share spent on (GRESE) child goods to depend on income. We show that this dependence is quantitatively important by examining the demand for children's clothing directly.

We assess how use of the income-dependent relative equivalence scales affects the measurement of inequality by estimating the level of expenditure inequality in Ontario, Canada in 1982 and 1992 using scales estimated given RESE, GAESE and GRESE. Although the Gini Coefficient for expenditure inequality using a RESE-restricted equivalence scale declines (significantly) from 0.228 in 1982 to 0.198 in 1992, when a GAESE-restricted scale is used, the coefficient is stable at approximately 0.285 in both years and, when a GRESE-restricted scale is used, the coefficient is stable at approximately 0.275 in both years. Thus, in this case, use of the more general equivalent-income functions and equivalence scales leads to higher measured inequality and a non-decreasing path over time.

Section 2 presents the model of household that we employ along with its relationship to equivalent-income functions and equivalence scales and Sections 3 and 4 discuss Generalized Absolute Equivalence-Scale Exactness (GAESE) and Generalized Relative Equivalence-Scale Exactness (GRESE). Section 5 provides our theorems on theoretical identification. Section 6 discusses the empirical model and the data, Section 7 presents our empirical results, and Section 8 concludes the paper.

2. Equivalent-Income Functions and Equivalence Scales

Suppose that $\mathcal{Z}$ is a set of vectors of possible household characteristics. Elements in $\mathcal{Z}$ may describe household characteristics exhaustively, including number and ages of household members, special needs and so on, or it may include only a subset of all possible characteristics. To use equivalent-income functions in a normative setting, it is necessary to assume that any two households with the same characteristics have identical preferences and that each household member enjoys the same level of utility.⁶

We restrict attention to the consumption of private goods only and use the indirect utility function $V: \mathcal{R}_{++}^m \times \mathcal{D} \times \mathcal{Z} \rightarrow \mathcal{R}$, $m \geq 2$, to represent household preferences and measure the levels of well-being of household members. $\mathcal{D}$ is the domain of definition for household income (x).⁷ For absolute scales and their generalizations, $\mathcal{D}$ is $\mathcal{R}_+$; for relative scales and their generalizations, $\mathcal{D}$ is $\mathcal{R}_{++}$. For a household with characteristics $z \in \mathcal{Z}$ facing prices $p \in \mathcal{R}_{++}^m$ with income $x \in \mathcal{D}$, the well-being of each household member is

$$u = V(p, x, z). \quad (2.1)$$

We assume that V is continuous, increasing in x and homogeneous of degree zero in (p, x) for each $z \in \mathcal{Z}$. The expenditure function corresponding to V is E where $E(u, p, z)$ is the minimum expenditure needed by a household with characteristics z facing prices p to attain utility level u for each of its members. E is continuous in (u, p) for each z , increasing in u and homogeneous of degree one in p . We do not assume that V and E are globally regular.

To define equivalent-income functions and equivalence scales, a reference household type is needed and, although other choices are possible, we use a childless single adult as the reference and denote his or her characteristics as z^r . For a household with characteristics z , equivalent income x^e is that income which, if enjoyed by a reference single adult facing the same prices, would result in a utility level equal to that of each household member. Thus, equivalent income is implicitly defined by

$$V(p, x, z) = V(p, x^e, z^r) = V^r(p, x^e) \quad (2.2)$$

⁶ See Blackorby and Donaldson [1993] for a short discussion of more complex formulations.

⁷ For empirical purposes, household expenditure is used instead of income.

where $V^r = V(\cdot, \cdot, z^r)$ is the indirect utility function for the reference household. We assume that, for every (p, x, z) in the domain of V , (2.2) can be solved for x^e , and we write

$$x^e = X(p, x, z). \quad (2.3)$$

The indirect utility function V is homogeneous of degree zero in (p, x) which implies that X is homogeneous of degree one in (p, x) . In addition, $X(p, x, z^r) = x$ for all (p, x) .

Because

$$V(p, x, z) = u \iff E(u, p, z) = x, \quad (2.4)$$

$$x^e = E(u, p, z^r) = E^r(u, p) \quad (2.5)$$

where $E^r = E(\cdot, \cdot, z^r)$ is the expenditure function of the reference household. Consequently,

$$X(p, x, z) = E^r(V(p, x, z), p). \quad (2.6)$$

X is increasing in x .

For welfare purposes, equivalent incomes permit the conversion of an economy with many household types into an economy of identical single individuals. If, for example, a household has four members, then it is equivalent, in terms of utilities, to four single adults, each of whom spends x^e . In addition, (2.2) clearly requires interpersonal comparisons of the utilities of people who belong to households with different characteristics.⁸

Two kinds of equivalence scales can be defined using equivalent incomes. In general, equivalence scales depend on income, but those that do not are called *exact*. Absolute equivalence scales measure the amount of income over and above reference-household income needed by a household with arbitrary characteristics to produce utility equality. If s^A is the value of an absolute equivalence scale S^A ,

$$u = V(p, x, z) = V^r(p, x - s^A), \quad (2.7)$$

and we write

$$s^A = S^A(p, x, z) = x - X(p, x, z), \quad (2.8)$$

with $S^A(p, x, z^r) = 0$ for all (p, x) . Because X is homogeneous of degree one in (p, x) , S^A is homogeneous of degree one as well. (2.7) implies that

$$s^A = E(u, p, z) - E^r(u, p). \quad (2.9)$$

The absolute equivalence scale S^A is income dependent or, in the formulation of (2.9), utility dependent. The scale is independent of these variables ($S^A(p, x, z) = \bar{S}^A(p, z)$) and,

⁸ Only comparisons of utility levels are needed; for discussions, see Blackorby and Donaldson [1993], Blackorby, Donaldson and Weymark [1984] and Sen [1977].

therefore, exact if and only if the expenditure function is additively decomposable (Blackorby and Donaldson [1994]), with

$$E(u, p, z) = E^r(u, p) + \bar{S}^A(p, z). \quad (2.10)$$

We call this condition on E Absolute Equivalence-Scale Exactness (AESE).

If s^R is the value of a relative equivalence scale S^R , then

$$u = V(p, x, z) = V^r(p, x/s^R). \quad (2.11)$$

If household income is divided by s^R , equivalent income results. Consequently, a relative scale is the ratio of household income to equivalent income (x/x^e). If $x^e \neq 0$, the scale is defined, and we can write

$$s^R = S^R(p, x, z) = \frac{x}{X(p, x, z)}. \quad (2.12)$$

Because V^r is homogeneous of degree zero, S^R is homogeneous of degree zero in (p, x) . In addition, $S^R(p, x, z^r) = 1$ for all (p, x) . (2.11) can be solved for s^R using expenditure functions, and

$$s^R = \frac{E(u, p, z)}{E^r(u, p)}. \quad (2.13)$$

The relative equivalence scale S^R is, in general, income dependent or, in the formulation of (2.13), utility dependent. The scale is independent of these variables ($S(p, x, z) = \bar{S}^R(p, z)$) and, therefore, exact if and only if the expenditure function is multiplicatively decomposable (Blackorby and Donaldson [1991, 1993], Lewbel [1991]), with

$$E(u, p, z) = \bar{S}^R(p, z)E^r(u, p). \quad (2.14)$$

We call this condition Relative Equivalence-Scale Exactness (RESE).⁹

AESE is equivalent to a condition on the way interpersonal comparisons are related to incomes called Income-Difference Comparability (Blackorby and Donaldson [1994]). If a household with arbitrary characteristics and a reference household facing the same prices have incomes such that their utilities are equal, common absolute increases in their incomes (which leave the income difference unchanged) preserve utility equality. Thus, an increase in a household's income of one dollar matched by an increase of one dollar in the reference household's income preserves equality of well-being.

RESE is equivalent to a condition on the way interpersonal comparisons are related to incomes called Income-Ratio Comparability (Blackorby and Donaldson [1991, 1993]). If a household with arbitrary characteristics and a reference household facing the same prices have incomes such that their utilities are equal, common scaling of their incomes (which leaves

⁹ Blackorby and Donaldson call the condition Equivalence-Scale Exactness (ESE) and Lewbel uses the term Independence of Base (IB).

the income ratio unchanged) preserves utility equality. Thus, an increase in a household's income of one per cent matched by an increase in the reference household's income of one per cent preserves equality of well-being.

Without additional assumptions, neither equivalent-income functions nor equivalence scales can be determined by household demand behaviour. The reason is that the same behaviour is implied by the utility functions V and V' where

$$V'(p, x, z) = \phi(V(p, x, z), z), \quad (2.15)$$

and ϕ is increasing in its first argument. This is equivalent to

$$E'(u, p, z) = E(\sigma(u, z), p, z) \quad (2.16)$$

where σ is increasing in its first argument. Although behaviour is the same in each case, the equivalent-income functions and equivalence scales are not.¹⁰

Blackorby and Donaldson [1993, 1994] have investigated the theoretical identification of equivalence scales when AESE or RESE is accepted as a maintained hypothesis. In the case of absolute scales, given one of two technical conditions, AESE permits the estimation of the equivalent-income function from demand behaviour if and only if the reference expenditure function is *not* affine (Blackorby and Donaldson [1994]). We say that E^r is affine if and only if it can be written as

$$E^r(u, p) = B(p)g(u) + F(p) \quad (2.17)$$

where B and F are homogeneous of degree one. In the case of RESE, given analogous technical conditions, estimation of the scale from demand behaviour is possible if and only if the reference expenditure function is *not* log-affine (Blackorby and Donaldson [1993, Theorems 5.1 and 5.3]). We say that E^r is log-affine if and only if it can be written as

$$\ln E^r(u, p) = C(p)f(u) + \ln D(p) \quad (2.18)$$

where C is homogeneous of degree zero, D is homogeneous of degree one, and f is increasing.¹¹

These results have the following implication. Suppose that the utility function V is such that (2.17) ((2.18)) does not hold and an absolute (relative) equivalence scale S^A (S^R) which satisfies AESE (RESE) is computed from it. Now suppose that the utility function V' is found using a characteristic-specific transform ϕ as in (2.15). The theorems show that, if the equivalence scale corresponding to V' is different from S^A (S^R), it cannot satisfy AESE (RESE) and, therefore, must depend on income. Consequently, given AESE (RESE) and a

¹⁰ See Pollak and Wales [1979, 1981, 1992].

¹¹ On the income domain $\mathcal{D} = \mathcal{R}_+$, (2.18) must be written as $E^r(u, p) = D(p)f(u)^{C(p)}$.

reference expenditure function that is not affine (log-affine), there is a unique exact absolute (relative) equivalence scale which corresponds to demand behaviour.

Because equivalent-income functions can be computed from relative or absolute equivalence scales, similar results apply. AESE (RESE) and a reference expenditure function that is not affine (log-affine) permit the estimation of a unique equivalent-income function from demand behaviour.

If $V(\cdot, \cdot, z)$ is differentiable for each $z \in \mathcal{Z}$ with $\partial V(p, x, z)/\partial x > 0$ for all (p, x, z) , AESE implies (using Roy's Theorem) that the demand function $D_j(\cdot, \cdot, z)$ is related to the reference demand function D_j^r by

$$D_j(p, x, z) = D_j^r(p, x - A(p, z)) + \frac{\partial A(p, z)}{\partial p_j}, \quad (2.19)$$

$j = 1, \dots, m$ (Pendakur [1998b]). RESE implies that the expenditure-share function $W_j(\cdot, \cdot, z)$ (written as a function of $\ln x$) is related to the expenditure-share function of the reference household W_j^r by

$$W_j(p, \ln x, z) = W_j^r(p, \ln x - \ln R(p, z)) + \frac{\partial \ln R(p, z)}{\partial \ln p_j}. \quad (2.20)$$

$j = 1, \dots, m$ (Pendakur [1999]). Blundell, Duncan and Pendakur [1998] argue that, because share equations satisfying RESE are integrable, semi-parametric estimation using this condition is a good strategy for estimation of share functions.

Both AESE and RESE suffer from an important weakness: they handle children's goods poorly (see Browning [1988, 1992]). If c is any children's good, reference demand $D_c^r(p, x) = 0$. In the case of AESE, this together with (2.19), implies that the demand function for good c by a household with characteristics z is

$$D_c(p, x, z) = \frac{\partial A(p, z)}{\partial p_c}; \quad (2.21)$$

the income elasticity of every children's good is zero. With RESE, the reference share $W_c^r(p, \ln x) = 0$ and this, together with (2.20) implies

$$W_c(p, \ln x, z) = \frac{\partial \ln R(p, z)}{\partial \ln p_c}, \quad (2.22)$$

which implies that the income elasticity of every children's good is one.

3. Generalized Absolute Equivalence-Scale Exactness (GAESE)

If Absolute Equivalence-Scale Exactness (AESE) is satisfied, equal absolute increases of income preserve utility equality across household types, and therefore

$$dx^e = dx. \quad (3.1)$$

Instead, suppose that we require the weaker condition that the change in the reference household's income that preserves utility equality is independent of income. This requires the existence of a function ρ such that

$$dx^e = \rho(p, z)dx. \quad (3.2)$$

A one dollar increase in household income requires an increase of $\rho(p, z)$ dollars to preserve equality of well-being. Integrating,

$$x^e = X(p, x, z) = \rho(p, z)x + \alpha(p, z) \quad (3.3)$$

for some function α . Because X is increasing in x and homogeneous of degree one in (p, x) , $\rho(p, z) > 0$ for all (p, z) , ρ is homogeneous of degree zero and α is homogeneous of degree one in p . In addition, because $X(p, x, z^r) = x$ for all (p, x) , $\rho(p, z^r) = 1$ and $\alpha(p, z^r) = 0$ for all p .

Defining $R(p, z) = 1/\rho(p, z)$ and $A(p, z) = -\alpha(p, z)/\rho(p, z)$, (3.3) can be rewritten as

$$x^e = X(p, x, z) = \frac{x - A(p, z)}{R(p, z)}. \quad (3.4)$$

Because $V(p, x, z) = V^r(p, x^e)$, the indirect utility function can be written as

$$V(p, x, z) = V^r\left(p, \frac{x - A(p, z)}{R(p, z)}\right), \quad (3.5)$$

and the expenditure function E is given by

$$E(u, p, z) = R(p, z)E^r(u, p) + A(p, z). \quad (3.6)$$

Because $\rho(p, z^r) = 1$ and $\alpha(p, z^r) = 0$, $R(p, z^r) = 1$ and $A(p, z^r) = 0$. We call the condition expressed in (3.3)–(3.6) Generalized Absolute Equivalence-Scale Exactness (GAESE). A and R are the absolute and relative components of the equivalent-income function.

Given GAESE, the absolute equivalence scale S^A is affine in income and is given by

$$S^A(p, x, z) = \frac{(R(p, z) - 1)x + A(p, z)}{R(p, z)}. \quad (3.7)$$

It is increasing (decreasing) in x if $R(p, z) > 1$ ($R(p, z) < 1$). The relative equivalence scale S^R is given by

$$S^R(p, x, z) = \frac{R(p, z)x}{x - A(p, z)}, \quad (3.8)$$

and it is increasing (decreasing) in x if $A(p, z) < 0$ ($A(p, z) > 0$). Because $\lim_{x \rightarrow 0} S^A(p, x, z) = A(p, z)/R(p, z)$ and $\lim_{x \rightarrow \infty} S^R(p, x, z) = R(p, z)$, AESE is approximately satisfied for small x and RESE is approximately satisfied for large x .

If $V(\cdot, \cdot, z)$ is differentiable for all z and $\partial V(p, x, z)/\partial x > 0$ for all (p, x, z) , the ordinary commodity demand function $D_j(\cdot, \cdot, z)$ is related to the reference demand function $D_j^r := D_j(\cdot, \cdot, z^r)$ by

$$D_j(p, x, z) = R(p, z)D_j^r\left(p, \frac{x - A(p, z)}{R(p, z)}\right) + \frac{\partial R(p, z)}{\partial p_j} \left(\frac{x - A(p, z)}{R(p, z)}\right) + \frac{\partial A(p, z)}{\partial p_j}, \quad (3.9)$$

$j = 1, \dots, m$. Given AESE, $R(p, z) = 1$, and the second term of (3.9) vanishes. Thus, the restrictions on the shapes of demand functions under GAESE are a generalization of the restrictions under AESE.

The problems with children's goods possessed by RESE and AESE disappear with GAESE. If good c is a children's good, (3.9) implies

$$D_c(p, x, z) = \frac{\partial R(p, z)}{\partial p_c} \left(\frac{x - A(p, z)}{R(p, z)}\right) + \frac{\partial A(p, z)}{\partial p_c} \quad (3.10)$$

which is affine in income. Because GAESE generalizes both AESE and GESE, the income elasticities are not forced to be zero or one.

4. Generalized Relative Equivalence-Scale Exactness (GRESE)

If Relative Equivalence-Scale Exactness (RESE) is satisfied, equal percentage increases in income preserve utility equality across household types and, therefore,

$$\frac{dx^e}{x^e} = \frac{dx}{x}. \quad (4.1)$$

Suppose that, instead of requiring an equal percentage increase in the reference household's income to preserve utility equality, we require the matching percentage increase to be independent of income only. This requires the existence of a function κ such that

$$\frac{dx^e}{x^e} = \kappa(p, z) \frac{dx}{x}, \quad (4.2)$$

$\kappa(p, z) > 0$, for all (p, x, z) . A one percent increase in household income requires an increase of $\kappa(p, z)$ percent to preserve equality of well-being. Integrating,

$$\ln x^e = \ln X(p, x, z) = \kappa(p, z) \ln x + \ln \gamma(p, z) \quad (4.3)$$

for some function γ . Consequently,

$$x^e = X(p, x, z) = \gamma(p, z) x^{\kappa(p, z)}. \quad (4.4)$$

Because $\kappa(p, z) > 0$ and X is increasing in x , $\gamma(p, z) > 0$ for all (p, z) . In addition, because $X(p, x, z^r) = x$ for all (p, x) , $\gamma(p, z^r) = 1$ and $\kappa(p, z^r) = 1$ for all p .

Equation (4.4) implies

$$X(p, \lambda x, z) = \lambda^{\kappa(p,z)} X(p, x, z) \quad (4.5)$$

for all $\lambda > 0$. Because X is homogeneous of degree one in (p, x) ,

$$X(\mu p, \mu \lambda x, z) = \mu X(p, \lambda x, z) = \mu \lambda^{\kappa(p,z)} X(p, x, z), \quad (4.6)$$

and

$$X(\mu p, \mu \lambda x, z) = \lambda^{\kappa(\mu p, z)} X(\mu p, \mu x, z) = \mu \lambda^{\kappa(\mu p, z)} X(p, x, z) \quad (4.7)$$

for all $\lambda > 0$, $\mu > 0$. Consequently, $\kappa(\mu p, z) = \kappa(p, z)$ for all $\mu > 0$, and κ is homogeneous of degree zero in p .

Defining $K(p, z) = 1/\kappa(p, z)$ and $G(p, z) = 1/\gamma(p, z)^{1/\kappa(p,z)}$, (4.3) and (4.4) can be rewritten as

$$\ln x^e = \ln X(p, x, z) = \frac{\ln x - \ln G(p, z)}{K(p, z)} \quad (4.8)$$

and

$$x^e = X(p, x, z) = \left[\frac{x}{G(p, z)} \right]^{\frac{1}{K(p, z)}}. \quad (4.9)$$

Because $V(p, x, z) = V^r(p, x^e)$, it follows that

$$V(p, x, z) = V^r \left(p, \left[\frac{x}{G(p, z)} \right]^{\frac{1}{K(p, z)}} \right) = V^r \left(p, \exp \left\{ \frac{\ln x - \ln G(p, z)}{K(p, z)} \right\} \right), \quad (4.10)$$

and the expenditure function E is given by

$$E(u, p, z) = G(p, z) E^r(u, p)^{K(p, z)} \quad (4.11)$$

or

$$\ln E(u, p, z) = K(p, z) \ln E^r(u, p) + \ln G(p, z). \quad (4.12)$$

Because $\gamma(p, z^r) = \kappa(p, z^r) = 1$, $G(p, z^r) = K(p, z^r) = 1$. In addition, because κ is homogeneous of degree zero in p , so is K . We call the condition expressed in (4.8)–(4.12) Generalized Relative Equivalence-Scale Exactness (GRESE). $K(p, z)$ is the income elasticity of the equivalent-income function.

Given GRESE, the absolute equivalence scale S^A is given by

$$S^A(p, x, z) = x - \left[\frac{x}{G(p, z)} \right]^{\frac{1}{K(p, z)}}, \quad (4.13)$$

and the relative equivalence scale S^R is given by

$$S^R(p, x, z) = x^{\frac{K(p, z)-1}{K(p, z)}} G(p, z)^{\frac{1}{K(p, z)}}. \quad (4.14)$$

As a consequence, the logarithm of the relative scale is affine in the logarithm of income, with

$$\ln S^R(p, x, z) = \frac{(K(p, z) - 1) \ln x + \ln G(p, z)}{K(p, z)}. \quad (4.15)$$

S^R is increasing (decreasing) in x if $K(p, z) > 1$ ($K(p, z) < 1$).

The functions G and K are not, in general, independent: if E^r , G and K are differentiable in p , K is functionally dependent on G but the converse is not true. In particular, Lemma 1 in the Appendix shows that

$$K(p, z) = 1 - \sum_{j=1}^m \frac{\partial \ln G(p, z)}{\partial \ln p_j}. \quad (4.16)$$

This means that K is functionally dependent on G : for every G there is exactly one K .

As an example suppose, for $m = 2$, that

$$G(p, z) = \Gamma(z) p_1^{\delta_1(z)} p_2^{\delta_2(z)}. \quad (4.17)$$

Then

$$K(p, z) = 1 - \delta_1(z) - \delta_2(z). \quad (4.18)$$

K is the same for every choice of the function Γ .

If $V(\cdot, \cdot, z)$ is differentiable for all z with $\partial V(p, x, z)/\partial x > 0$ for all (p, x, z) , Roy's Theorem implies that, given GRESE, the share equations W_j , $j = 1, \dots, m$, satisfy

$$\begin{aligned} W_j(p, \ln x, z) = & K(p, z) W_j^r \left(p, \frac{\ln x - \ln G(p, z)}{K(p, z)} \right) \\ & + \frac{\partial K(p, z)}{\partial \ln p_j} \left(\frac{\ln x - \ln G(p, z)}{K(p, z)} \right) + \frac{\partial \ln G(p, z)}{\partial \ln p_j}, \end{aligned} \quad (4.19)$$

where $W_j^r := W_j(\cdot, \cdot, z^r)$, $j = 1, \dots, m$ is the reference household's share equation. Given RESE, $K(p, z) = 1$, and the second term of (4.19) vanishes. The restrictions on share equations implied by GRESE are thus a generalization of those implied by RESE.

The problems with children's goods possessed by RESE and AESE disappear with GRESE. If good c is a children's good, $W_c^r(p, \ln x) = 0$ and (4.19) implies

$$W_c(p, \ln x, z) = \frac{\partial K(p, z)}{\partial \ln p_j} \left(\frac{\ln x - \ln G(p, z)}{K(p, z)} \right) + \frac{\partial \ln G(p, z)}{\partial \ln p_j}, \quad (4.20)$$

$j = 1, \dots, m$. These share equations are affine in the logarithm of income and, as a consequence, income elasticities are not forced to zero or one.

5. Theoretical Identification

The behaviour implied by the indirect utility function V or the expenditure function E has no implications, by itself, for equivalent-income functions and equivalence scales. The reason is that, if two utility functions satisfy (2.15) or, equivalently, if two expenditure functions satisfy (2.16), behaviour is the same but equivalent-income functions and equivalence scales may be different. Without additional restriction, these functions are not identified.

In this section, we investigate the relationship of behaviour and equivalent-income functions when GAESE or GRESE is satisfied and show that the results of Blackorby and Donaldson [1991, 1993, 1994] concerning RESE and AESE can be generalized. Assuming that there is a characteristic, such as age, that is a continuous variable and that V is a continuous function of it, we show that the equivalent-income function and equivalence scales are uniquely identified by behaviour if: (i) GAESE holds and the reference expenditure function is neither affine (quasi-homothetic preferences) nor log-affine; (ii) GRESE holds and the reference expenditure function is not log-affine. These conditions are closely related to the ones found by Blackorby and Donaldson for AESE (Blackorby and Donaldson [1994]) and RESE (Blackorby and Donaldson [1993]).

The two range conditions used for the theorems in this section are global: they apply to all $p \in \mathcal{R}_{++}^m$. They need not hold globally, however. Each theorem needs only two price vectors at which the relevant range condition holds and, if the condition holds for one price vector, continuity ensures that it holds in a neighborhood. Consequently, the theorems can be applied locally.

These results have strong implications. Theorems 1–3 show that if an expenditure function satisfies GAESE and the reference expenditure function is neither affine nor log-affine, then any other expenditure function that represents the same preferences necessarily does not satisfy GAESE. Similarly, Theorems 4 and 5 imply that if an expenditure function satisfies GRESE and the reference expenditure function is not log-affine, any other expenditure function that represents the same preferences cannot satisfy GRESE. Consequently, if the reference expenditure function is neither affine nor log-affine and GAESE or GRESE is assumed as a maintained hypothesis, the equivalent-income function and equivalence scales are uniquely determined by behaviour.

5.1. Theoretical Identification and GAESE

The indirect utility functions $\hat{V}$ and $\tilde{V}$ represent the same behaviour if and only if there exists a function ϕ , increasing in its first argument, such that

$$\hat{V}(p, x, z) = \phi\left(\tilde{V}(p, x, z), z\right) \quad (5.1)$$

for all (p, x, z) . Equivalently, the expenditure functions $\hat{E}$ and $\tilde{E}$ represent the same behaviour if and only if there exists a function σ , increasing in its first argument, such that, for all (u, p, z) ,

$$\hat{E}(u, p, z) = \tilde{E}(\sigma(u, z), p, z). \quad (5.2)$$

Because $\hat{V}(\cdot, \cdot, z)$, $\tilde{V}(\cdot, \cdot, z)$, $\hat{E}(\cdot, \cdot, z)$ and $\tilde{E}(\cdot, \cdot, z)$ are continuous by assumption, $\phi(\cdot, z)$ and $\sigma(\cdot, z)$ are continuous for all $z \in \mathcal{Z}$.

If GAESE is satisfied,

$$V(p, x, z) = V^r\left(p, \frac{x - A(p, z)}{R(p, z)}\right) \quad (5.3)$$

The functions $\hat{V}$ and $\tilde{V}$ represent the same preferences when GAESE is satisfied if and only if

$$\hat{V}^r\left(p, \frac{x - \hat{A}(p, z)}{\hat{R}(p, z)}\right) = \phi\left(\tilde{V}^r\left(p, \frac{x - \tilde{A}(p, z)}{\tilde{R}(p, z)}\right), z\right). \quad (5.4)$$

Because ϕ is increasing in its first argument and because $\hat{V}^r$ and $\tilde{V}^r$ are ordinally equivalent, we can choose $\hat{V}^r = \tilde{V}^r$, and we call the common function V^r .

If the functions $\hat{E}$ and $\tilde{E}$ satisfy GAESE, then, using (3.6) with $\hat{E}^r = \tilde{E}^r = E^r$, $\hat{E}(u, p, z) = \hat{R}(p, z)E^r(u, p) + \hat{A}(p, z)$ and $\tilde{E}(\sigma(u, z), p, z) = \tilde{R}(p, z)E^r(\sigma(u, z), p, z) + \tilde{A}(p, z)$. Using the above argument, it follows that they represent the same preferences if and only if

$$\hat{R}(p, z)E^r(u, p) + \hat{A}(p, z) = \tilde{R}(p, z)E^r(\sigma(u, z), p) + \tilde{A}(p, z) \quad (5.5)$$

for all (u, p, z) where E^r is the expenditure function corresponding to V^r . Rearranging terms,

$$E^r(\sigma(u, z), p) = \frac{\hat{R}(p, z)}{\tilde{R}(p, z)}E^r(u, p) + \frac{\hat{A}(p, z) - \tilde{A}(p, z)}{\tilde{R}(p, z)}. \quad (5.6)$$

Defining $J(p, z) = \hat{R}(p, z)/\tilde{R}(p, z)$ and $T(p, z) = (\hat{A}(p, z) - \tilde{A}(p, z))/\tilde{R}(p, z)$, (5.6) becomes

$$E^r(\sigma(u, z), p) = J(p, z)E^r(u, p) + T(p, z). \quad (5.7)$$

We employ a range condition for Theorems 1–3. We assume that $\mathcal{Z}$ contains at least one continuous variable such as age and, in the case in which $\hat{R}$ is different from $\tilde{R}$ and $\hat{A}$ is different from $\tilde{A}$, $J(p, z)$ can be moved independently of $T(p, z)$.

Range Condition A: $\mathcal{Z}$ contains a subset $\overset{\circ}{\mathcal{Z}}$ of at least one continuous variable such as age and, for every $p \in \mathcal{R}_{++}^m$, the functions $\hat{R}$, $\tilde{R}$, $\hat{A}$ and $\tilde{A}$ are continuous in and sensitive to $\overset{\circ}{z} \in \overset{\circ}{\mathcal{Z}}$. In addition, $J(p, z)$ can be moved independently of $T(p, z)$ by changing $\overset{\circ}{z}$.

Note that $J(p, z^r) = 1$ and $T(p, z^r) = 0$. If $\hat{R} \neq \tilde{R}$, then $J(p, z) \neq 1$ for some z . The range condition implies that $J(p, z)$ can be moved through an interval by changing z . A similar consideration applies to $T(p, z)$.

Theorems 1–3 prove that the equivalent-income function and equivalence scales are uniquely identified by behaviour if GAESE holds and the reference expenditure function is neither affine (quasi-homothetic preferences) nor log-affine. Theorem 1 is concerned with the case in which $\hat{R} \neq \tilde{R}$ and $\hat{A} \neq \tilde{A}$. It requires the second part of the range condition which allows $J(p, z)$ and $T(p, z)$ to be moved independently by changing z .

Theorem 1: *Given GAESE, two pairs of functions $\hat{R}$ and $\tilde{R}$ and $\hat{A}$ and $\tilde{A}$ with $\hat{R} \neq \tilde{R}$ and $\hat{A} \neq \tilde{A}$ satisfying Range Condition A are consistent with the same household behaviour if and only if there exist functions $B: \mathcal{R}_{++}^m \rightarrow \mathcal{R}_{++}$, $F: \mathcal{R}_{++}^m \rightarrow \mathcal{R}_{++}$, $g: \mathcal{R} \rightarrow \mathcal{R}$, $j: \mathcal{Z} \rightarrow \mathcal{R}_{++}$ and $t: \mathcal{Z} \rightarrow \mathcal{R}$ such that, for all (u, p, z) ,*

$$E^r(u, p) = B(p)g(u) + F(p), \quad (5.8)$$

$$\hat{R}(p, z) = j(z)\tilde{R}(p, z) \quad (5.9)$$

and

$$\hat{A}(p, z) = \tilde{A}(p, z) + \tilde{R}(p, z) \left[B(p)t(z) + F(p)(1 - j(z)) \right], \quad (5.10)$$

where B and F are homogeneous of degree one, g is increasing and continuous, $j(z^r) = 1$ and $t(z^r) = 0$.

Proof: See the Appendix.

Theorem 2 considers the case in which there are two different A s, $\hat{A}$ and $\tilde{A}$ but only one function R . In that case, the function j in (5.9) and (5.10) satisfies $j(z) = 1$ for all $z \in \mathcal{Z}$.

Theorem 2: *Given GAESE, two pairs of functions $\hat{R}$ and $\tilde{R}$ and $\hat{A}$ and $\tilde{A}$ with $\hat{R} = \tilde{R} =: R$ and $\hat{A} \neq \tilde{A}$ satisfying Range Condition A are consistent with the same household behaviour if and only if there exist functions $B: \mathcal{R}_{++}^m \rightarrow \mathcal{R}_{++}$, $F: \mathcal{R}_{++}^m \rightarrow \mathcal{R}_{++}$, $g: \mathcal{R} \rightarrow \mathcal{R}$ and $t: \mathcal{Z} \rightarrow \mathcal{R}$ such that, for all (u, p, z) ,*

$$E^r(u, p) = B(p)g(u) + F(p) \quad (5.11)$$

and

$$\hat{A}(p, z) = \tilde{A}(p, z) + R(p, z)B(p)t(z), \quad (5.12)$$

where B and F are homogeneous of degree one, g is increasing and continuous, and $t(z^r) = 0$.

Proof: See the Appendix.

Note that equation (5.11) is identical to (5.8) and (5.12) is (5.10) with $j(z) = 1$. Reference preferences are quasi-homothetic in both cases.

A different possibility arises when $\hat{R} \neq \tilde{R}$ and $\hat{A} = \tilde{A}$. The second term disappears from equation (5.7) and the functional equation investigated by Blackorby and Donaldson [1989, Theorem 5.1] results. This case resembles RESE, and the condition on reference preferences is the same: the reference expenditure function must be log-affine.

Theorem 3: *Given GAESE, two pairs of functions $\hat{R}$ and $\tilde{R}$ and $\hat{A}$ and $\tilde{A}$ with $\hat{R} \neq \tilde{R}$ and $\hat{A} = \tilde{A}$ satisfying Range Condition A are consistent with the same household behaviour if and only if there exist functions $C: \mathcal{R}_{++}^m \rightarrow \mathcal{R}_{++}$, $D: \mathcal{R}_{++}^m \rightarrow \mathcal{R}_{++}$, $g: \mathcal{R} \rightarrow \mathcal{R}$ and $r: \mathcal{Z} \rightarrow \mathcal{R}$ such that, for all (u, p, z) ,*

$$E^r(u, p) = D(p)g(u)^{C(p)} \quad (5.13)$$

and

$$\hat{R}(p, z) = r(z)\tilde{R}(p, z), \quad (5.14)$$

where C is homogeneous of degree zero, D is homogeneous of degree one, f is increasing and continuous, and $r(z^r) = 1$.

Proof: See the Appendix.

For positive incomes, (5.13) can be rewritten as

$$\ln E^r(u, p) = C(p) \ln g(u) + \ln D(p). \quad (5.15)$$

With $f(u) = \ln g(u)$, this is the condition for non-identification of the equivalent-income function for GRESE (see Theorems 4 and 5 below).

It follows from these theorems that, in order to be able to identify equivalent-income functions and equivalence scales from behaviour alone, given GAESE, the reference expenditure function must be neither affine nor log-affine. If this condition is met, there is a unique indirect utility function V (and corresponding expenditure function E), consistent with demand behaviour, satisfying GAESE.

5.2. Theoretical Identification and GRESE

GRESE implies that $\ln X(p, x, z) = K(p, z)x + \ln G(p, z)$. To discover when the functions G and K are identified from behaviour alone, assuming that GRESE is true, (5.2) and the argument of the previous subsection imply that it must be impossible for there to be functions $\hat{G}$, $\tilde{G}$, $\hat{K}$, and $\tilde{K}$ such that

$$\hat{K}(p, z) \ln E^r(u, p) + \ln \hat{G}(p, z) = \tilde{K}(p, z) \ln E^r(\sigma(u, z), p) + \ln \tilde{G}(p, z) \quad (5.16)$$

for all (u, p, z) , where σ is increasing in its first argument. (Equation (5.16) is analogous to equation (5.5).)

Rearranging terms,

$$\ln E^r(\sigma(u, z), p) = \frac{\hat{K}(p, z)}{\tilde{K}(p, z)} \ln E^r(u, p) + \left[\frac{\ln \hat{G}(p, z) - \ln \tilde{G}(p, z)}{\tilde{K}(p, z)} \right]. \quad (5.17)$$

Defining the functions H and Q by $H(p, z) = \hat{K}(p, z)/\tilde{K}(p, z)$ and $Q(p, z) = (\ln \hat{G}(p, z) - \ln \tilde{G}(p, z))/\tilde{K}(p, z)$, (5.17) can be rewritten as

$$\ln E^r(\sigma(u, z), p) = H(p, z) \ln E^r(u, p) + Q(p, z). \quad (5.18)$$

Although $\hat{K}$ and $\tilde{K}$ are determined, in the differentiable case, by $\hat{G}$ and $\tilde{G}$, the relationship, as noted in Section 4 is not one-to-one. We investigate the problem using a condition on the ranges of the functions $\hat{G}$, $\tilde{G}$, $\hat{K}$, and $\tilde{K}$ that is analogous to Range Condition A.

Range Condition R: $\mathcal{Z}$ contains a subset $\overset{\circ}{\mathcal{Z}}$ of at least one continuous variable such as age and, for every $p \in \mathcal{R}_{++}^m$, the functions $\hat{G}$, $\tilde{G}$, $\hat{K}$, and $\tilde{K}$ are continuous in and sensitive to $\overset{\circ}{z} \in \overset{\circ}{\mathcal{Z}}$. In addition, $Q(p, z)$ can be moved independently of $H(p, z)$ by changing $\overset{\circ}{z}$.

Theorems 4 and 5 prove that the equivalent-income function and equivalence scales are uniquely identified by behaviour if GRESE holds and the reference expenditure function is not log-affine. Theorem 4 investigates the identification problem when $\hat{G} \neq \tilde{G}$ and $\hat{K} \neq \tilde{K}$.

Theorem 4: Given GRESE, two pairs of functions $\hat{G}$ and $\tilde{G}$ and $\hat{K}$ and $\tilde{K}$ with $\hat{G} \neq \tilde{G}$ and $\hat{K} \neq \tilde{K}$ satisfying Range Condition R are consistent with the same household behaviour if and only if there exist functions $C: \mathcal{R}_{++}^m \rightarrow \mathcal{R}_{++}$, $D: \mathcal{R}_{++}^m \rightarrow \mathcal{R}_{++}$, $f: \mathcal{R} \rightarrow \mathcal{R}$, $h: \mathcal{Z} \rightarrow \mathcal{R}_{++}$ and $q: \mathcal{Z} \rightarrow \mathcal{R}$ such that, for all (u, p, z) ,

$$\ln E^r(u, p) = C(p)f(u) + \ln D(p), \quad (5.19)$$

$$\hat{K}(p, z) = h(z)\tilde{K}(p, z), \quad (5.20)$$

and

$$\ln \hat{G}(p, z) = \ln \tilde{G}(p, z) + \tilde{K}(p, z) \left[C(p)q(z) + \ln D(p)(1 - h(z)) \right], \quad (5.21)$$

where C is homogeneous of degree zero, D is homogeneous of degree one, f is increasing and continuous, $h(z^r) = 1$ and $q(z^r) = 0$.

Proof: See the Appendix.

Theorem 5 considers the case in which there are two different G s, $\hat{G}$ and $\tilde{G}$, but only one function K . In that case, $H(p, z) = 1$ for all (p, z) .

Theorem 5: *Given GRESE, two pairs of functions $\hat{G}$ and $\tilde{G}$ and $\hat{K}$ and $\tilde{K}$ with $\hat{G} \neq \tilde{G}$ and $\hat{K} = \tilde{K} := K$ satisfying Range Condition R are consistent with the same household behaviour if and only if there exist functions $C: \mathcal{R}_{++}^m \rightarrow \mathcal{R}_{++}$, $D: \mathcal{R}_{++}^m \rightarrow \mathcal{R}_{++}$, $f: \mathcal{R} \rightarrow \mathcal{R}$, $h: \mathcal{Z} \rightarrow \mathcal{R}_{++}$ and $q: \mathcal{Z} \rightarrow \mathcal{R}$ such that, for all (u, p, z) ,*

$$\ln E^r(u, p) = C(p)f(u) + \ln D(p) \quad (5.22)$$

and

$$\ln \hat{G}(p, z) = \ln \tilde{G}(p, z) + K(p, z)C(p)q(z), \quad (5.23)$$

where C is homogeneous of degree zero, D is homogeneous of degree one, f is increasing and continuous, and $q(z^r) = 0$.

Proof: See the Appendix.

Because, in the differentiable case, K is functionally dependent on G , we need not consider the case where $\hat{G} = \tilde{G}$ and $\hat{K} \neq \tilde{K}$. It follows from Theorems 4 and 5 that, in order to identify equivalent-income functions and equivalence scales from behaviour alone, the reference expenditure function must not be log-affine. If this condition is met, there is a unique indirect utility function V (and corresponding expenditure function E), consistent with demand behaviour, satisfying GRESE.

6. Empirical Model

Estimation of equivalent-income functions requires estimation of a demand system. Recently, many scholars have chosen non-parametric or semi-parametric approaches to the estimation of consumer demand and equivalence scales (see, for example, Blundell, Duncan and Pendakur [1998]; Pendakur [1999]; Gozalo [1997]) because such approaches restrict the shapes of Engel curves less than parametric approaches do. Unfortunately, non-parametric and semi-parametric methods are not well-suited to the problem at hand for two reasons. First, as we shall see below, identification of equivalent-income functions comes from *both* price and income responses. Non- and semi-parametric methods are not well-developed for estimation of consumer demand as a function of both prices and income; rather the strategy is usually to hold prices constant and look at variation across income only. Second, in the case of

GRESE, the parameters of the log-affine equivalent-income function are not independent— K is a function of G . Estimating functionally dependent parameters in a semiparametric environment is extremely cumbersome.

Parametric estimation of equivalent-income functions requires specification of the demand system and parametric expressions of the restrictions embodying GAESE and GRESE. The econometric strategy we employ exploits the following convenient characteristic of GAESE and GRESE. Given GAESE, if any household type has commodity demands that are quadratic in income, then all household types have commodity demands that are quadratic in income. Similarly, given GRESE, if any household type has expenditure-share equations that are quadratic in the logarithm of income, then all household types have expenditure-share equations that are quadratic in the logarithm of income. Thus, we estimate equivalent-income functions given GAESE using the Quadratic Expenditure System (QES) due to Pollack and Wales [1979, 1992] and equivalent-income functions given GRESE using the Quadratic Almost Ideal demand system (QAI) due to Banks, Blundell and Lewbel [1997].

6.1. *The Data*

We use expenditure data from the 1978, 1982, 1984, 1986, 1990 and 1992 Canadian Family Expenditure Surveys (Statistics Canada [1990, 1992]) to estimate demand systems given GAESE and GRESE and recover equivalent-income functions. These data contain annual expenditures in approximately one hundred categories for five to ten thousand households per year. We use only (1) households residing in the fourteen largest Census Metropolitan Areas of Canada (to minimize the effects of home production), (2) households with rental tenure (to avoid rent imputation) and (3) households whose members are all under the age of 65.

We estimate a demand system composed of the following seven expenditure categories: (1) food purchased from stores; (2) total rent, including utilities; (3) clothing for adults; (4) clothing for children; (5) transportation, excluding capital expenditures; (6) household operation; and (7) personal care. Personal care is the ‘left-out’ equation in all estimation. Price data for these commodity groups are available for the six survey years in five regions of Canada: (1) Atlantic Canada; (2) Quebec; (3) Ontario; (4) the Prairies; and (5) British Columbia. In the following estimation, prices are normalized so that residents of Ontario, who form the largest population subgroup in the sample, face the prices (100, ..., 100) in 1986.

We use several household demographic characteristics in our estimation. For each household, we include information on the number of young children aged 0–4, the number of older children aged 5–15, the number of teenagers aged 16–17, the number of adults aged 18–64, the age of the household head, the education of the household head and the labour force

activity of the household head and spouse. We estimate three versions of all demand systems: (1) estimates conditioned on the number of household members and their ages; (2) estimates conditioned on the number of household members, their ages and the education of the household head; and (3) estimates conditioned on the number of household members, their ages, the education of the household head and the labour force activity of the head and spouse. In all of the above cases, the number of household members and their ages are included in z , the list of exogenous household characteristics, for the purpose of estimating equivalent-income functions. The first demand system ignores the effects of education and labour force activity on utility, the second allows education (treated as an exogenous variable) to affect it, and the third allows education and labour force activity (treated as exogenous variables) to affect it.

Summary statistics on the data are given in Table 1.

Table 1: The Data

No. of households: 8191	Mean	Std Dev	Min	Max
Total expenditure	13289	6677	1623	64539
Food share	0.240	0.102	0.001	0.834
Rent share	0.393	0.116	0.001	0.933
Adult clothing share	0.096	0.072	0.000	0.551
Children's clothing share	0.011	0.021	0.000	0.236
Transport share	0.129	0.091	0.000	0.667
Household operation share	0.089	0.052	0.000	0.653
Personal care share	0.041	0.024	0.000	0.207
Age of head (years)	38.6	11.2	25	64
Education of head (years)	12.5	2.2	8	16
Head unemployed	0.17	0.37	0	1
Spouse unemployed	0.16	0.37	0	1
Members aged 18–64 (adults)	1.62	0.68	1	7
Members aged 0–5 (young children)	0.26	0.58	0	4
Members aged 6–15 (older children)	0.34	0.72	0	6
Members aged 16–17 (teenagers)	0.06	0.26	0	2

6.2. Parametric Demand-System Specifications

For estimation given GAESE, we use the Quadratic Expenditure System (QES) proposed by Pollack and Wales [1979, 1992], in which

$$V(p, x, z) = \left(\left(\frac{x - a(p, z)}{b(p, z)} \right)^{-1} - q(p, z) \right)^{-1}, \quad (6.1)$$

where a and b are homogeneous of degree one in p and q is homogeneous of degree zero in p . Denoting $a^r(p) = a(p, z^r)$, $b^r(p) = b(p, z^r)$ and $q^r(p) = q(p, z^r)$ and assuming that the reference indirect utility function is QES, GAESE implies that

$$\begin{aligned} V(p, x, z) &= V^r(p, x^e) = V^r \left(p, \frac{x - A(p, z)}{R(p, z)} \right) \\ &= \left(\left(\frac{x - A(p, z) - R(p, z)a^r(p)}{R(p, z)b^r(p)} \right)^{-1} - q^r(p) \right)^{-1}. \end{aligned} \quad (6.2)$$

Thus, if reference preferences satisfy QES then, given GAESE, all households have QES preferences. In this case, GAESE implies that

$$a(p, z) = R(p, z)a^r(p) + A(p, z), \quad (6.3)$$

$$b(p, z) = R(p, z)b^r(p), \quad (6.4)$$

and

$$q(p, z) = q^r(p). \quad (6.5)$$

Thus, we can estimate equivalent-income functions given GAESE by requiring $q(p, z) = q^r(p)$ and calculating

$$R(p, z) = \frac{b(p, z)}{b^r(p)} \quad (6.6)$$

and

$$A(p, z) = a(p, z) - R(p, z)a^r(p). \quad (6.7)$$

As noted above, both R and A are functions which depend on prices.

In addition to estimating equivalent-income functions under GAESE, use of the QES allows a simple parametric test of GAESE against an unrestricted QES alternative. In particular, if preferences do not satisfy $q(p, z) = q^r(p)$, then GAESE cannot hold. It is also possible to test down from GAESE in this framework. We can test AESE against GAESE by asking whether $R(p, z) = 1$ or, equivalently, $b(p, z) = b^r(p)$. Although it is more complex,

we can also test RESE against GAESE by asking whether $A(p, z) = 0$, which is equivalent to

$$\frac{a(p, z)}{a^r(p)} = \frac{b(p, z)}{b^r(p)}. \quad (6.8)$$

To estimate the demand systems, we specify the functions a , b and q as follows:

$$\ln a(p, z) = a_0(z) + \sum_{k=1}^m a_k(z) \ln p_k + \frac{1}{2} \sum_{k=1}^m \sum_{l=1}^m a_{kl}(z) \ln p_k \ln p_l, \quad (6.9)$$

where $\sum_{k=1}^m a_k(z) = 1$, $\sum_{l=1}^m a_{kl}(z) = 0$ for all k , and $a_{kl}(z) = a_{lk}(z)$ for all k, l ;

$$\ln b(p, z) = b_0(z) + \sum_{k=1}^m b_k(z) \ln p_k, \quad (6.10)$$

where $\sum_{k=1}^m b_k(z) = 1$; and

$$q(p, z) = q_0(z) + \sum_{k=1}^m q_k(z) \ln p_k, \quad (6.11)$$

where $\sum_{k=1}^m q_k(z) = 0$. The functions a_k , a_{kl} , b_k and q_k depend on z , and we assume that

$$a_k(z) = a_k^r + \sum_{s=1}^4 a_k^s n^s + a_k^{age} age + a_k^{educ} educ + a_k^{lfh} lf h + a_k^{lfs} lfs, \quad (6.12)$$

$$a_{kl}(z) = a_{kl}^r + \sum_{s=1}^4 a_{kl}^s n^s + a_{kl}^{age} age + a_{kl}^{educ} educ + a_{kl}^{lfh} lf h + a_{kl}^{lfs} lfs, \quad (6.13)$$

$$b_k(z) = b_k^r + \sum_{s=1}^4 b_k^s n^s + b_k^{age} age + b_k^{educ} educ + b_k^{lfh} lf h + b_k^{lfs} lfs, \quad (6.14)$$

and

$$q_k(z) = q_k^r + \sum_{s=1}^4 q_k^s n^s + q_k^{age} age + q_k^{educ} educ + q_k^{lfh} lf h + q_k^{lfs} lfs. \quad (6.15)$$

The characteristics contained in z are: n^1 , the number of adults aged 18–64 minus one; n^2 , the number of young children aged 0–4; n^3 , the number of older children aged 5–15; n^4 , the number of teenagers aged 16–17; age , the age of the household head minus 40; $educ$, the number of years of education of the household head minus 12; lfh , a dummy indicating that the household head is not employed; and lfs , a dummy indicating that there is a spouse and that the spouse is not employed. For some estimates, the coefficients on $educ$, lfh and lfs are set to zero. For households consisting of an employed single childless adult aged forty with twelve years of education (the reference household type), $a_k(z) = a_k^r$, $a_{kl}(z) = a_{kl}^r$, $b_k(z) = b_k^r$ and $q_k(z) = q_k^r$.

We note that, with the QES specification, the GAESE functions A and R take on relatively simple forms in terms of the parameters if evaluated at an m -vector of equal prices $\hat{p}\mathbf{1}_m = (\hat{p}, \dots, \hat{p})$, with

$$R(\hat{p}\mathbf{1}_m, z) = \exp(b_0(z)) \quad (6.16)$$

and

$$A(\hat{p}\mathbf{1}_m, z) = \exp(a_0(z) + \ln \hat{p}) - \exp(b_0(z) + a_0^r + \ln \hat{p}). \quad (6.17)$$

With these specifications for a , b and q , A can be either positive or negative but R is positive.

Application of Roy's Theorem to (6.1) generates the quadratic commodity-demand equations

$$D_j(p, x, z) = \frac{\partial a(p, z)}{\partial p_j} + \frac{\partial \ln b(p, z)}{\partial p_j} (x - a(p, z)) + \frac{\partial q(p, z)}{\partial p_j} \frac{(x - a(p, z))^2}{b(p, z)}, \quad (6.18)$$

$j = 1, \dots, m$. Substituting (6.9)–(6.11) and (6.12)–(6.15) into (6.18), we get

$$\begin{aligned} D_j(p, x, z) = & \frac{a(p, z)}{p_j} \left(a_j(z) + \sum_{k=1}^m a_{jk}(z) \ln p_k \right) \\ & + \frac{b_j(z)}{p_j} (x - a(p, z)) + \frac{q_j(z)}{p_j b(p, z)} (x - a(p, z))^2, \end{aligned} \quad (6.19)$$

$j = 1, \dots, m$. Equation (6.19) does not necessarily satisfy GAESE, which requires $q_k(z) = q_k^r$. These demand equations are homogeneous of degree zero in $(b(p, z), q_k(z), \dots, q_m(z))$. So, for identification of $q_k(z)$ in the estimation of unrestricted quadratic commodity demand equations (used to test up from GAESE), we impose the restriction $b_0(z) = 0$. However, given GAESE, this restriction is not necessary because $q_k(z) = q_k^r$. Here, we use the weaker restriction that $b_0(z^r) = b_0^r = 0$. Thus, GAESE restricts preferences in such a way that, given quadratic commodity demand equations, the second-order terms are proportional across household types.

As noted above, under the QES model, GAESE can be tested in three directions. First, it can be tested against an unrestricted alternative by testing whether $q(p, z) = q^r(p)$. This amounts to testing whether $q_k^s = q_k^{age} = q_k^{educ} = q_k^{lfh} = q_k^{lfs} = 0$. Second, AESE can be tested against a GAESE alternative by testing whether $b(p, z) = b^r(p)$ in a GAESE-restricted model. This amounts to testing whether $b_k^s = b_k^{age} = b_k^{educ} = b_k^{lfh} = b_k^{lfs} = 0$. Third, RESE can be tested against a GAESE alternative by testing whether $a(p, z)/a^r(p) = b(p, z)/b^r(p)$ in a GAESE-restricted model. This amounts to testing whether $b_0^s = a_0^s$, $b_0^{age} = a_0^{age}$, $b_0^{educ} = a_0^{educ}$, $b_0^{lfh} = a_0^{lfh}$, $b_0^{lfs} = a_0^{lfs}$, $b_k^s = a_k^s$, $b_k^{age} = a_k^{age}$, $b_k^{educ} = a_k^{educ}$, $b_k^{lfh} = a_k^{lfh}$, $b_k^{lfs} = a_k^{lfs}$, and $a_{kl}^s = a_{kl}^{age} = a_{kl}^{educ} = a_{kl}^{lfh} = a_{kl}^{lfs} = 0$.

The QES corresponds to an affine expenditure function if and only if $q(p, z) = 0$ and corresponds to a log-affine expenditure function if and only if $b(p, z) = 0$ and $q(p, z) = 0$.

Since equivalent-income functions under GAESE are uniquely identifiable if the expenditure function is neither affine nor log-affine, we are able to test the identification restrictions of GAESE by testing whether $q_k^r = q_k^s = q_k^{age} = q_k^{educ} = q_k^{lfh} = q_k^{lfs} = 0$ and $b_k^r = b_k^s = b_k^{age} = b_k^{educ} = b_k^{lfh} = b_k^{lfs} = 0$.

For estimation given GRESE, we use the Quadratic Almost Ideal (QAI) model of Banks, Blundell and Lewbel [1997], in which

$$V(p, x, z) = \left(\left(\frac{\ln x - \ln \bar{a}(p, z)}{\bar{b}(p, z)} \right)^{-1} - \bar{q}(p, z) \right)^{-1}, \quad (6.20)$$

where $\bar{a}$ is homogeneous of degree one in p and $\bar{b}$ and $\bar{q}$ are homogeneous of degree zero in p . Denoting $\bar{a}^r(p) = \bar{a}(p, z^r)$, $\bar{b}^r(p) = \bar{b}(p, z^r)$ and $\bar{q}^r(p) = \bar{q}(p, z^r)$, and assuming that the reference indirect utility function is QAI, GRESE implies that

$$\begin{aligned} V(p, x, z) &= V^r(p, x^e) = V^r \left(p, \exp \left\{ \frac{\ln x^e - \ln G(p, z)}{K(p, z)} \right\} \right) \\ &= \left(\left(\frac{\ln x - \ln G(p, z) - K(p, z) \ln \bar{a}^r(p)}{K(p, z) \bar{b}^r(p)} \right)^{-1} - \bar{q}^r(p) \right)^{-1}. \end{aligned} \quad (6.21)$$

Thus, if reference preferences satisfy QAI, then under GRESE all households have QAI preferences. In this case, GRESE implies that

$$\ln \bar{a}(p, z) = K(p, z) \ln \bar{a}^r(p) + \ln G(p, z), \quad (6.22)$$

$$\bar{b}(p, z) = K(p, z) \bar{b}^r(p), \quad (6.23)$$

and

$$\bar{q}(p, z) = \bar{q}^r(p). \quad (6.24)$$

Thus, we can estimate equivalent-income functions given GRESE by requiring $\bar{q}(p, z) = \bar{q}^r(p)$ and calculating

$$K(p, z) = \frac{\bar{b}(p, z)}{\bar{b}^r(p)} \quad (6.25)$$

and

$$\ln G(p, z) = \ln \bar{a}(p, z) - K(p, z) \ln \bar{a}^r(p). \quad (6.26)$$

We note that, with these specifications for $\bar{a}$, $\bar{b}$ and $\bar{q}$, G and K are positive. As noted above, K and G are functionally related and depend on prices.

In addition to estimating equivalent-income functions under GRESE, use of the QAI allows a simple parametric test of GRESE against an unrestricted QAI alternative. In particular, if preferences do not satisfy $\bar{q}(p, z) = \bar{q}^r(p)$, then GRESE cannot hold. It is also

possible to test down from GRESE in this framework. We can test RESE against GRESE by asking whether $K(p, z) = 1$ or, equivalently, $\bar{b}(p, z) = \bar{b}^r(p)$.

To estimate the demand systems, we specify the functions $\bar{a}$, $\bar{b}$ and $\bar{q}$ as follows:

$$\ln \bar{a}(p, z) = \bar{a}_0(z) + \sum_{k=1}^m \bar{a}_k(z) \ln p_k + \frac{1}{2} \sum_{k=1}^m \sum_{l=1}^m \bar{a}_{kl}(z) \ln p_k \ln p_l, \quad (6.27)$$

where $\sum_{k=1}^m \bar{a}_k(z) = 1$, $\sum_{l=1}^m \bar{a}_{kl}(z) = 0$ for all k , and $\bar{a}_{kl}(z) = \bar{a}_{lk}(z)$ for all k, l ;

$$\ln \bar{b}(p, z) = \bar{b}_0(z) + \sum_{k=1}^m \bar{b}_k(z) \ln p_k, \quad (6.28)$$

where $\sum_{k=1}^m \bar{b}_k(z) = 0$; and

$$\bar{q}(p, z) = \bar{q}_0(z) + \sum_{k=1}^m \bar{q}_k(z) \ln p_k, \quad (6.29)$$

where $\sum_{k=1}^m \bar{q}_k(z) = 0$. The functions $\bar{a}_k$, $\bar{a}_{kl}$, $\bar{b}_k$ and $\bar{q}_k$ depend on z , and we assume that

$$\bar{a}_k(z) = \bar{a}_k^r + \sum_{s=1}^4 \bar{a}_k^s n^s + \bar{a}_k^{age} age + \bar{a}_k^{educ} educ + \bar{a}_k^{lfh} lfh + \bar{a}_k^{lfs} lfs, \quad (6.30)$$

$$\bar{a}_{kl}(z) = \bar{a}_{kl}^r + \sum_{s=1}^4 \bar{a}_{kl}^s n^s + \bar{a}_{kl}^{age} age + \bar{a}_{kl}^{educ} educ + \bar{a}_{kl}^{lfh} lfh + \bar{a}_{kl}^{lfs} lfs, \quad (6.31)$$

$$\bar{b}_k(z) = \bar{b}_k^r + \sum_{s=1}^4 \bar{b}_k^s n^s + \bar{b}_k^{age} age + \bar{b}_k^{educ} educ + \bar{b}_k^{lfh} lfh + \bar{b}_k^{lfs} lfs, \quad (6.32)$$

and

$$\bar{q}_k(z) = \bar{q}_k^r + \sum_{s=1}^4 \bar{q}_k^s n^s + \bar{q}_k^{age} age + \bar{q}_k^{educ} educ + \bar{q}_k^{lfh} lfh + \bar{q}_k^{lfs} lfs. \quad (6.33)$$

For some estimates, the coefficients on *educ*, *lfh* and *lfs* are set equal to zero.

With the QAI system, the GRESE functions K and G take on relatively simple forms in terms of the parameters if evaluated at an m -vector of equal prices $\hat{p}\mathbf{1}_m = (\hat{p}, \dots, \hat{p})$, with

$$K(\hat{p}\mathbf{1}_m, z) = \exp(\bar{b}_0(z)) \quad (6.34)$$

and

$$\ln G(\hat{p}\mathbf{1}_m, z) = \bar{a}_0(z) + \ln \hat{p} - \exp(\bar{b}_0(z))(\bar{a}_0^r + \ln \hat{p}). \quad (6.35)$$

Application of Roy's Theorem to (6.20) generates the log-quadratic expenditure-share equations

$$W_j(p, \ln x, z) = \frac{\partial \ln \bar{a}(p, z)}{\partial \ln p_j} + \frac{\partial \ln \bar{b}(p, z)}{\partial \ln p_j} (\ln x - \ln \bar{a}(p, z)) + \frac{\partial \bar{q}(p, z)}{\partial \ln p_j} \frac{(\ln x - \ln \bar{a}(p, z))^2}{\bar{b}(p, z)}, \quad (6.36)$$

$j = 1, \dots, m$. Substituting (6.27)–(6.33) into (6.36), we get

$$W_j(p, \ln x, z) = \left(\bar{a}_j(z) + \sum_{k=1}^m \bar{a}_{jk}(z) \ln p_k \right) + \bar{b}_j(z) (\ln x - \ln \bar{a}(p, z)) + \bar{q}_j(z) \frac{(\ln x - \ln \bar{a}(p, z))^2}{\bar{b}(p, z)}, \quad (6.37)$$

$j = 1, \dots, m$. Estimation given GRESE requires $\bar{q}_k(z) = \bar{q}_k^r$.

Here, the share equations are homogeneous of degree zero in $(\bar{b}_0(z), \bar{q}_1(z), \dots, \bar{q}_m(z))$, so that these functions are not separately identifiable. To identify $\bar{q}_k(z)$ in the unrestricted QAI (used to test up from GRESE), we impose the restriction $\bar{b}_0(z) = 0$. However, under GRESE, this restriction is not necessary because $\bar{q}_k(z) = \bar{q}_k^r$ and we use the weaker restriction that $\bar{b}_0(z^r) = \bar{b}_0^r = 0$. Thus, given QAI demands, GRESE restricts preferences in such a way that the second-order terms are proportional across household types.

Under the QAI model, GRESE can be tested in two directions. First, GRESE can be tested against an unrestricted alternative by testing whether $\bar{q}(p, z) = \bar{q}^r(p)$. This amounts to testing whether $\bar{q}_k^s = \bar{q}_k^{age} = \bar{q}_k^{educ} = \bar{q}_k^{lfh} = \bar{q}_k^{lfs} = 0$. Second, RESE can be tested against a GRESE alternative by testing whether $\bar{b}(p, z) = \bar{b}^r(p)$ in a GAESE restricted model. This amounts to testing whether $\bar{b}_k^s = \bar{b}_k^{age} = \bar{b}_k^{educ} = \bar{b}_k^{lfh} = \bar{b}_k^{lfs} = 0$.

Equivalent-income functions under GRESE are uniquely identifiable if and only if the reference expenditure function is not log-affine in income. The QAI model corresponds to log-affine expenditures if and only if $\bar{q}(p, z) = 0$. Thus, we are able to test the identification restrictions of GRESE by testing whether $\bar{q}_k^r = \bar{q}_k^s = \bar{q}_k^{age} = \bar{q}_k^{educ} = \bar{q}_k^{lfh} = \bar{q}_k^{lfs} = 0$.

To assess the properties of GAESE and GRESE with respect to child goods, we pay special attention to the children's-clothing share equations. For childless households, we code children's clothing expenditures as zero (4% of childless households have nonzero children's clothing expenditures, presumably due to gift-giving) and restrict the demand system so that the demands for children's clothing are zero in childless households.¹²

¹² Denote children's clothing as c and define a dummy variable *child* equal to one for all households with children. For estimation with the QES, we restrict the functions a_c , a_{jc} , b_c and q_c as follows:

$$a_c(z) = child \left(a_c^r + \sum_{s=1}^4 a_c^s n^s + a_c^{age} age + a_c^{educ} educ + a_c^{lfh} lfh + a_c^{lfs} lfs \right);$$

$$a_{kc}(z) = child \left(a_{kc}^r + \sum_{s=1}^4 a_{kc}^s n^s + a_{kc}^{age} age + a_{kc}^{educ} educ + a_{kc}^{lfh} lfh + a_{kc}^{lfs} lfs \right);$$

For all estimation, x is total expenditure within the demand system. For the sake of continuity and brevity, we will continue to refer to x as ‘income’. All reported estimates are by standard maximum likelihood (ML) and do not correct for the possible endogeneity of total expenditure. Estimation using U.K. data (e.g. Banks, Blundell and Lewbel [1997], Blundell, Duncan and Pendakur [1998]) commonly correct for endogeneity of total expenditure because those data are for two week time-spans, leading to problems of lumpy durables consumption, which produces measurement error in total expenditure. In contrast, Canadian expenditure data are reported at the annual level which greatly mitigates this problem. In addition, we do not include owned-accommodation expenses or automobile-purchase expenses in the demand system. Thus, the more efficient maximum likelihood estimators (under the exogeneity null) are appropriate in the current circumstances. However, in addition to the reported material, we estimated all the demand systems by GMM instrumenting for potentially endogenous total expenditure.¹³ Point estimates were very similar to those under ML, but variances were larger. In several cases, total expenditure did not pass the exogeneity test but, for all regressions that included education of the household head, the exogeneity test passed at the 1% level.

Because we are interested in equivalent-income functions corresponding to unconditional rather than conditional expenditure functions, it is important to take possible endogeneity of household characteristics seriously (see Browning [1983] and Browning and Meghir [1991] for a full treatment of conditional versus unconditional expenditure functions). We take household composition (age and number of members) as exogenous because children are not simply consumption goods, parents do not have complete control of fertility and are not fully informed about the long-term consequences of fertility decisions, and because evolutionary theory predicts that self-interest may have little to do with the desire to have children. We cautiously take the education of the household head as exogenous because education decisions for most household heads were made in the past without the benefit of current information about income and prices. We are more reticent to take the labour-force activity of the head and spouse as exogenous because leisure and consumption decisions are made jointly, and in this case, we are left with two alternatives. Either we condition on labour force activity and take equivalent-income functions as conditional (and therefore difficult to interpret),

$$b_c(z) = child \left(b_c^r + \sum_{s=1}^4 b_c^s n^s + b_c^{age} age + b_c^{educ} educ + b_c^{lfh} lf h + b_c^{lfs} lfs \right);$$

and

$$q_c(z) = child \left(q_c^r + \sum_{s=1}^4 q_c^s n^s + q_c^{age} age + q_c^{educ} educ + q_c^{lfh} lf h + q_c^{lfs} lfs \right).$$

For estimation with the QAI, analogous restrictions on $\bar{a}_c$, $\bar{a}_{jc}$, $\bar{b}_c$ and $\bar{q}_c$ were used.

¹³ We used a variety of instruments in GMM estimation, including all excluded elements of z , labour income, labour income squared, net income and net income squared.

or drop labour-supply information from the demand system and interpret equivalent-income functions as unconditional. Neither alternative is particularly attractive, but we lean towards the latter option. Thus, although we present estimates from demand systems using all three sets of household characteristics noted above, we focus mainly on results from estimation conditioning on household composition and the education of the household head.

7. Results

All results in this section are based on estimation where the reference demand system is given by (6.19) or (6.37). We multiply (6.19) by p_j/x to get QES expenditure-share equations so that the dependent variables for all estimation are expenditure shares. Selected coefficients are presented in the tables and equivalent-income functions and relative equivalence scales for several household types in the figures. Complete and detailed parameter estimates are available from the authors on request. The reference household type for all estimation is an employed childless single adult aged 40 with 12 years of education. Asymptotic standard errors are provided in Tables 3–6, 8 and 9.

Table 2 presents model statistics for nine models, each with three specifications of the list of elements included in z . The nine models are: (1) unrestricted QES; (2) GAESE-restricted QES; (3) affine (quasi-homothetic) QES; (4) AESE-restricted QES; (5) RESE-restricted QES; (6) unrestricted QAI; (7) GRESE-restricted QAI; (8) log-affine (almost ideal) QAI; and (9) RESE-restricted QAI. Specification 1 uses only information on the ages and numbers of household members, Specification 2 adds the education of the household head and Specification 3 adds the labour force activity of the household head and spouse.

Table 2 can be summarized simply: every testable hypothesis is rejected at conventional levels of significance. Under the QES model, we reject GAESE against an unrestricted QES alternative, we reject the hypothesis that the GAESE equivalent-income function is not identified, we reject AESE against GAESE and we reject RESE against GAESE. Similarly, under the QAI model, we reject GRESE against an unrestricted QAI alternative, we reject the hypothesis that the GRESE equivalent-income function is not identified and we reject RESE against GRESE. In addition, for both QES and QAI models, we reject Specification 1 against a Specification 2 alternative and we reject Specification 2 against a Specification 3 alternative.

We see three important observations in these tests of hypotheses. First, GAESE and GRESE fit the data better than AESE and RESE. This does not surprise us, given the severe restrictions that AESE and RESE put on the shapes of Engel curves and on the demand for children’s goods in particular. Second, we note that the likelihood-ratio test statistics associated with testing down from GAESE and GRESE are an order of magnitude larger than the likelihood-ratio test statistics associated with testing up from GAESE and GRESE. This

suggests that the shape restrictions inherent in GAESE and GRESE do less violence to the data than the stricter restrictions inherent in AESE and RESE. Third, adding characteristics to the list of those included always improves the fit. However, due to difficulties in interpreting conditional equivalent-income functions, this should not be taken as an invitation to include all available variables.

Table 2: Model Statistics

System	Restrictions	Specification 1		Specification 2		Specification 3	
		Log Likelihood	df	Log Likelihood	df	Log Likelihood	df
QES	Unrestricted	86965.5	245	87111.3	286	87426.1	368
	GAESE	86855.9	214	86953.5	249	87254.8	319
	Affine [†]	86386.8	209	86491.2	244	86832.1	314
	AESE	85534.7	178	85652.7	206	85980.0	262
	RESE	85896.2	53	86017.7	60	86210.8	74
QAI	Unrestricted	87261.5	245	87353.8	286	87636.3	368
	GRESE	87140.8	214	87209.1	249	87473.1	319
	Log-Affine [‡]	87050.2	209	87098.8	244	87351.1	314
	RESE	86635.6	178	86703.7	206	86945.4	262

[†] quasi-homothetic

[‡] almost ideal

7.1. Equivalent-Income Functions Given GAESE

Table 3 shows selected parameter estimates from GAESE-restricted QES demand systems. As noted above, the selected parameters are sufficient to estimate equivalent-income functions and equivalence scales at a vector of equal prices. We consider the vector of equal prices $p = 100(\mathbf{1}_7) = (100, 100, 100, 100, 100, 100, 100)$ which is the price vector for Ontario in 1986. Figures 1 and 2 show equivalent-income functions and relative equivalence scales computed from the Specification 2 coefficients in Table 3.

The parameter estimates are broadly similar across the three specifications. Looking first at R (where $R(100(\mathbf{1}_7), z) = \exp(b_0(z))$), which reflects relative costs, we see that R is increasing in the numbers of household members, particularly for young children. Further, R is increasing in age and education. Turning to A (where $A(100(\mathbf{1}_7), z) = \exp(a_0(z) + \ln 100) - \exp(b_0(z) + a_0^r + \ln 100)$), which reflects absolute costs, we see that A is for the most part decreasing in the number of household members, especially adults and teenagers.

Table 3: Estimates given QES and GAESE

	Specification 1		Specification 2		Specification 3	
	Coefficient	Std Error	Coefficient	Std Error	Coefficient	Std Error
a_0^r	2.396***	(0.132)	1.474***	(0.161)	1.606***	(0.164)
a_0^1	-0.305***	(0.045)	-0.335***	(0.048)	-0.309***	(0.051)
a_0^2	0.097***	(0.032)	0.096***	(0.033)	-0.817***	(0.088)
a_0^3	-0.262***	(0.038)	-0.226***	(0.038)	-0.090***	(0.035)
a_0^4	-0.625***	(0.173)	-0.722***	(0.195)	-0.247**	(0.108)
a_0^{age}	-0.010***	(0.003)	-0.010***	(0.003)	0.024**	(0.010)
a_0^{educ}			0.023**	(0.010)	-0.004*	(0.002)
a_0^{lfh}					0.115**	(0.053)
a_0^{lfs}					-0.546***	(0.124)
b_0^1	0.332***	(0.024)	0.336***	(0.024)	0.354***	(0.026)
b_0^2	0.676***	(0.050)	0.653***	(0.048)	0.577***	(0.045)
b_0^3	0.441***	(0.036)	0.451***	(0.035)	0.438***	(0.037)
b_0^4	0.104	(0.064)	0.120*	(0.063)	0.128*	(0.066)
b_0^{age}	0.015***	(0.002)	0.039***	(0.007)	0.032***	(0.007)
b_0^{educ}			0.014***	(0.002)	0.015***	(0.002)
b_0^{lfh}					0.112	(0.070)
b_0^{lfs}					-0.074	(0.051)

*** significant at the 1% level
** significant at the 5% level
* significant at the 10% level

These estimates can be more easily understood graphically. Figure 1 shows equivalent income given GAESE for six household types: childless couples; single parents with one older child; dual parents with one younger child; dual parents with one older child; dual parents with one teenager; and dual parents with an older child and a teenager. Note that the equivalent-income functions for single parents with an older child and dual parents with a younger child lie almost exactly on top of each other (and so are difficult to distinguish). In terms of the equivalent-income function, A gives the intercept and R gives the slope. As is evident from the figure, these equivalent-income functions are affine in income, with different slopes and intercepts depending on household characteristics. Comparing equivalent incomes

for childless couples and dual parents with an older child, we see that the presence of the children decreases the slopes. Single parents with an older child have an equivalent-income function that is higher and steeper than that of dual parents with an older child.

Figure 2 shows income-dependent relative equivalence scales given GAESE for the same six household types. These relative equivalence scales lie between one and the number of household members, satisfying our intuition that there are scale economies in household consumption. As noted above, the econometric specification requires R to be positive but does not restrict A . Thus, relative equivalence scales are not restricted to any particular range. For example, for A large and positive, relative equivalence scales can be less than one or even negative. That estimated relative equivalence scales lie in the relatively narrow plausible range is encouraging.

Relative equivalence scales are increasing in the number of household members. However, the age of household members seems to matter. For example, the scale for a single parent with one older child is about 0.10 higher than that of a childless couple, so that costs for the former are higher. Teenagers are the highest-cost group of children—the relative equivalence scale for dual parents with one teenager is about 0.50 higher than that of dual parents with one older child.

If $A(p, z) = 0$, then relative equivalence scales under GAESE are independent of income, and are equal to $R(p, z)$. For $A(p, z) < 0$, relative equivalence scales decline with income, and for $A(p, z) > 0$, relative equivalence scales rise with income. In our view, the most important feature of Figure 2 is not how relative equivalence scales change with demographic composition, but rather that all these relative equivalence scales increase with income. For example, the scale for childless couples is 1.32 at an income of \$5,000 and 1.39 at an income of \$30,000. For dual parents with one teenager, the relative equivalence scale increases substantially with income—it is 2.30 at an income of \$5,000 and 2.61 at an income of \$30,000. That these scales increase with income suggests that scale economies in household consumption are larger for poor households than for rich households. This is the situation that would emerge if highly shareable goods were necessities. In the demand systems estimated in this paper, expenditure shares on rent decline with income. Since housing is shared, this is consistent with increasing relative equivalence scales.

7.2. *Equivalent-Income Functions Given GRESE*

Table 4 shows selected parameter estimates from GRESE-restricted QAI demand systems. Figures 3 and 4 show equivalent-income functions and relative equivalence scales computed from the Specification 2 coefficients in Table 4. The parameter estimates are broadly similar across the three specifications. Looking first at K (where $K(100(\mathbf{1}_7), z) = \exp(\bar{b}_0(z))$), which is the income elasticity of the equivalent-income function, we see that K is increasing in the number of household members, particularly for young children and teenagers. Further, K

is decreasing in age and increasing in education. Turning to G (where $\ln G(100(\mathbf{17}), z) = \bar{a}_0(z) + \ln 100 - \exp(\bar{b}_0(z))(\bar{a}_0^r + \ln 100)$), which reflects relative costs at low income, we see that G is, for the most part, increasing in the number of household members, especially adults. Further, G seems to be increasing in age. The way G responds to z , given GRESE, is very similar to the way R responds to z , given GAESE. We note that the interpretation of G under GRESE is similar to that of R under GAESE—it reflects relative cost differentials.

Table 4: Estimates given QAI and GRESE

	Specification 1		Specification 2		Specification 3	
	Coefficient	Std Error	Coefficient	Std Error	Coefficient	Std Error
$\bar{a}_0^r$	1.114	(0.413)	5.189***	(0.368)	5.380***	(0.444)
$\bar{a}_0^1$	0.725***	(0.036)	0.660***	(0.030)	0.673***	(0.035)
$\bar{a}_0^2$	0.592***	(0.044)	0.524***	(0.038)	0.485***	(0.038)
$\bar{a}_0^3$	0.504***	(0.036)	0.426***	(0.029)	0.438***	(0.032)
$\bar{a}_0^4$	0.421***	(0.086)	0.349***	(0.076)	0.409***	(0.083)
$\bar{a}_0^{age}$	0.015***	(0.003)	0.010***	(0.003)	0.006**	(0.003)
$\bar{a}_0^{educ}$			−0.011	(0.010)	0.002	(0.009)
$\bar{a}_0^{lfh}$					0.340***	(0.065)
$\bar{a}_0^{lfs}$					0.342***	(0.064)
$\bar{b}_0^1$	0.009	(0.017)	0.082***	(0.026)	0.048**	(0.021)
$\bar{b}_0^2$	0.249***	(0.045)	0.127***	(0.044)	−0.002	(0.033)
$\bar{b}_0^3$	0.015	(0.020)	0.003	(0.025)	0.024	(0.023)
$\bar{b}_0^4$	−0.070	(0.050)	0.135*	(0.072)	0.171***	(0.060)
$\bar{b}_0^{age}$	−0.005***	(0.002)	−0.007***	(0.003)	−0.009***	(0.003)
$\bar{b}_0^{educ}$			0.029***	(0.010)	0.007	(0.008)
$\bar{b}_0^{lfh}$					−0.125*	(0.070)
$\bar{b}_0^{lfs}$					0.214***	(0.064)

*** significant at the 1% level

** significant at the 5% level

* significant at the 10% level

Figure 3 shows the equivalent-income function given GRESE. K determines the curvature of the equivalent-income function, and G determines the slope (at low income). Although it is difficult to see in the figure, all of these equivalent-income functions are slightly

concave, which means that equivalent income rises more slowly for rich households than for poor households. Equivalent income also declines with the number of household members, but is not monotonically declining in the age of children. The equivalent-income function for dual parents with one teenager lies for the most part above that for dual parents with one older child and entirely below that for dual parents with one younger child.

Figure 4 shows relative equivalence scales under GRESE. Here, the effect of K is more obvious. If $K(p, z) = 1$, equivalence scales are independent of income and equal to $G(p, z)$. For $K(p, z) < 1$, relative equivalence scales decrease with income, and for $K(p, z) > 1$, relative equivalence scales increase with income. Consequently, scale economies diminish as income rises, as they do under GAESE. Figure 4 shows that all scales increase with income. For a single parent with one older child, the increase in the relative equivalence scale is very small—it is 1.52 at an income of \$5,000 and 1.53 at an income of \$30,000. For dual parents with one teenager, the increase is much more dramatic. The relative equivalence scale is 2.06 at an income of \$5,000 and 2.88 at an income of \$30,000. For dual parents with one teenager, economies of household consumption are substantial at low incomes and negligible at high incomes.

A comparison of Figures 2 and 4 reveals some consistency across GAESE and GRESE in the estimated equivalence scales. In particular, given either assumption about the equivalent-income function, relative equivalence scales rise with income. However, the estimated relative equivalence scales given GRESE increase more with income than those given GAESE. This is due to the fact that under GAESE, the dependence of the relative equivalence scale on income comes through the function A , and therefore approaches zero as income becomes large. In contrast, under GRESE, the dependence of the relative equivalence scale on income comes through the function K , and therefore matters even at high income.

7.3. *Equivalent-Income Functions Given AESE and RESE*

Table 5 shows selected parameter estimates for AESE-restricted and RESE-restricted QES demand systems. Looking first at the results, given AESE, for A (where $A(100(\mathbf{1}_7), z) = \exp(a_0(z) + \ln 100)$), we see that absolute equivalence scales do not satisfy our a priori assumption that child costs are positive. The coefficients on the number of household members in all age groups are either negative or insignificantly different from zero (Table 2). As noted above, AESE is rejected against a GAESE alternative. Therefore, sensible equivalence scales must depend on income.

Turning now to the results for R given RESE (where $R(100(\mathbf{1}_7), z) = \exp(b_0(z))$), the estimates are much more plausible than those under AESE. Since, given RESE, the relative equivalence scale is independent of income and equal to $R(p, z)$, the coefficients may be interpreted as the response of cost to z . We see that costs are increasing in the number of

household members, though less so for teenagers. Further, costs increase in the age of the household head and the education level of the household head.

Table 5: Estimates given QES and AESE or RESE

	Specification 1		Specification 2		Specification 3	
	Coefficient	Std Error	Coefficient	Std Error	Coefficient	Std Error
AESE						
a_0^r	2.861***	(0.123)	2.801***	(0.133)	2.751***	(0.126)
a_0^1	-0.019	(0.016)	-0.019	(0.016)	0.004	(0.017)
a_0^2	-0.198***	(0.020)	-0.195***	(0.020)	-0.162***	(0.020)
a_0^3	-0.091***	(0.014)	-0.090***	(0.014)	-0.057***	(0.014)
a_0^4	-0.059	(0.040)	-0.054	(0.041)	-0.065	(0.041)
a_0^{age}	-0.003**	(0.001)	0.012**	(0.005)	0.004	(0.005)
a_0^{educ}			-0.003**	(0.001)	-0.003**	(0.001)
a_0^{lfh}					-0.204***	(0.025)
a_0^{lfs}					-0.295***	(0.035)
RESE						
b_0^1	0.393***	(0.020)	0.387***	(0.021)	0.385***	(0.023)
b_0^2	0.271***	(0.031)	0.282***	(0.032)	0.366***	(0.034)
b_0^3	0.204***	(0.020)	0.228***	(0.021)	0.242***	(0.021)
b_0^4	0.147***	(0.054)	0.135**	(0.056)	0.102*	(0.056)
b_0^{age}	0.005***	(0.001)	0.006***	(0.002)	0.010***	(0.002)
b_0^{educ}			0.059***	(0.007)	0.047***	(0.007)
b_0^{lfh}					-0.061	(0.050)
b_0^{lfs}					-0.015	(0.050)
*** significant at the 1% level						
** significant at the 5% level						
* significant at the 10% level						

Table 6 shows selected parameter estimates for RESE-restricted QAI demand systems. Looking at the results for R (where $R(100(\mathbf{17}), z) = \exp(\bar{a}_0(z))$), we see that costs are increasing in the number of household members, though not very much for young children. Further, costs increase in the age of the household head, but are insensitive to the education

level of the household head. We may compare these results with those of Table 5 for RESE-restricted QES demand systems. The estimates, given RESE, are broadly robust to choice of demand system (QES or QAI) in that costs are increasing in the number of household members and the age of the household head in both cases. However, the relative cost of young children versus teenagers is different under the two demand systems.

Table 6: Estimates given QAI and RESE

	Specification 1		Specification 2		Specification 3	
	Coefficient	Std Error	Coefficient	Std Error	Coefficient	Std Error
$\bar{a}_0^1$	0.282***	(0.023)	0.279***	(0.022)	0.284***	(0.024)
$\bar{a}_0^2$	0.042	(0.034)	0.064*	(0.033)	0.046	(0.033)
$\bar{a}_0^3$	0.242***	(0.025)	0.249***	(0.025)	0.241***	(0.025)
$\bar{a}_0^4$	0.187***	(0.066)	0.178***	(0.064)	0.214***	(0.062)
$\bar{a}_0^{age}$	0.011***	(0.002)	0.011***	(0.002)	0.012***	(0.002)
$\bar{a}_0^{educ}$			0.009	(0.008)	0.011	(0.008)
$\bar{a}_0^{lfh}$					0.212***	(0.066)
$\bar{a}_0^{lfs}$					0.084	(0.052)

*** significant at the 1% level
** significant at the 5% level
* significant at the 10% level

Estimates given RESE are different from those given GAESE and GRESE. To facilitate comparison with Figures 2 and 4, Table 7 shows relative equivalence scales given RESE for the six household types used in the figures. We show results for estimation under QES and QAI for Specification 2. For the most part, these estimates suggest smaller relative equivalence scales than those given GAESE and GRESE. For example, given RESE and QAI, the relative equivalence scale for dual parents with one teenager is 1.58. In contrast, given GRESE and QAI, the relative equivalence scale for this household type ranges from 2.06 to 2.88 depending on the income of the household.

Table 7: Relative Equivalence Scale Sizes Given RESE

Adults:	Two	One	Two	Two	Two	Two
Children:	None	Older	Younger	Older	Teen	Older & Teen
QES	1.47	1.26	1.95	1.85	1.69	2.12
QAI	1.32	1.28	1.41	1.70	1.58	2.03

That the estimated relative equivalence scale given RESE lies outside the range of the estimated relative equivalence scale given GRESE may at first seem counterintuitive. However, the generalization of RESE to GRESE affects more than just the equivalence-scale function; as noted above, it affects the way commodity demands vary across household types. Thus, if the restrictions that RESE puts on commodity demands are violated by the data, then equivalence scales estimated given RESE need not have any relationship to those estimated given GRESE. In Table 2, we showed that RESE does put binding restrictions on commodity demands in comparison with GAESE and GRESE. In the next subsection, we explore how GAESE and GRESE allow for more general characterizations of the demand for children’s goods.

7.4. The Demand for Children’s Goods

Given AESE or RESE, the shape of the Engel curve for children’s goods is highly restricted. In particular, under AESE the income elasticity of demand for children’s goods is zero and under RESE the income elasticity of demand for children’s goods is one. In contrast, the income elasticity of demand for children’s goods under GAESE and GRESE need not be zero or one. Given GAESE, the demand for children’s goods is affine in income and, given GRESE, the expenditure share on children’s goods is affine in the logarithm of income.

Table 8 shows selected parameter estimates, using Specification 2, that reveal how the demand for children’s clothing—denoted c —changes with income under GAESE and GRESE. The table shows that the demand for children’s clothing responds strongly to income. The results given GAESE and QES suggest that the demand for children’s clothing rises with income as one adds young and older children to the household and falls with income as one adds teenagers and adults to the household. Similarly, the results given GRESE and QAI suggest that the expenditure share on children’s clothing rises as one adds younger and older children to the household and falls as one adds teenagers and adults to the household. Thus, the restrictions that AESE and RESE put on commodity demands are binding, particularly in the context of children’s goods.

Table 8: Income Responses in the Demand for Children's Clothing

GAESE and QES			GRESE and QAI		
x term			$\ln x$ term		
Coefficient	Estimate	Std Error	Coefficient	Estimate	Std Error
b_c^{ref}	0.0024	(0.0030)	$\bar{b}_c^{ref}$	-0.0024	(0.0032)
b_c^1	-0.0025***	(0.0007)	$\bar{b}_c^1$	-0.0017***	(0.0004)
b_c^2	0.128***	(0.0008)	$\bar{b}_c^2$	0.0024***	(0.0005)
b_c^3	0.0100***	(0.0006)	$\bar{b}_c^3$	0.0049***	(0.0004)
b_c^4	-0.0060***	(0.0013)	$\bar{b}_c^4$	-0.0043***	(0.0010)
b_c^{age}	-0.0007***	(0.0002)	$\bar{b}_c^{age}$	0.0003*	(0.0002)
b_c^{educ}	0.0000	(0.0001)	$\bar{b}_c^{educ}$	0.0000	(0.0001)

*** significant at the 1% level

** significant at the 5% level

* significant at the 10% level

7.5. GAESE, GRESE and Measured Inequality

Table 9 shows the estimated Gini Coefficient given seven different equivalent-income functions for Ontario 1982 and 1992. The sample frame, income variable and equivalent-income functions are from the demand estimation above, using Specification 2. In the table, asymptotic standard errors are given in parentheses and are calculated following Barrett and Pendakur [1995]. We present results for this region so that there is no inter-regional price variation within years. Thus, the variation in equivalent-income functions comes solely from variation in z and x .

With either no scaling (where the relative equivalence scale is equal to one) or naive scaling (where the relative scale is the number of people in the household), the estimated Gini Coefficients suggest that inequality rose slightly (and insignificantly) between 1982 and 1992. This is consistent with previous research on the Canadian consumption distribution which uses naive scaling (Pendakur [1998a]). Estimated Gini Coefficients given RESE equivalent-income functions, however, show a large and statistically significant decrease in inequality over the period.

Table 9: GAESE, GRESE and Measured Inequality
Ontario: 1982 and 1992

Equivalent-Income Function	Type	1982		1992	
x	no scaling	0.218	(0.008)	0.220	(0.009)
$x/(\text{household size})^{1/2}$	naive scaling	0.185	(0.007)	0.194	(0.008)
$x - A(p, z)$	AESE-QES	0.220	(0.008)	0.216	(0.009)
$x/R(p, z)$	RESE-QES	0.261	(0.012)	0.220	(0.008)
$x/R(p, z)$	RESE-QAI	0.228	(0.009)	0.198	(0.007)
$((x - A(p, z))/R(p, z))$	GAESE-QES	0.287	(0.008)	0.284	(0.009)
$\exp [(\ln x - \ln G(p, z))/K(p, z)]$	GRESE/QAI	0.276	(0.008)	0.275	(0.009)

The results given GAESE and GRESE equivalent-income functions reveal a different pattern. First, the level of inequality is much higher than that given RESE equivalent-income functions. Second, these results show no change in inequality over the period. Because these equivalent-income functions could have satisfied RESE if the data had supported it, the inequality estimates given GAESE and GRESE equivalent-income functions are more accurate representations of the distribution of well-being. Thus, from the point of view of inequality measurement, GAESE and GRESE provide important generalizations of conventional equivalence scales. They make a difference to the level of and trend in measured inequality.

8. Conclusion

The two classes of equivalent-income functions introduced in this paper are generalizations of those associated with exact absolute and relative equivalence scales. Generalized Absolute Equivalence-Scale Exactness (GAESE) obtains if and only if the equivalent-income function and associated absolute equivalence scale are affine in income. Generalized Relative Equivalence-Scale Exactness (GRESE) obtains if and only if the equivalent-income function and the associated relative equivalence scale are log-affine in income. GAESE and GRESE impose fewer restrictions on household commodity demands than income-independent scales do, particularly in the case of children's goods.

We show that, if reference indirect utility functions are neither affine nor log-affine in income and GAESE or GRESE is accepted as a maintained hypothesis, equivalent-income functions and associated income-dependent equivalence scales can be uniquely estimated

from demand data. Equivalent-income functions and equivalence scales are estimated using Canadian data and the estimated equivalence scales are both plausible and income-dependent. We find that GAESE and GRESE fit the data better than income-independent formulations do, particularly in the demand equations for children's goods. In addition, the use of GAESE and GRESE scales makes a significant difference to the level of and trend in measured inequality.

APPENDIX

Lemma 1: *If E^r , G and K are differentiable in p ,*

$$K(p, z) = 1 - \sum_{j=1}^m \frac{\partial \ln G(p, z)}{\partial \ln p_j} \quad (\text{A.1})$$

for all $p \in \mathcal{R}_{++}^m$, $z \in \mathcal{Z}$.

Proof: Recall that if a function f is homogeneous of degree ρ ,

$$\sum_{j=1}^m w_j \frac{\partial f(w)}{\partial w_j} = \rho f(w), \quad (\text{A.2})$$

which implies

$$\sum_{j=1}^m w_j \frac{\partial \ln f(w)}{\partial w_j} = \sum_{j=1}^m \frac{\partial \ln f(w)}{\partial \ln w_j} = \rho. \quad (\text{A.3})$$

GRESE implies that

$$\ln E(u, p, z) = K(p, z) \ln E^r(u, p) + \ln G(p, z). \quad (\text{A.4})$$

Differentiating,

$$\begin{aligned} \sum_{j=1}^m p_j \frac{\partial \ln E(u, p, z)}{\partial p_j} &= \sum_{j=1}^m p_j \frac{\partial K(p, z)}{\partial p_j} \ln E^r(u, p) + K(p, z) \sum_{j=1}^m p_j \frac{\partial \ln E^r(u, p)}{\partial p_j} \\ &\quad + \sum_{j=1}^m p_j \frac{\partial \ln G(p, z)}{\partial p_j}. \end{aligned} \quad (\text{A.5})$$

Because $E(u, \cdot, z)$ and $E^r(u, \cdot)$ are homogeneous of degree one and $K(\cdot, z)$ is homogeneous of degree zero, (A.5) becomes

$$1 = K(p, z) + \sum_{j=1}^m p_j \frac{\partial \ln G(p, z)}{\partial p_j} \quad (\text{A.6})$$

and, using (A.3), (A.1) results. ■

It is mathematically convenient to prove Theorem 4 first.

Proof of Theorem 4: In (5.18), set $p = \tilde{p}$ to get

$$\ln E^r(\sigma(u, z), \tilde{p}) = H(\tilde{p}, z) \ln E^r(u, \tilde{p}) + Q(\tilde{p}, z). \quad (\text{A.7})$$

Defining $h(z) = H(\tilde{p}, z)$, $f(u) = \ln E^r(u, \tilde{p})$, and $q(z) = Q(\tilde{p}, z)$,

$$f(\sigma(u, z)) = h(z)f(u) + q(z), \quad (\text{A.8})$$

so that

$$\sigma(u, z) = f^{-1}(h(z)f(u) + q(z)) = \phi(h(z)f(u) + q(z)) \quad (\text{A.9})$$

where $\phi := f^{-1}$. Therefore (5.18) becomes

$$\ln E^r(\phi(h(z)f(u) + q(z)), p) = H(p, z) \ln E^r(u, p) + Q(p, z). \quad (\text{A.10})$$

Defining $w = f(u)$ and $F(\cdot, p) = \ln E^r(\phi(\cdot), p)$,

$$\ln E^r(u, p) = \ln E^r(f^{-1}(w), p) = \ln E^r(\phi(w), p) = F(w, p), \quad (\text{A.11})$$

and (A.10) becomes

$$F(h(z)w + q(z), p) = H(p, z)F(w, p) + Q(p, z) \quad (\text{A.12})$$

for any $p \in \mathcal{R}_{++}^m$, $z \in \mathcal{Z}$.

Suppressing p in (A.12) and defining the functions $\check{f}$, $\check{h}$, and $\check{q}$ in the obvious way, (A.12) is

$$\check{f}(h(z)w + q(z)) = \check{h}(z)\check{f}(w) + \check{q}(z). \quad (\text{A.13})$$

Defining $a = h(z)$, $b = q(z)$, (A.13) becomes

$$\check{f}(aw + b) = \check{h}(z)\check{f}(w) + \check{q}(z). \quad (\text{A.14})$$

Set $w = 0$ in (A.14) to get

$$\check{f}(b) = \check{h}(z)\check{f}(0) + \check{q}(z) \quad (\text{A.15})$$

so

$$\check{q}(z) = \check{f}(b) - \bar{c}\check{h}(z) \quad (\text{A.16})$$

where $\bar{c} := \check{f}(0)$. Consequently, (A.14) can be written as

$$\check{f}(aw + b) = \check{h}(z)\check{f}(w) + \check{f}(b) - \bar{c}\check{h}(z) = \check{h}(z)(\check{f}(w) - \bar{c}) + \check{f}(b) = \check{h}(z)\bar{f}(w) + \check{f}(b) \quad (\text{A.17})$$

where $\bar{f}(w) = \check{f}(w) - \bar{c}$. Now set $w = 1$ in (A.17) to get

$$\check{f}(a + b) = \check{h}(z)\bar{f}(1) + \check{f}(b) \quad (\text{A.18})$$

which implies

$$\check{h}(z) = \overset{\circ}{h}(a, b) \quad (\text{A.19})$$

for some function $\overset{\circ}{h}$. Consequently, (A.17) is

$$\check{f}(aw + b) = \overset{\circ}{h}(a, b)\bar{f}(w) + \check{f}(b). \quad (\text{A.20})$$

For any b , define

$$\hat{f}_b(aw) = \check{f}(aw + b) - \check{f}(b) \quad (\text{A.21})$$

and

$$\hat{h}_b(a) = \overset{\circ}{h}(a, b). \quad (\text{A.22})$$

Then (A.20) can be written as

$$\hat{f}_b(aw) = \hat{h}_b(a)\bar{f}(w), \quad (\text{A.23})$$

a Pexider equation (Eichhorn [1987]) whose solution is

$$\hat{h}_b(a) = \hat{C}(b)a^{R(b)}, \quad (\text{A.24})$$

$$\bar{f}(w) = \bar{C}(b)w^{R(b)}, \quad (\text{A.25})$$

and

$$\hat{f}_b(t) = \hat{C}(b)\bar{C}(b)t^{R(b)} = \tilde{C}(b)t^{R(b)} \quad (\text{A.26})$$

where $\tilde{C}(b) = \bar{C}(b)\hat{C}(b)$.

$\bar{f}$ is independent of b . Setting $w = 1$ in (A.25), this implies that $\bar{C}$ is independent of b which, in turn, implies that R is independent of b as well. Defining $\rho = R(b)$ for all b , (A.21) and (A.26) imply

$$\check{f}(aw + b) = \hat{f}_b(aw) + \check{f}(b) = \tilde{C}(b)(aw)^\rho + \check{f}(b). \quad (\text{A.27})$$

Now set $z = z^r$, so that $a = h(z^r) = H(p, z^r) = 1$ and $b = q(z^r) = Q(p, z^r) = 0$ so that (A.27) implies

$$\check{f}(w) = \tilde{C}(0)w^\rho + \check{f}(0) = cw^\rho + d \quad (\text{A.28})$$

where $c := \tilde{C}(0)$ and $d := \check{f}(0)$. Because $\check{f}$ is increasing, $c \neq 0$ and $\rho \neq 0$.

Define $\zeta = aw$ so that (A.27) can be written as

$$\check{f}(\zeta + b) = \tilde{C}(b)\zeta^\rho + \check{f}(b). \quad (\text{A.29})$$

This implies, given (A.28), that

$$c(\zeta + b)^\rho + d = \tilde{C}(b)\zeta^\rho + cb^\rho + d \quad (\text{A.30})$$

and

$$c(\zeta + b)^\rho = \tilde{C}(b)\zeta^\rho + cb^\rho. \quad (\text{A.31})$$

Because $c \neq 0$, (A.31) can be rewritten as

$$\frac{\tilde{C}(b)}{c} = \frac{(\zeta + b)^\rho - b^\rho}{\zeta^\rho} \quad (\text{A.32})$$

for all $\zeta \neq 0$. Because the left side of (A.32) is independent of ζ , the right side must be as well. It is if $\rho = 1$ and, in that case, $\tilde{C}(b) = c$. To show that $\rho = 1$ is necessary, define $\psi(\zeta)$ to be the right side of (A.32) for any fixed value of b , $b \neq 0$. ψ is differentiable and, because it is independent of ζ , $\psi'(\zeta) = 0$ for all ζ . Because $\rho \neq 0$,

$$\begin{aligned} \psi'(\zeta) = 0 &\iff \rho(\zeta + b)^{\rho-1}\zeta^\rho - ((\zeta + b)^\rho - b^\rho)\rho\zeta^{\rho-1} = 0 \\ &\iff (\zeta + b)^{\rho-1}\zeta - (\zeta + b)^\rho + b^\rho = 0 \\ &\iff (\zeta + b)^{\rho-1}(\zeta - \zeta - b) + b^\rho = 0 \\ &\iff (\zeta + b)^{\rho-1} = b^{\rho-1}. \end{aligned} \quad (\text{A.33})$$

Because (A.33) is true for all ζ , $\rho = 1$. As a consequence, $\tilde{C}(b) = c$. This, together with (A.29), implies

$$\check{f}(w) = cw + d. \quad (\text{A.34})$$

Because $\check{f}$ is increasing, $c > 0$.

In the functional equation, p was suppressed so c and d can depend on p . Because $w = f(u)$ and $\check{f}(w) = F(w, p) = \ln E^r(f^{-1}(w), p) = \ln E^r(u, p)$, (A.34) implies that

$$\ln E^r(u, p) = C(p)f(u) + \ln D(p) \quad (\text{A.35})$$

which is (5.19), and its necessity is established. Because $c > 0$ and f is increasing, $C(p)$ must be positive. Homogeneity properties of C and D follow from the fact that E^r is homogeneous of degree one in p .

Equation (A.10) requires

$$\ln E^r\left(\phi(h(z)f(u) + q(z)), p\right) = H(p, z) \ln E^r(u, p) + Q(p, z) \quad (\text{A.36})$$

Because of (A.35) and the fact that $\phi = f^{-1}$, this becomes

$$C(p) \left[h(z)f(u) + q(z) \right] + \ln D(p) = H(p, z) \left[C(p)f(u) + \ln D(p) \right] + Q(p, z), \quad (\text{A.37})$$

so

$$C(p)h(z)f(u) + C(p)q(z) + \ln D(p) = H(p, z)C(p)f(u) + H(p, z)\ln D(p) + Q(p, z). \quad (\text{A.38})$$

It follows that

$$H(p, z) = h(z) \quad (\text{A.39})$$

which results in (5.20). Because $K(p, z) \in \mathcal{R}_{++}$, $h(z) \in \mathcal{R}_{++}$ for all $z \in \mathcal{Z}$ and, because $\hat{K}(p, z^r) = \tilde{K}(p, z^r) = 1$, $h(z^r) = 1$.

We know from (A.14) that

$$\check{f}(aw + b) = \check{h}(z)\check{f}(w) + \check{q}(z). \quad (\text{A.40})$$

Because $\check{f}(t) = ct + d$ and the fact that $\check{h}(a) = h(z) = a$ (from (A.39)),

$$c[aw + b] + d = a[cw + d] + \check{q}(z), \quad (\text{A.41})$$

so, because $b = q(z)$,

$$\check{q}(z) = cq(z) + d[1 - a], \quad (\text{A.42})$$

and, because $\check{q}(z) = Q(p, z)$ with p suppressed,

$$Q(p, z) = C(p)q(z) + \ln D(p)[1 - h(z)] \quad (\text{A.43})$$

which implies (5.21). Because, in (5.21), $\hat{G}(p, z^r) = \tilde{G}(p, z^r) = 1$, $\tilde{K}(p, z^r) = 1$ and $h(z^r) = 1$, $q(z^r) = 0$.

Sufficiency can be checked as follows. Suppose that (5.20) and GRESE are satisfied, with

$$\begin{aligned} \ln \tilde{E}(u, p, z) &= \tilde{K}(p, z) \ln E^r(u, p) + \ln \tilde{G}(p, z) \\ &= \tilde{K}(p, z) \left[C(p)f(u) + \ln D(p) \right] + \ln \tilde{G}(p, z). \end{aligned} \quad (\text{A.44})$$

Now suppose that (5.20) and (5.21) are satisfied, with

$$\begin{aligned} \ln \hat{E}^r(u, p, z) &= \hat{K}(p, z) \ln E^r(u, p) + \ln \hat{G}(p, z) \\ &= h(z)\tilde{K}(p, z) \left[C(p)f(u) + \ln D(p) \right] + \ln \tilde{G}(p, z) \\ &\quad + \tilde{K}(p, z) \left[C(p)q(z) + \ln D(p)(1 - h(z)) \right] \\ &= \tilde{K}(p, z) \left[C(p) \left[h(z)f(u) + q(z) \right] + \ln D(p) \right] + \ln \tilde{G}(p, z) \\ &= \ln \tilde{E} \left(\left[h(z)f(u) + q(z) \right], p, z \right), \end{aligned} \quad (\text{A.45})$$

which implies that $\tilde{E}$ and $\hat{E}$ represent the same preferences for all $z \in \mathcal{Z}$. ■

Proof of Theorem 5: Because $\hat{K} = \tilde{K}$, $H(p, z) = 1$ for all $z \in \mathcal{Z}$. In the proof of Theorem 4, this implies $h(z) = 1$ for all $z \in \mathcal{Z}$, and equation (A.14) becomes

$$\check{f}(w + b) = \check{f}(w) + \check{q}(z). \quad (\text{A.46})$$

Set $w = 0$ to get

$$\check{q}(z) = \check{f}(b) - \check{f}(0), \quad (\text{A.47})$$

and rewrite (A.46) as

$$\check{f}(w + b) = \check{f}(w) + \check{f}(b) - \check{f}(0), \quad (\text{A.48})$$

which is equivalent to

$$[\check{f}(w + b) - \check{f}(0)] = [\check{f}(w) - \check{f}(0)] + [\check{f}(b) - \check{f}(0)]. \quad (\text{A.49})$$

Defining $\overset{\circ}{f}(t) = \check{f}(t) - \check{f}(0)$, (A.49) becomes

$$\overset{\circ}{f}(w + b) = \overset{\circ}{f}(w) + \overset{\circ}{f}(b), \quad (\text{A.50})$$

a Cauchy equation whose solution is (Eichhorn [1977])

$$\overset{\circ}{f}(w) = cw \quad (\text{A.51})$$

for some $c \in \mathcal{R}$. As a consequence,

$$\check{f}(w) = cw + \check{f}(0) =: cw + d \quad (\text{A.52})$$

as in equation (A.34). The rest of the proof, including sufficiency, follows from the proof of Theorem 4. ■

Proof of Theorem 1: Equation (5.7) is analogous to equation (5.18) with $\ln E^r(\sigma(u, z), p)$ and $\ln E^r(u, p)$ replaced by $E^r(\sigma(u, p), p)$ and $E^r(u, p)$. The result, including sufficiency, follows from the proof of Theorem 4. Setting $w = 0$ in (A.15) is possible because the domain for income is $\mathcal{R}_+$. ■

Proof of Theorem 2: The result follows from Theorem 1 using the proof of Theorem 2. ■

Proof of Theorem 3: When $\hat{R} \neq \tilde{R}$ and $\hat{A} = \tilde{A}$, (5.7) becomes

$$E^r(\sigma(u, p), p) = J(p, z)E^r(u, p) \quad (\text{A.53})$$

which is equation (A.24) in Blackorby and Donaldson [1993, p.354]. On the income domain $\mathcal{R}_{++}$, their Theorem 5.1 proves that (5.13) and (5.14) are necessary. The result is extended to the larger domain $(\mathcal{R}_+)$ by continuity. Sufficiency is easily checked. ■

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Figure 1: Equivalent Income Functions Given GAESE

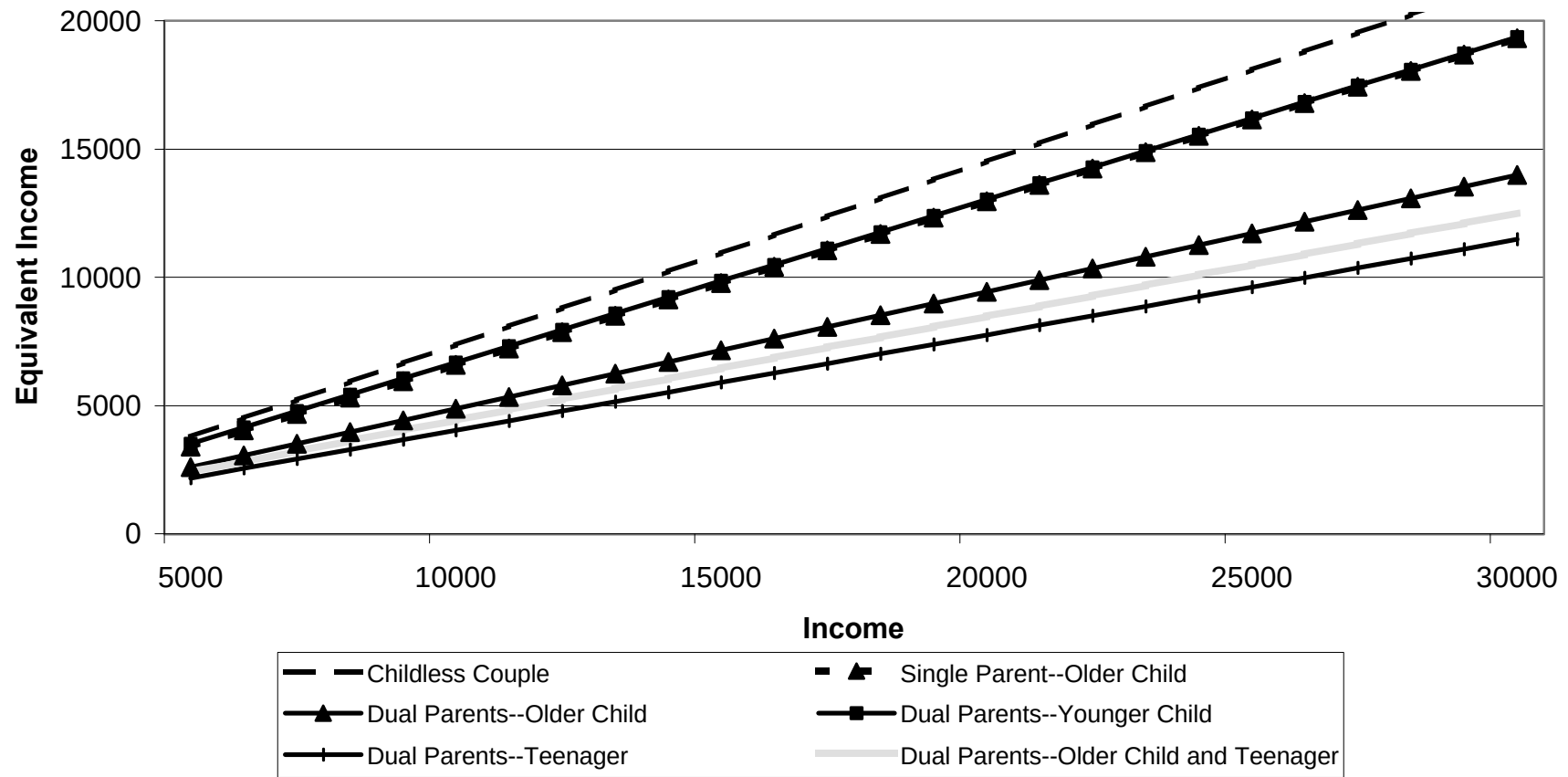


Figure 2: Relative Equivalence Scales Given GAESE

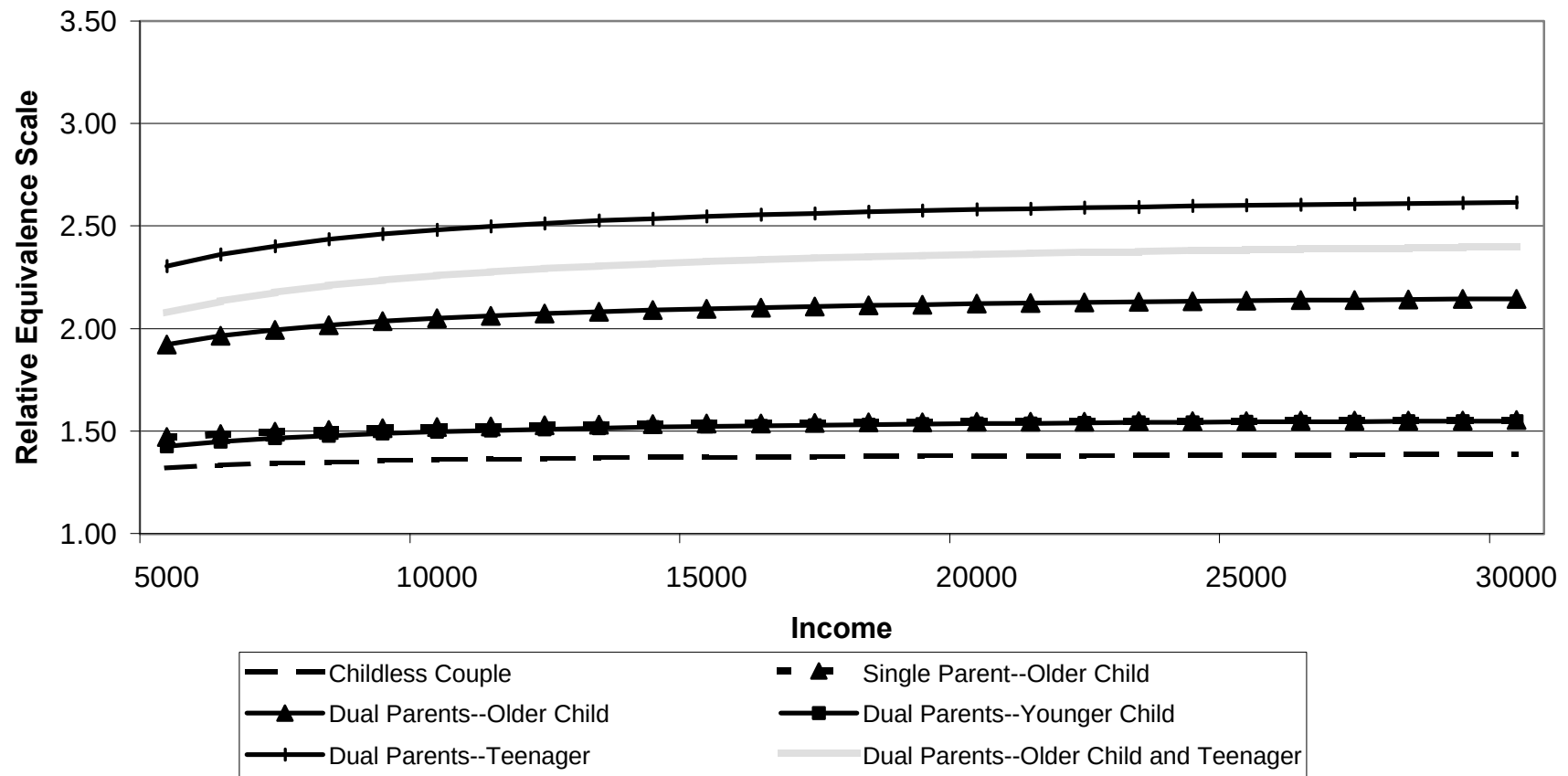


Figure 3: Equivalent Income Functions Given GRESE

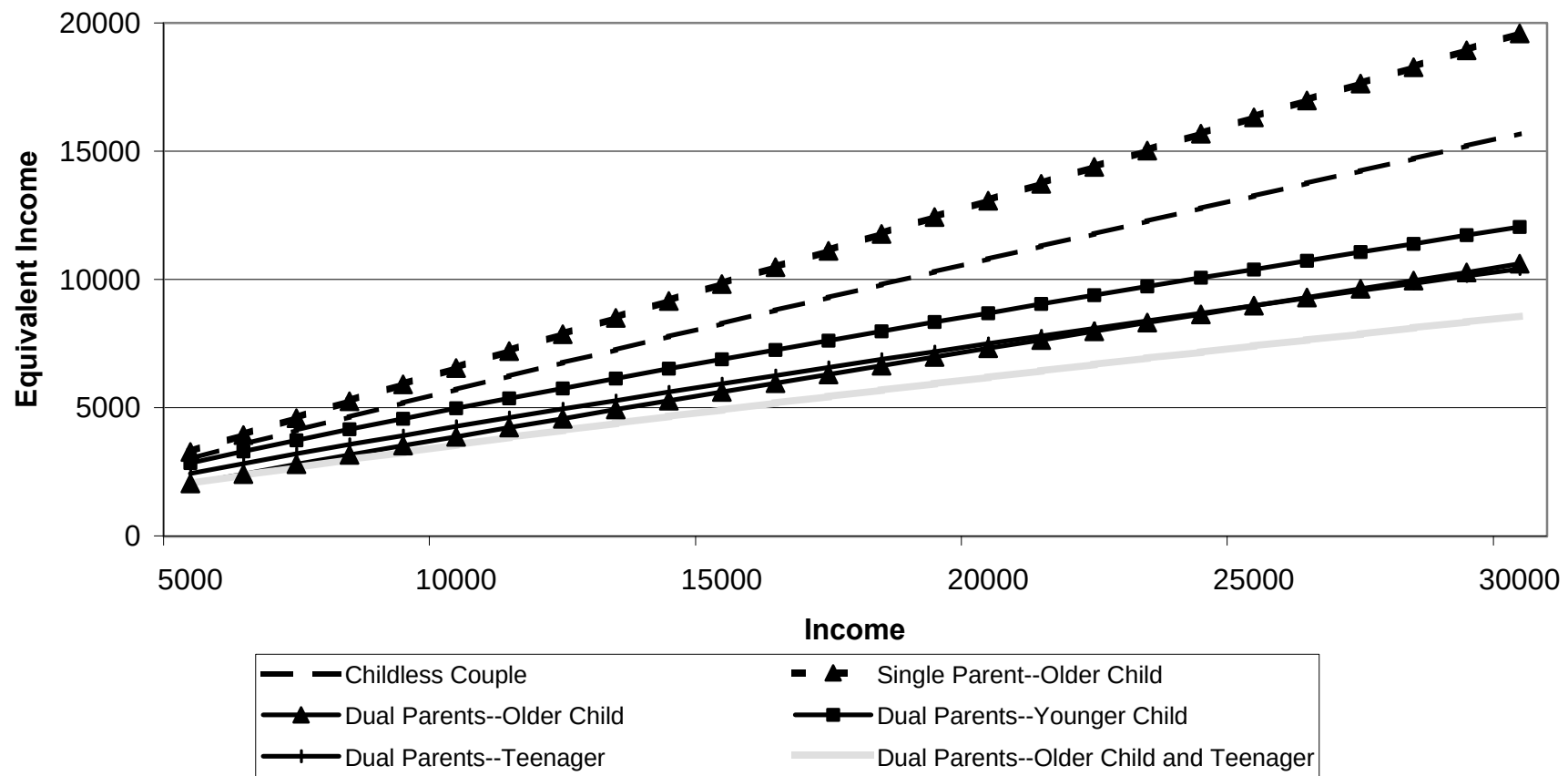


Figure 4: Relative Equivalence Scales Given GRESE

