Complex Numbers

Complex numbers $z = a + ib, \quad i = \sqrt{-1}$

have a real part and an imaginary part: $\mathrm{Re}\{z\} = a,\ \mathrm{Im}\{z\} = b$

Basic Algebra:

$$(a+ib)+(c+id)=(a+c)+i(b+d)$$

$$(a+ib)(c+id)=(ac-bd)+i(bc+ad)$$

Complex Conjugate: $z^* = (a+ib)^* = a - ib$

$$z + z^* = 2a$$

$$z - z^* = 2ib$$

Absolute Value: $|z|$

$$zz^* = a^2 + b^2$$

(modulus, amplitude, magnitude) $|z| = \sqrt{zz^*} = \left(a^2 + b^2\right)^{1/2}$

Polar Form: $z = r[\cos\theta + i\sin\theta] = r\,\mathrm{cis}\,\theta = r\,\mathrm{e}^{i\theta}$

$$\left.\begin{aligned} \mathrm{e}^{i\theta} &= \cos\theta + i\sin\theta \\ \mathrm{e}^{-i\theta} &= \cos\theta - i\sin\theta \end{aligned}\right\} \qquad \left.\begin{aligned} \cos\theta &= \tfrac{1}{2}\left(\mathrm{e}^{i\theta} + \mathrm{e}^{-i\theta}\right) \\ \sin\theta &= \tfrac{1}{2i}\left(\mathrm{e}^{i\theta} - \mathrm{e}^{-i\theta}\right) \end{aligned}\right\}$$

e.g. If $z_1 = r_1\,\mathrm{e}^{i\theta_1}$ and $z_2 = r_2\,\mathrm{e}^{i\theta_2}$

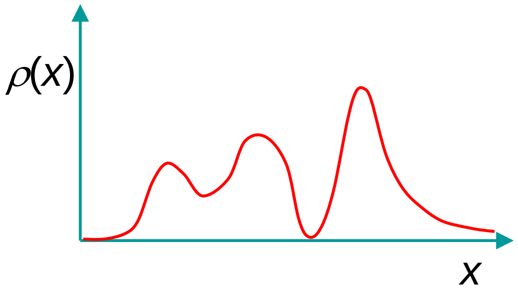
$$z_1 z_2 = r_1 r_2\,\mathrm{e}^{i(\theta_1+\theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\mathrm{e}^{i(\theta_1-\theta_2)}$$

$$z^m = \left(r\mathrm{e}^{i\theta}\right)^m = r^m \mathrm{e}^{im\theta}$$

Statistics for Continuous Variables

Suppose the distribution is continuous, e.g. temperature variation.



The probability that a value chosen at random lies between *x* and *x* + *dx*

$$= \rho(x)\,dx$$

Probability density

The probability that a value chosen at random lies between a and b i.e. in a finite interval

$$P_{ab} = \int_a^b \rho(x)dx$$

The probability "sums" to 1:

$$\int_{-\infty}^{\infty} \rho(x)dx = 1$$

The average value

$$\langle x \rangle = \int_{-\infty}^{\infty} x\rho(x)dx$$

Expectation value

Similarly

$$\langle f(x) \rangle = \int_{-\infty}^{\infty} f(x)\rho(x)dx$$

Variance

$$\sigma^2 = \langle (\Delta x)^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2$$

Miscellaneous Calculus

To differentiate a product $$\frac{d}{dx}(fg) = f\frac{dg}{dx} + \frac{df}{dx}g$$

To integrate a product $$\int_a^b f\frac{dg}{dx}\,\mathrm{d}x = -\int_a^b \frac{df}{dx}g\,\mathrm{d}x + fg\Big|_a^b$$

$$\int_a^b f(x)g(x)\,\mathrm{d}x = -\int_a^b \frac{df}{dx}\left(\int g\right)\mathrm{d}x + \left[f\left(\int g\right)\right]_a^b$$

Some common integrals:

$$\int_0^\infty x^n\,\mathrm{e}^{-ax}\,\mathrm{d}x = \frac{n!}{a^{n+1}}$$

$$\int_0^\infty \mathrm{e}^{-ax^2}\,\mathrm{d}x = \left(\frac{\pi}{4a}\right)^{1/2}$$

Some Common Integrals

$$\int \sin^2(ax+b)\,\mathrm{d}x = \frac{x}{2} - \frac{\sin(2(ax+b))}{4a} + \text{constant}$$

$$\int \cos^2(ax+b)\,\mathrm{d}x = \frac{x}{2} + \frac{\sin(2(ax+b))}{4a} + \text{constant}$$

$$\int_0^\infty \mathrm{e}^{-ax}\,\mathrm{d}x = \frac{1}{a}$$

$$\int_0^\infty x^n\,\mathrm{e}^{-ax}\,\mathrm{d}x = \frac{n!}{a^{n+1}}$$

$$\int_0^\infty \mathrm{e}^{-ax^2}\,\mathrm{d}x = \left(\frac{\pi}{4a}\right)^{1/2}$$

$$\int_0^\infty x\,\mathrm{e}^{-ax^2}\,\mathrm{d}x = \frac{1}{2a}$$

Even and Odd Functions

Suppose $I = \int_{-\infty}^{+\infty} F(x)\mathrm{d}x$

Then $I = \int_{-\infty}^{0} F(x)\mathrm{d}x + \int_{0}^{+\infty} F(x)\mathrm{d}x$

$= \int_{0}^{+\infty} F(-x)\mathrm{d}x + \int_{0}^{+\infty} F(x)\mathrm{d}x$

For $F(x)$ odd $\quad F(-x) = -F(x) \quad \Rightarrow \quad I = 0$

For $F(x)$ even $\quad F(-x) = F(x) \quad \Rightarrow \quad I \neq 0$

Most functions have no symmetry, but some may be broken into symmetric parts:

$$\left.\begin{array}{l}\text{Since } \mathrm{e}^{ix} = \cos x + i \sin x \\ \text{and } \mathrm{e}^{-ix} = \cos x - i \sin x\end{array}\right\}$$

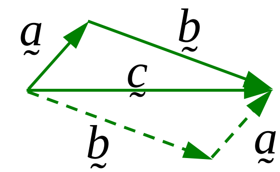
$\mathrm{Re}\left\{\mathrm{e}^{ix}\right\}$ is even, and $\mathrm{Im}\left\{\mathrm{e}^{ix}\right\}$ is odd

Vectors

A scalar has magnitude (only) but a vector has magnitude and direction.

It can be expressed in components $\underset{\sim}{r} = \vec{r} = \vec{r} = \boldsymbol{r} = (x, y, z) = x\,\vec{\mathrm{i}} + y\,\vec{\mathrm{j}} + z\,\vec{\mathrm{k}}$

Addition $\underset{\sim}{c} = \underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{b} + \underset{\sim}{a} = (a_x + b_x, a_y + b_y, a_z + b_z)$



Scalar product
$$\underset{\sim}{a} \bullet \underset{\sim}{b} = \underset{\sim}{b} \bullet \underset{\sim}{a} = a_x b_x + a_y b_y + a_z b_z$$
$$= |a||b| \cos\theta_{ab}$$

$$\begin{cases} \vec{\mathrm{i}} \bullet \vec{\mathrm{i}} = \vec{\mathrm{j}} \bullet \vec{\mathrm{j}} = \vec{\mathrm{k}} \bullet \vec{\mathrm{k}} = 1 \\ \vec{\mathrm{i}} \bullet \vec{\mathrm{j}} = \vec{\mathrm{j}} \bullet \vec{\mathrm{k}} = \vec{\mathrm{k}} \bullet \vec{\mathrm{i}} = 0 \end{cases}$$

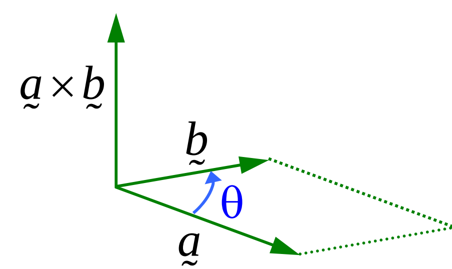
$$\underset{\sim}{a} \bullet \underset{\sim}{a} = a_x a_x + a_y a_y + a_z a_z = |a|^2$$
$$|a| = \left(a_x^2 + a_y^2 + a_z^2\right)^{1/2}$$

Vector product
$$\underset{\sim}{a} \times \underset{\sim}{b} = |a||b| \sin\theta_{ab}\, \underset{\sim}{n}$$

where $\underset{\sim}{n}$ is the unit vector $\perp$ to the plane of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

$$= -\underset{\sim}{b} \times \underset{\sim}{a}$$
$$= \begin{vmatrix} \underset{\sim}{\mathrm{i}} & \underset{\sim}{\mathrm{j}} & \underset{\sim}{\mathrm{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$\vec{\mathrm{i}} \times \vec{\mathrm{i}} = \vec{\mathrm{j}} \times \vec{\mathrm{j}} = \vec{\mathrm{k}} \times \vec{\mathrm{k}} = 0$$
$$\vec{\mathrm{i}} \times \vec{\mathrm{j}} = \vec{\mathrm{k}}, \quad \vec{\mathrm{j}} \times \vec{\mathrm{k}} = \vec{\mathrm{i}}, \quad \vec{\mathrm{k}} \times \vec{\mathrm{i}} = \vec{\mathrm{j}}$$



del $\nabla = \frac{\partial}{\partial x}\underset{\sim}{\mathrm{i}} + \frac{\partial}{\partial y}\underset{\sim}{\mathrm{j}} + \frac{\partial}{\partial z}\underset{\sim}{\mathrm{k}}$

del squared $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$