Series

Arithmetic: $a+(a+d)+(a+2d)+(a+3d)+\ldots+[a+(N-1)d]$

$$\sum_{j=1}^{N} a+(j-1)d = \frac{N}{2}[2a+(N-1)d]$$

add to a copy in reverse order

Geometric: $a+ar+ar^2+ar^3+\ldots+ar^{N-1}$

$$\sum_{j=1}^{N} ar^{j-1} = \frac{a\left(1-r^N\right)}{1-r}$$

subtract a copy multiplied by *r*

$$\sum_{j=1}^{N} ar^{j-1} = \frac{a}{1-r} \quad \text{for} \quad |r|<1$$

Taylor : $f(x)=\sum_{n=0}^{N}\frac{f^n(x_0)}{n!}(x-x_0)^n = f(x_0)+\Delta f'(x_0)+\frac{\Delta^2}{2!}f''(x_0)+\ldots,\qquad \Delta = x-x_0$

e.g. $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$ and $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots$

Taylor-MacLaurin Expansion

Suppose

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots \qquad f(x_0) = a_0$$

$$f'(x) = a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + \dots \qquad f'(x_0) = a_1$$

$$f''(x) = 2a_2 + 6a_3(x - x_0) + \dots \qquad f''(x_0) = 2a_2$$

Taylor :

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!} f''(x_0)(x - x_0)^2 + \dots$$

MacLaurin :

$$f(x) = f(0) + f'(0)x + \frac{1}{2!} f''(0)x^2 + \dots$$

e.g.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

Complex Numbers

Complex numbers $\quad z = a + ib, \quad i = \sqrt{-1}$

have a real part and an imaginary part: $\quad \mathrm{Re}\{z\} = a, \ \mathrm{Im}\{z\} = b$

Basic Algebra:

$$(a+ib)+(c+id)=(a+c)+i(b+d)$$

$$(a+ib)(c+id)=(ac-bd)+i(bc+ad)$$

Complex Conjugate: $\quad z^* = (a+ib)^* = a - ib$

$$z + z^* = 2a$$

$$z - z^* = 2ib$$

Absolute Value: $|z|$

$$zz^* = a^2 + b^2$$

(modulus, amplitude, magnitude)

$$|z| = \sqrt{zz^*} = \left(a^2 + b^2\right)^{1/2}$$

Polar Form: $\quad z = r[\cos\theta + i\sin\theta] = r\,\mathrm{cis}\,\theta = r\,e^{i\theta}$

$$\left.\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{-i\theta} &= \cos\theta - i\sin\theta \end{aligned}\right\} \qquad \left.\begin{aligned} \cos\theta &= \tfrac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \\ \sin\theta &= \tfrac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) \end{aligned}\right\}$$

e.g. If $\quad z_1 = r_1\,e^{i\theta_1}$ and $z_2 = r_2\,e^{i\theta_2}$

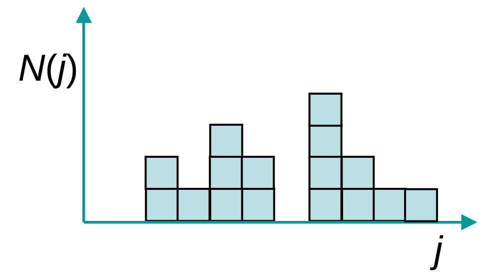
$$z_1 z_2 = r_1 r_2\,e^{i(\theta_1+\theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\,e^{i(\theta_1-\theta_2)}$$

$$z^m = \left(re^{i\theta}\right)^m = r^m e^{im\theta}$$

Discrete Statistics

Take a histogram of data, e.g. the age distribution of a set of people



The total number $N = \sum_{j=0}^{\infty} N(j)$

If picked at random, what is the probability of a particular age? $P(j) = \frac{N(j)}{N}$ $\sum_{j=0}^{\infty} P(j) = 1$

What is the most probable value? The one with the highest column

What is the median value? $P(> \text{median}) = P(< \text{median})$

The average value $\langle j \rangle = \frac{\sum jN(j)}{N} = \sum_{j=0}^{\infty} jP(j)$ Expectation value

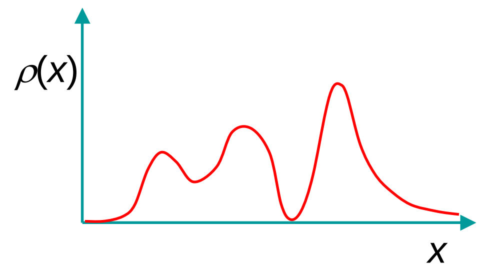
The average square value $\langle j^2 \rangle = \sum_{j=0}^{\infty} j^2 P(j)$ In general, $\langle f(j) \rangle = \sum_{j=0}^{\infty} f(j) P(j)$

The variance $\sigma^2 = \langle (\Delta j)^2 \rangle = \langle (j - \langle j \rangle)^2 \rangle = \sum (j - \langle j \rangle)^2 P(j) = \langle j^2 \rangle - \langle j \rangle^2$

$\sigma =$ the standard deviation

Statistics for Continuous Variables

Suppose the distribution is continuous, e.g. temperature variation.



The probability that a value chosen at random lies between x and $x + dx$

$$= \rho(x)\,dx$$

Probability density

The probability that a value chosen at random lies between a and b i.e. in a finite interval

$$P_{ab} = \int_a^b \rho(x)dx$$

The probability "sums" to 1:

$$\int_{-\infty}^{\infty} \rho(x)dx = 1$$

The average value

$$\langle x \rangle = \int_{-\infty}^{\infty} x\rho(x)dx$$

Expectation value

Similarly

$$\langle f(x) \rangle = \int_{-\infty}^{\infty} f(x)\rho(x)dx$$

Variance

$$\sigma^2 = \langle (\Delta x)^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2$$

Micellaneous Calculus

To differentiate a product $$\frac{d}{dx}(fg) = f\frac{dg}{dx} + \frac{df}{dx}g$$

To integrate a product $$\int_a^b f\frac{dg}{dx}\,\mathrm{d}x = -\int_a^b \frac{df}{dx}g\,\mathrm{d}x + fg\big|_a^b$$

$$\int_a^b f(x)g(x)\,\mathrm{d}x = -\int_a^b \frac{df}{dx}\left(\int g\right)\mathrm{d}x + \left[f\left(\int g\right)\right]_a^b$$

Some common integrals:

$$\int_0^\infty x^n\,\mathrm{e}^{-ax}\,\mathrm{d}x = \frac{n!}{a^{n+1}}$$

$$\int_0^\infty \mathrm{e}^{-ax^2}\,\mathrm{d}x = \left(\frac{\pi}{4a}\right)^{1/2}$$

Even and Odd Functions

Suppose $I = \int_{-\infty}^{+\infty} F(x)\mathrm{d}x$

Then $I = \int_{-\infty}^{0} F(x)\mathrm{d}x + \int_{0}^{+\infty} F(x)\mathrm{d}x$

$= \int_{0}^{+\infty} F(-x)\mathrm{d}x + \int_{0}^{+\infty} F(x)\mathrm{d}x$

For $F(x)$ odd $\quad F(-x) = -F(x) \quad \Rightarrow \quad I = 0$

For $F(x)$ even $\quad F(-x) = F(x) \quad \Rightarrow \quad I \neq 0$

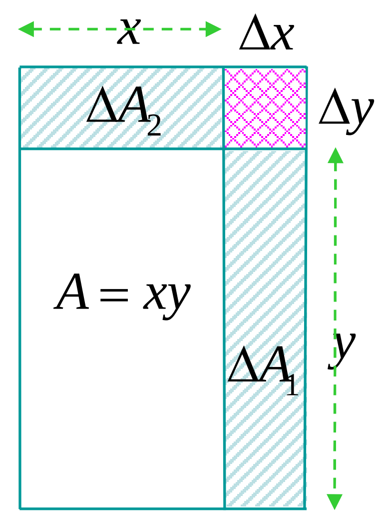
Most functions have no symmetry, but some may be broken into symmetric parts:

Since $\mathrm{e}^{ix} = \cos x + i \sin x$
and $\mathrm{e}^{-ix} = \cos x - i \sin x$

$\mathrm{Re}\{\mathrm{e}^{ix}\}$ is even, and $\mathrm{Im}\{\mathrm{e}^{ix}\}$ is odd

Partial Differential Calculus

for functions of more than one variable: $f=f(x, y, \ldots)$



Take area as an example $A = xy$

For an increase Δx in x, $\Delta A_1 = y\Delta x$ y constant

For an increase Δy in y, $\Delta A_2 = x\Delta y$ x constant

For a simultaneous increase

$$\Delta A = (x+\Delta x)(y+\Delta y) - xy$$
$$= y\Delta x + x\Delta y + \Delta x\Delta y$$
$$= \frac{\Delta A_1}{\Delta x}\Delta x + \frac{\Delta A_2}{\Delta y}\Delta y + \Delta x\Delta y$$

In the limits $\Delta x \to 0,\ \Delta y \to 0$

$$\Delta A \to dA = \left(\frac{\partial A}{\partial x}\right)_y dx + \left(\frac{\partial A}{\partial y}\right)_x dy$$

total differential

partial differential

For a real single-value function f of two independent variables,

$$\left(\frac{\partial f}{\partial x}\right)_y = \lim_{\delta x \to 0}\left\{\frac{f(x+\delta x, y) - f(x, y)}{\delta x}\right\}$$

Partial Derivative Relations

Consider $f(x, y, z) = 0$, so $z = z(x, y)$,

$$dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy$$

- Partial derivatives can be taken in any order.

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

$$\left[\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)_x\right]_y = \left[\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)_y\right]_x$$

- Taking the inverse:

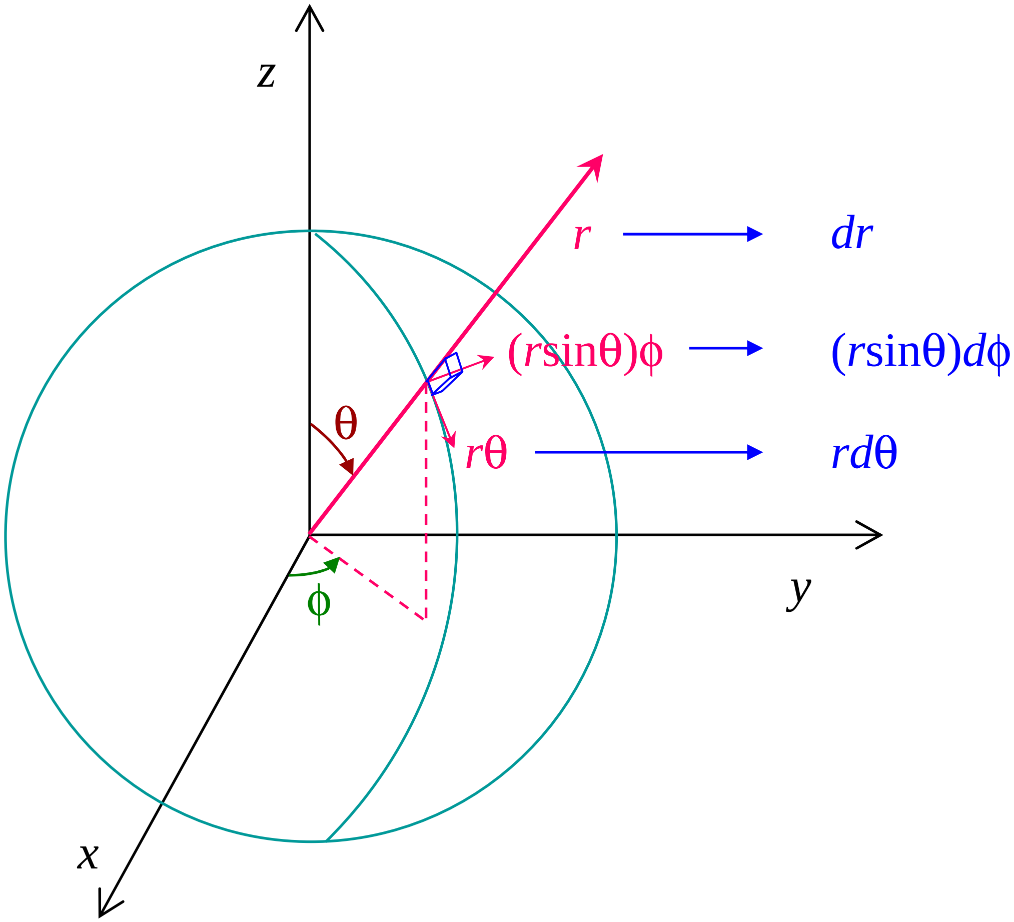
$$\left[\left(\frac{\partial z}{\partial x}\right)_y\right]^{-1} = \left(\frac{\partial x}{\partial z}\right)_y$$

- To find the third partial derivative:

$$dz = 0 \;\Rightarrow\; \left(\frac{\partial z}{\partial y}\right)_x dy = -\left(\frac{\partial z}{\partial x}\right)_y dx \;\Rightarrow\; \left(\frac{\partial x}{\partial y}\right)_z = -\frac{(\partial z/\partial y)_x}{(\partial z/\partial x)_y} = -\left(\frac{\partial z}{\partial y}\right)_x \left(\frac{\partial x}{\partial z}\right)_y$$

- Chain Rule: $\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1$ and $\left(\frac{\partial y}{\partial x}\right)_z \left(\frac{\partial x}{\partial z}\right)_y \left(\frac{\partial z}{\partial y}\right)_x = -1$

Spherical Polar Coordinates



$$\left.\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}\right\}$$

$$0 \leqslant r \leqslant R$$
$$0 \leqslant \phi \leqslant 2\pi$$
$$0 \leqslant \theta \leqslant \pi$$

$$\begin{aligned} d\tau &= dx\,dy\,dz \\ &= r^2 \sin\theta\, d\theta\, d\phi\, dr \end{aligned}$$

surface area

volume of sphere

$$\int d\tau = \int_0^R \int_0^{2\pi} \int_0^{\pi} r^2 \sin\theta\, d\theta\, d\phi\, dr = \int_0^R r^2 dr \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta = \int_0^R 4\pi r^2 dr = \frac{4}{3}\pi R^3$$

Spherical Polar Volume Element by Jacobean

$$d\tau = dxdydz = \frac{\partial(x,y,z)}{\partial(\theta,\phi,r)} d\theta d\phi dr$$

$$\left.\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}\right\}$$

$$\frac{\partial(x,y,z)}{\partial(\theta,\phi,r)} = \begin{vmatrix} \partial x/\partial\theta & \partial y/\partial\theta & \partial z/\partial\theta \\ \partial x/\partial\phi & \partial y/\partial\phi & \partial z/\partial\phi \\ \partial x/\partial r & \partial y/\partial r & \partial z/\partial r \end{vmatrix}$$

$$= \begin{vmatrix} r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{vmatrix}$$

$$= r^2\sin\theta$$

$$d\tau = r^2\sin\theta\, d\theta\, d\phi\, dr$$

Laplacian in Various Coordinate Systems

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$ cartesian

$$= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$ cylindrical

$$= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Lambda^2$$ spherical

where $$\Lambda^2 = \frac{\partial^2}{\partial \theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$ Legendrian