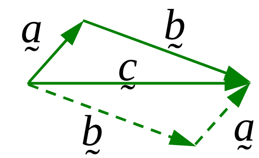
Vectors

A scalar has magnitude (only) but a vector has magnitude and direction.

It can be expressed in components $\quad \underset{\sim}{r} = \vec{r} = \vec{r} = \boldsymbol{r} = (x, y, z) = x\,\vec{\mathrm{i}} + y\,\vec{\mathrm{j}} + z\,\vec{\mathrm{k}}$

Addition $\quad \underset{\sim}{c} = \underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{b} + \underset{\sim}{a} = (a_x + b_x, a_y + b_y, a_z + b_z)$



Scalar product

$$\underset{\sim}{a} \bullet \underset{\sim}{b} = \underset{\sim}{b} \bullet \underset{\sim}{a} = a_x b_x + a_y b_y + a_z b_z$$
$$= |a||b| \cos\theta_{ab}$$

$$\begin{cases} \vec{\mathrm{i}} \bullet \vec{\mathrm{i}} = \vec{\mathrm{j}} \bullet \vec{\mathrm{j}} = \vec{\mathrm{k}} \bullet \vec{\mathrm{k}} = 1 \\ \vec{\mathrm{i}} \bullet \vec{\mathrm{j}} = \vec{\mathrm{j}} \bullet \vec{\mathrm{k}} = \vec{\mathrm{k}} \bullet \vec{\mathrm{i}} = 0 \end{cases}$$

$$\underset{\sim}{a} \bullet \underset{\sim}{a} = a_x a_x + a_y a_y + a_z a_z = |a|^2$$

$$|a| = \left(a_x^2 + a_y^2 + a_z^2\right)^{1/2}$$

Vector product

$$\underset{\sim}{a} \times \underset{\sim}{b} = |a||b| \sin\theta_{ab}\, \underset{\sim}{n}$$

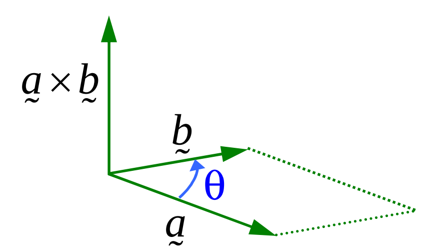
where $\underset{\sim}{n}$ is the unit vector $\perp$ to the plane of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

$$= -\underset{\sim}{b} \times \underset{\sim}{a}$$

$$= \begin{vmatrix} \underset{\sim}{\mathrm{i}} & \underset{\sim}{\mathrm{j}} & \underset{\sim}{\mathrm{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$\vec{\mathrm{i}} \times \vec{\mathrm{i}} = \vec{\mathrm{j}} \times \vec{\mathrm{j}} = \vec{\mathrm{k}} \times \vec{\mathrm{k}} = 0$$

$$\vec{\mathrm{i}} \times \vec{\mathrm{j}} = \vec{\mathrm{k}}, \quad \vec{\mathrm{j}} \times \vec{\mathrm{k}} = \vec{\mathrm{i}}, \quad \vec{\mathrm{k}} \times \vec{\mathrm{i}} = \vec{\mathrm{j}}$$



del $\quad \nabla = \frac{\partial}{\partial x}\underset{\sim}{\mathrm{i}} + \frac{\partial}{\partial y}\underset{\sim}{\mathrm{j}} + \frac{\partial}{\partial z}\underset{\sim}{\mathrm{k}}$

del squared $\quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

Determinants – 1

Associated with every square matrix is a real number called a determinant.

Evaluation:
$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = a_1 \left(b_2 c_3 - b_3 c_2 \right) - b_1 \left(a_2 c_3 - a_3 c_2 \right) + c_1 \left(a_2 b_3 - a_3 b_2 \right)$$

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \\ a_4 & c_4 & d_4 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \\ a_4 & b_4 & d_4 \end{vmatrix} - d_1 \begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix} = \ldots$$

Diagonal or block diagonal determinants are convenient.

$$\begin{vmatrix} a_1 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 \\ 0 & 0 & c_3 & 0 \\ 0 & 0 & 0 & d_4 \end{vmatrix} = a_1 b_2 c_3 d_4$$

$$\begin{vmatrix} a_1 & 0 & 0 & 0 \\ 0 & b_2 & c_2 & 0 \\ 0 & b_3 & c_3 & 0 \\ 0 & 0 & 0 & d_4 \end{vmatrix} = a_1 d_4 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$$

Determinants – 2

The value is unchanged if all rows and columns are swapped.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Swapping individual rows or columns changes the sign.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_1 & c_1 & b_1 \\ a_2 & c_2 & b_2 \\ a_3 & c_3 & b_3 \end{vmatrix} \qquad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

The value is zero if a pair of rows or columns is equal.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0 \qquad \begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix} = 0$$

Factorization:

$$\begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & kc_1 \\ a_2 & b_2 & kc_2 \\ a_3 & b_3 & kc_3 \end{vmatrix}$$

The value is unchanged if a row or column is combined with another.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} a_1 + kb_1 & b_1 & c_1 \\ a_2 + kb_2 & b_2 & c_2 \\ a_2 + kb_3 & b_3 & c_3 \end{vmatrix}$$

To Solve Simultaneous Equations by Cramer's Rule

3 equations for
3 unknowns
x, y and z

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

matrix version

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

$$x = \frac{1}{\Delta} \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \qquad y = \frac{1}{\Delta} \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \qquad z = \frac{1}{\Delta} \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

where

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Matrices – 1

The transpose of a row matrix: $\begin{pmatrix} a_1 & a_2 & \dots & a_n \end{pmatrix}$ is a column matrix: $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$

For square matrix $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$ transpose: $A^{\mathrm{T}} = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{pmatrix}$

The matrix is symmetric if $A = A^T \equiv \tilde{A}$

The matrix is real if $A = A^*$

The matrix is Hermitian if $A^* = A^T$

All real symmetric matrices are Hermitian.

Addition: if $C = A + B,\quad C_{ij} = A_{ij} + B_{ij}$

Multiplication: if $C = AB,\quad C_{ij} = \sum_k A_{ik} B_{kj}$

In general, $AB \neq BA$

Scalar multiplication: if $C = \alpha B,\quad C_{ij} = \alpha B_{ij}$

i.e. multiply all elements

The inverse matrix is defined by $AA^{-1} = A^{-1}A = I$ where I is the unit matrix

$$I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

Matrices – 2

When a matrix is applied to a vector, it generates a new vector (≡ a column matrix).

$$\underset{\approx}{A}\,\underset{\sim}{X} = \underset{\sim}{Y} \quad \text{e.g.} \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

where $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix}$

The matrix transforms X into Y.

If A transforms X into itself, with a scale factor, this is an eigenvalue equation

$$\underset{\approx}{A}\,\underset{\sim}{X} = \lambda\,\underset{\sim}{X}$$

λ is an eigenvalue

$\underset{\sim}{X}$ is an eigenvector

It follows from $\left(\underset{\approx}{A} - \lambda \underset{\approx}{I}\right)\underset{\sim}{X} = 0$ that $\det\left(\underset{\approx}{A} - \lambda \underset{\approx}{I}\right) = 0$

Expansion of the determinant results in the characteristic equation, an *n*-th order polynomial in λ, where *n* is the dimension of the (square) matrix.

Hermitian matrices have real eigenvalues,
and the eigenvectors are orthogonal: $\underset{\sim}{X}_i \bullet \underset{\sim}{X}_j = 0$

The product of eigenvalues $\lambda_1\lambda_2\lambda_3\ldots\lambda_n = \det\left(\underset{\approx}{A}\right)$

The sum of eigenvalues $\lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_n = \mathrm{trace}\left(\underset{\approx}{A}\right)$ = sum of diagonal elements

Diagonalization of Matrices – Example

Find the eigenvalues and eigenvectors of $\begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

Solve the characteristic equation

$$\begin{pmatrix} 1-\lambda & 0 & 1 \\ 0 & -1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{pmatrix} = 0$$

$$-(1-\lambda)^2(1+\lambda) + (1+\lambda) = 0$$

$$\lambda = -1 \text{ or } 0 \text{ or } 2$$

$$\mathrm{L} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

normalize

For $\lambda = 0$,

$$x + z = 0 \qquad y = 0 \qquad \phi_0 = \begin{pmatrix} m \\ 0 \\ -m \end{pmatrix} \Rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

For $\lambda = -1$,

$$2x + z = 0 \qquad x + 2z = 0 \qquad \phi_{-1} = \begin{pmatrix} 0 \\ p \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

For $\lambda = 2$,

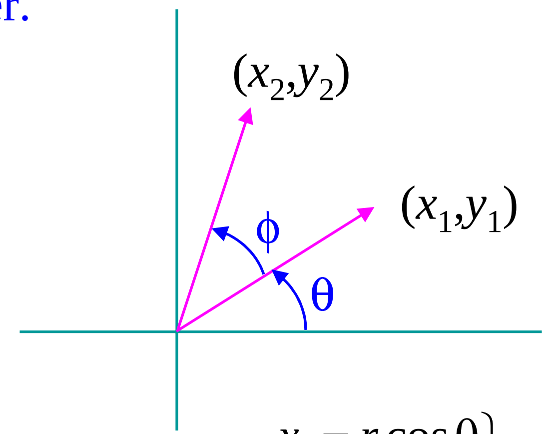
$$-x + z = 0 \qquad y = 0 \qquad \phi_2 = \begin{pmatrix} q \\ 0 \\ q \end{pmatrix} \Rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\tilde{\mathrm{L}}\,\mathrm{M}\,\mathrm{L} = \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Transformation as Matrix Operation

The operation of a matrix transforms one vector into another.

e.g. Rotation through an angle in a 2-D plane



$$\left.\begin{aligned} x_1 &= r\cos\theta \\ y_1 &= r\sin\theta \end{aligned}\right\}$$

$$\begin{aligned} x_2 = r\cos(\theta+\phi) &= r\cos\theta\cos\phi - r\sin\theta\sin\phi \\ &= x_1\cos\phi - y_1\sin\phi \end{aligned}$$

$$\begin{aligned} y_2 = r\sin(\theta+\phi) &= r\sin\theta\cos\phi + r\cos\theta\sin\phi \\ &= y_1\cos\phi + x_1\sin\phi \\ &= x_1\sin\phi + y_1\cos\phi \end{aligned}$$

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

$$\begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

$$\Rightarrow \qquad \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}^{-1}$$