Discretization and Solution of Convection-Diffusion Problems

Howard Elman

University of Maryland

Overview

1. The convection-diffusion equation
   — Introduction and examples

2. Discretization strategies
   — Finite element methods
   — Inadequacy of Galerkin methods
   — Stabilization: streamline diffusion methods

3. Iterative solution algorithms
   — Krylov subspace methods
   — Splitting methods
   — Multigrid

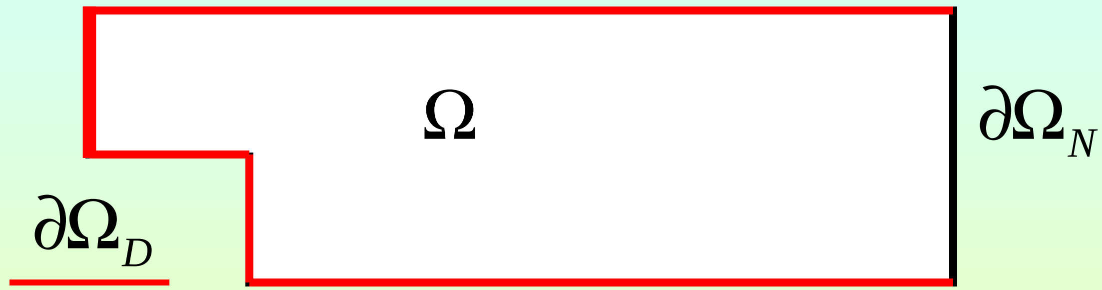
The Convection-Diffusion Equation

$$-\varepsilon \nabla^2 u + w \cdot \nabla u = f \text{ in } \Omega \subset \mathbb{R}^d, \ d = 1,2,3$$

Boundary conditions:

$$u = g_D \text{ on } \partial\Omega_D$$

$$\frac{\partial u}{\partial n} = g_N \text{ on } \partial\Omega_N$$



Inflow boundary:

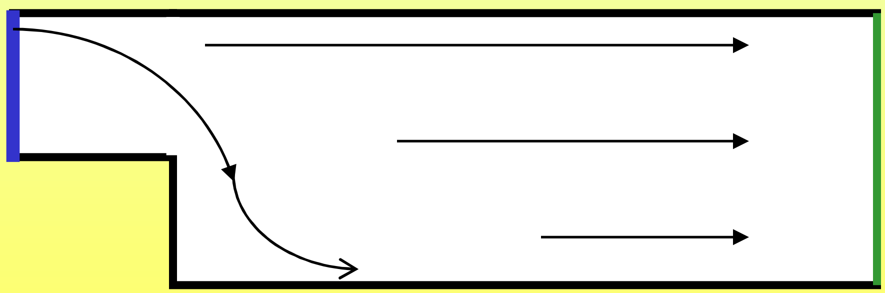
$\partial\Omega_+ = \{x \in \partial\Omega \mid w \cdot n > 0\}$

Characteristic boundary:

$\partial\Omega_0 = \{x \in \partial\Omega \mid w \cdot n = 0\}$

Outflow boundary:

$\partial\Omega_- = \{x \in \partial\Omega \mid w \cdot n < 0\}$



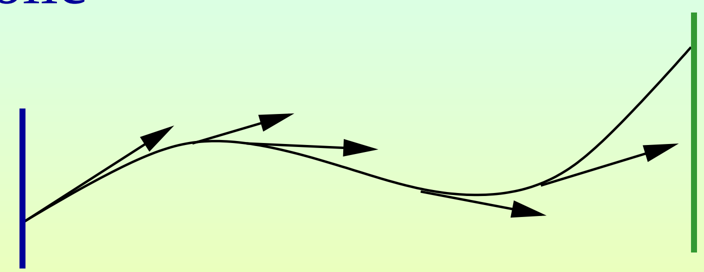
The Convection-Diffusion Equation

$$-\varepsilon \nabla^2 u + w \cdot \nabla u = f$$

Challenging / interesting case: $\varepsilon \longrightarrow 0$

Reduced problem: $w \cdot \nabla u = f$, hyperbolic

Streamlines: parameterized curves
c(s) in Ω s.t. c(s) has tangent vector
w(c(s)) on c



$$\Rightarrow (\nabla u) \cdot w = \boxed{\frac{d}{ds} u(c(s)) = f}$$

Solution u to reduced problem = solution to ODE

If $u(s_0) \in$ inflow boundary $\partial\Omega_+$, and $u(s_1) \in \partial\Omega$, say outflow $\partial\Omega_-$,
then boundary values are determined by ODE

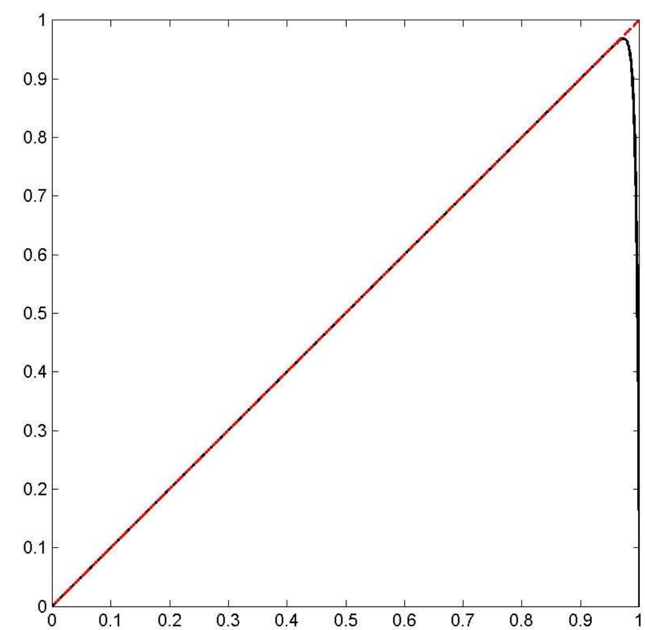
Consequence $-\varepsilon\nabla^2 u + w\cdot\nabla u = f$

For small ε, solution to convection-diffusion equation often has ***boundary layers,*** steep gradients near parts of ∂Ω.

Also: discontinuities at inflow propagate into Ω along streamlines

Simple (1D) example of first phenomenon:

$$-\varepsilon u'' + u' = 1 \text{ on } (0,1),$$
$$u(0) = 0,\ u(1) = 0$$



Solution

$$u(x) = x - e^{-(1/\varepsilon)(1-x)}\left(\frac{1-e^{-x/\varepsilon}}{1-e^{1/\varepsilon}}\right)$$
$$\approx x \text{ except near } x = 1$$

Solution to reduced equation

Additional Consequence

These layers (steep gradients) are difficult to resolve with discretization

Conventions of Notation

L = characteristic length scale in boundary
e.g. length of inflow boundary

$\hat{x}$ = x/L in normalized domain

W = normalization for velocity (wind) w,
e.g. $w = W\, w_*$, where $\|w_*\|=1$

In normalized variables:

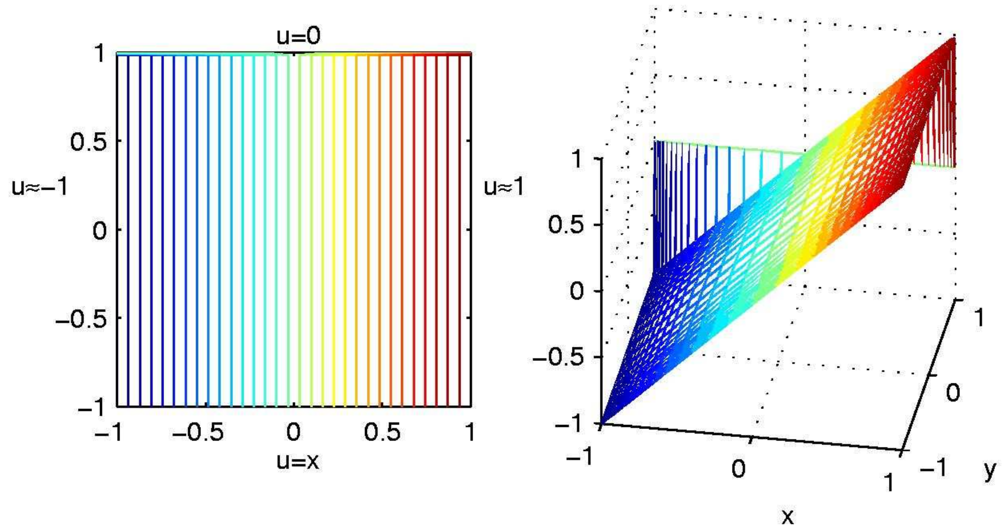
$$-\nabla^2 u_* + \left(\frac{WL}{\varepsilon}\right) w_* \cdot \nabla u_* = \left(\frac{L^2}{\varepsilon}\right) f$$

$P \equiv \dfrac{WL}{\varepsilon}$, ***Peclet number***, characterizes relative contributions of convection and diffusion

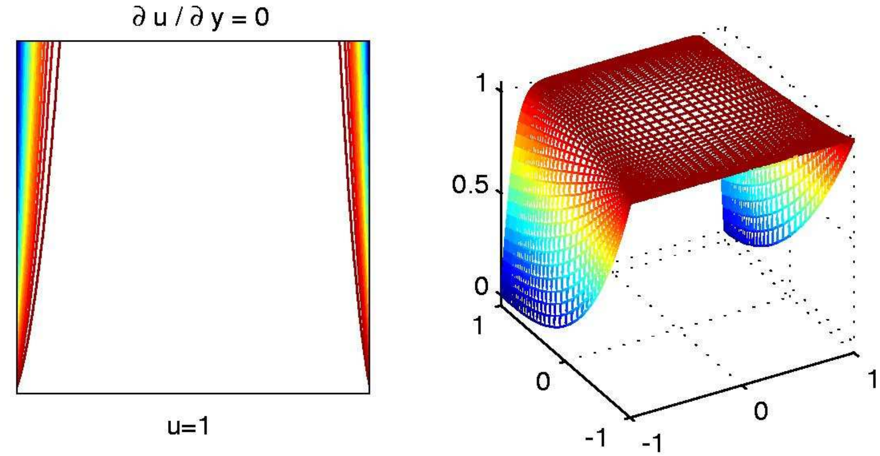
Reference Problems $-\varepsilon\nabla^2 u + w\cdot\nabla u = f$

1. w=(0,1)
   Dirichlet b.c.
   analytic solution

$$u(x,y) = x\left(\frac{1-e^{-(1-y)/\varepsilon}}{1-e^{-2/\varepsilon}}\right)$$

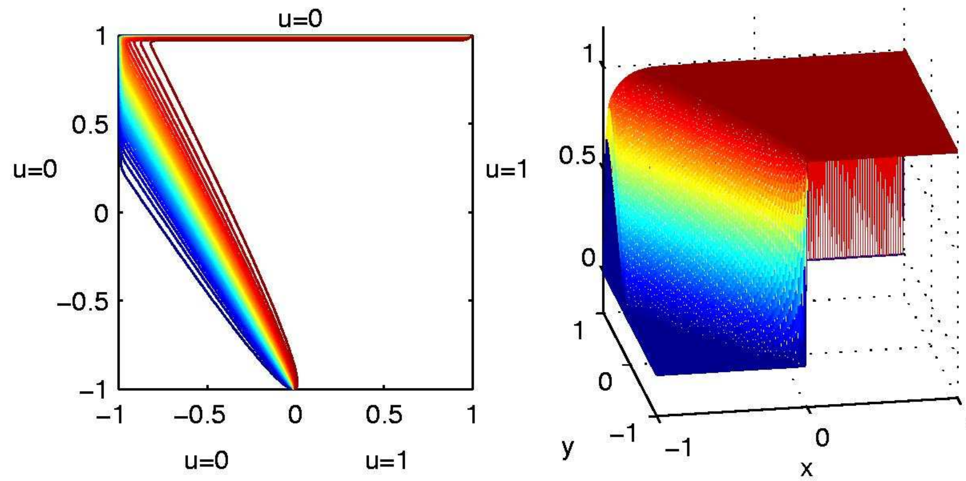


2. w=(0,(1+(x+1)^2/4))
   Neumann b.c. at outflow
   characteristic boundary layers

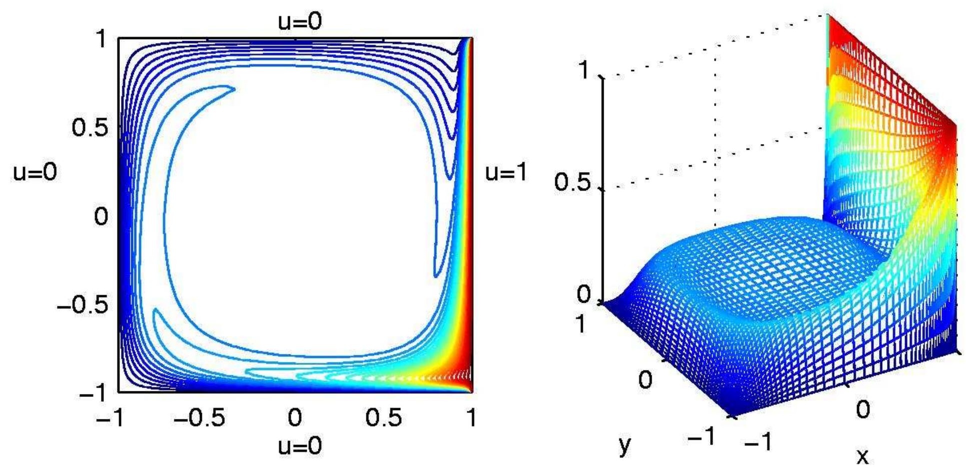


Reference Problems $-\varepsilon\nabla^2 u + w\cdot\nabla u = f$

3. *w*: 30° left of vertical
   interior layer from
   discontinuous b.c.
   downstream boundary layer



4. *w*=recirculating
   $(2y(1-x^2),-2x(1-y^2))$
   characteristic boundary layers
   discontinuous b.c.



Weak Formulation $-\varepsilon\nabla^2 u + w\cdot\nabla u = f$

$$\text{Find } u\in H^1_E(\Omega) \text{ s.t. for all } v\in H^1_{E_0}(\Omega),$$

$$\int_\Omega \varepsilon\nabla u\cdot\nabla v + (w\cdot\nabla u)v = \int_\Omega fv + \int_{\partial\Omega_N} v g_N$$

$$H^1_E(\Omega) = \{v \mid v = g_D \text{ on } \partial\Omega_D\}$$

$$H^1_{E_0}(\Omega) = \{v \mid v = 0 \quad \text{on } \partial\Omega_D\}$$

Shorthand notation: $a(u,v) = l(v)$ for all v

Can show:

$$\left.\begin{aligned} a(u,u) &\geq \varepsilon\, \|\nabla u\|^2 = \varepsilon\int_\Omega \nabla u\cdot\nabla u \\ a(u,v) &\leq (\varepsilon + \|w\|_\infty\, L)\, \|\nabla u\|\, \|\nabla v\| \\ l(v) &\leq C\, \|\nabla v\| \end{aligned}\right\}$$

Lax-Milgram lemma ⟶ existence and uniqueness of solution

Approximation by Finite Elements

Given finite dimensional $S_E^h \subset H_E^1, \ S_0^h \subset H_{E_0}^1$,

find $u_h \in S_E^h$ such that for all $v_h \in S_0^h$,

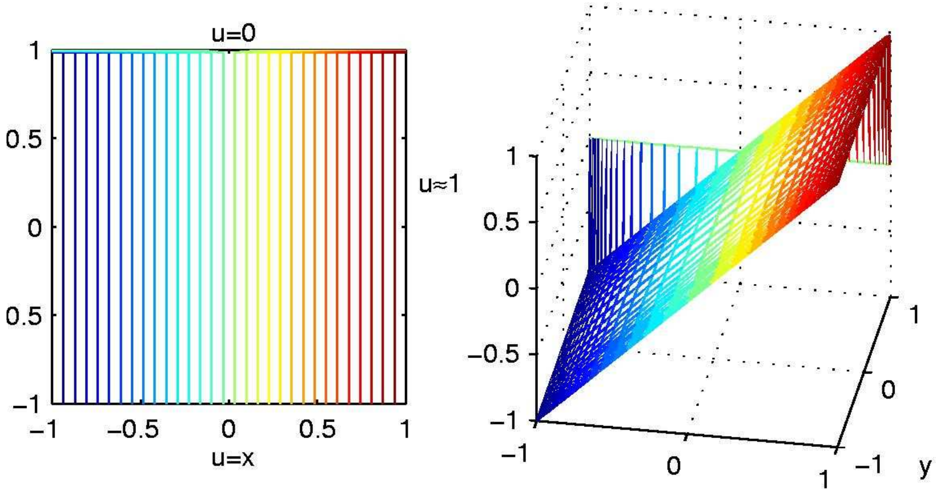
$$\varepsilon \int_\Omega \nabla u_h \cdot \nabla u_h + \int_\Omega (w \cdot \nabla u_h) v_h = \int_\Omega f v_h + \int_{\partial\Omega_N} v_h g_N$$

$a(u_h, v_\mathrm{h}) = l(v_\mathrm{h})$ for all v_h

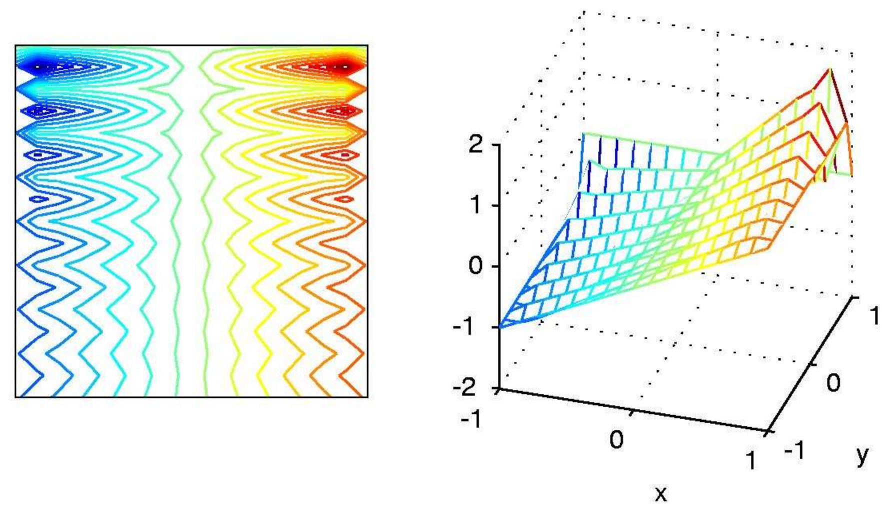
Typically: finite element spaces are defined by low-order basis functions, e.g.
— linear or quadratic functions on triangles
— bilinear or biquadratic functions on quadrilaterals

What happens in such cases?

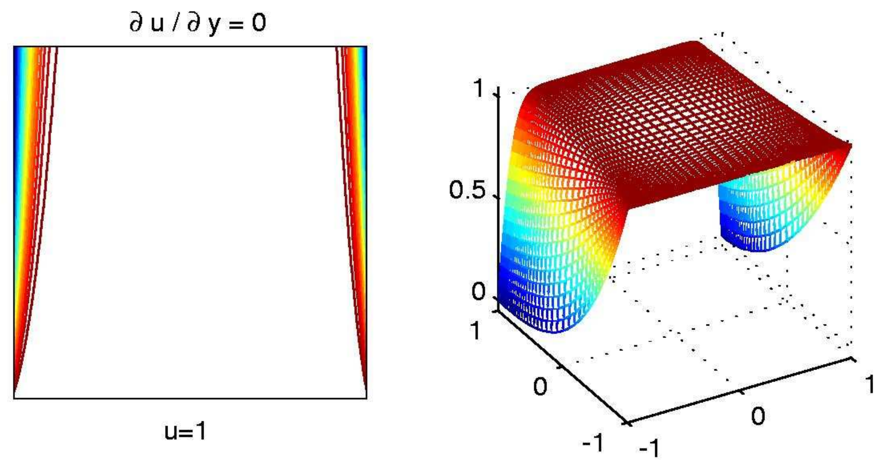
Problem 1, accurate



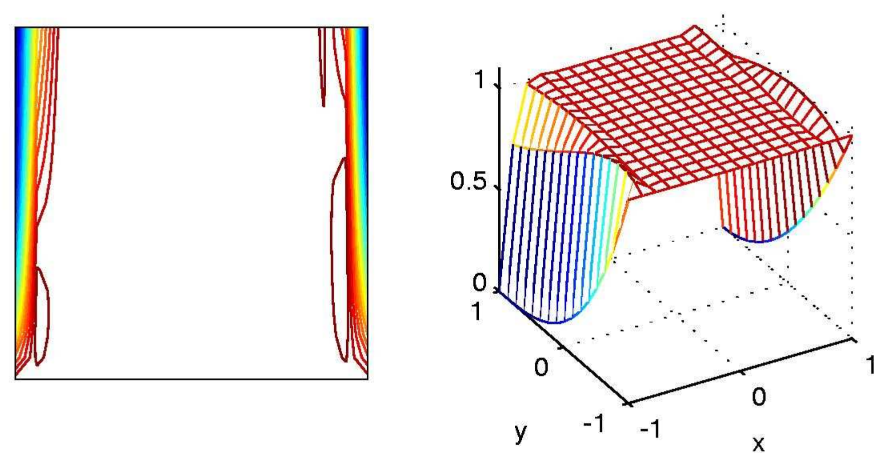
Problem 1, inaccurate



Problem 2, accurate



Problem 2, inaccurate



Explanations

1. Error analysis: discrete solution is *quasi-optimal*:

$$\|\nabla(u-u_h)\| \le \frac{\Gamma_w}{\varepsilon} \inf_{v_h \in S_E^h} \|\nabla(u-v_h)\|,$$

$$\Gamma_w = 1 + \frac{WL}{\varepsilon} = 1 + P, \text{ large if } \varepsilon \text{ is small}$$

2. ***Mesh Peclet number***: $P_h = \frac{Ph}{2L} = \frac{Wh}{2\varepsilon}$

If $P_h > 1$, then

— there are oscillations in the discrete solution

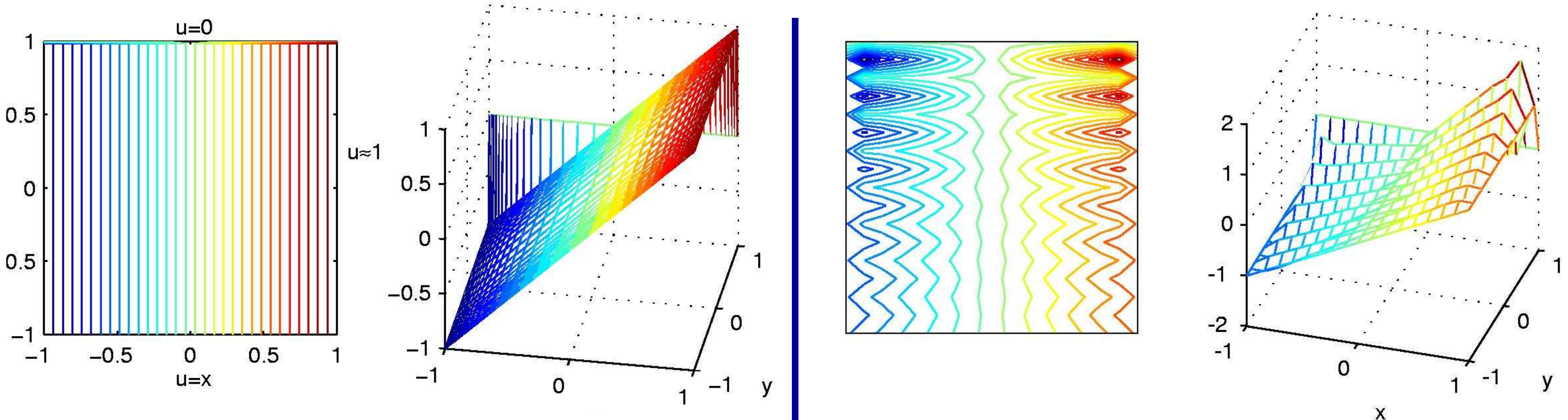
— these become pronounced if mesh does not resolve layers

— oscillations propagate into regions where solution is smooth

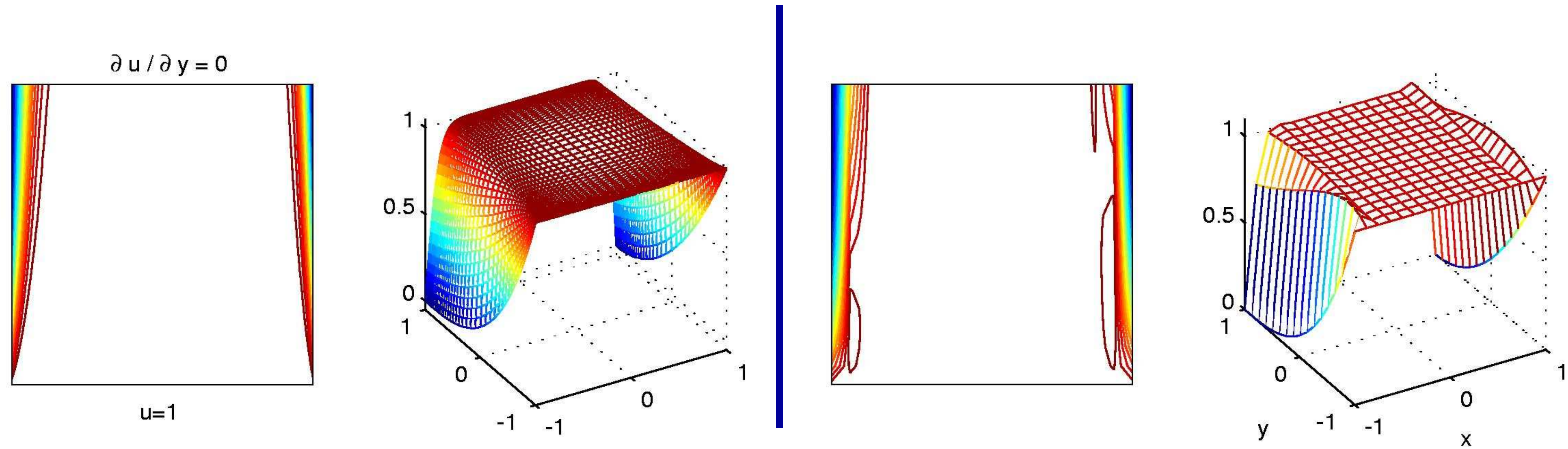
— problem is most severe for *exponential* boundary layers

Revisit two examples

Problem 1, exponential layer, width ~ ε



Problem 2, characteristic layer, width ~ $\varepsilon^{1/2}$



Fix: The Streamline Diffusion Method

Petrov-Galerkin method: change the test functions

Galerkin: $a(u_h, v_h) = l(v_h)$ for all v_h

Petrov-Galerkin: $a(u_h, v_h + \delta w \cdot \nabla v_h) = l(v_h + \delta w \cdot \nabla v_h)$ for all v_h

δ is a parameter

Result: $a_{sd}(u_h, v_h) = l_{sd}(v_h)$

Streamline diffusion term

$$a_{sd}(u_h, v_h) = \varepsilon \int_\Omega \nabla u_h \cdot \nabla u_h + \int_\Omega (w \cdot \nabla u_h) v_h + \delta \int_\Omega (w \cdot \nabla u_h)(w \cdot \nabla v_h)$$

$$- \delta \varepsilon \sum_k \int_{\Delta_k} (\nabla^2 u_h)(w \cdot \nabla v_h)$$

0 for linear/bilinear

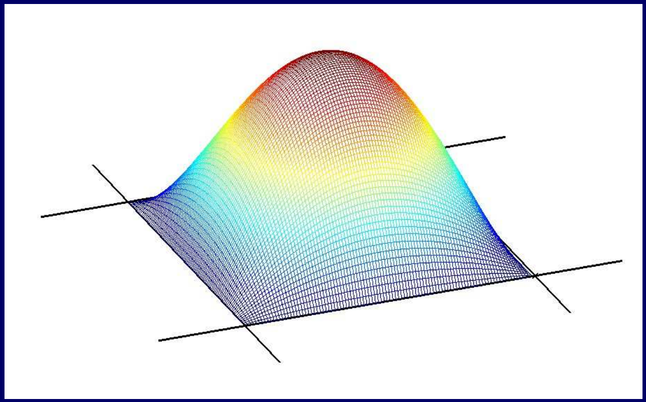
$$l(v_h) = \int_\Omega f v_h + \delta \int_\Omega f (w \cdot \nabla v_h) + \int_{\partial \Omega_N} (v_h + \delta w \cdot \nabla v_h) g_N$$

The Streamline Diffusion Method Explained

Augment finite element space:

$$\hat{S}^{\mathrm{h}} = S^{\mathrm{h}} + B^{h}$$

$\boldsymbol{B}^{h}$: *bubble functions*, with support local to element



Principle: augmented space $\hat{S}^{\mathrm{h}}$ places basis functions in layers not resolved by the grid

We could pose the problem on the augmented space:

find u_h in $\hat{S}^{\mathrm{h}}$ s.t. $a(u_h,v_h) = l(v_h)$ for all v_h in $\hat{S}^{\mathrm{h}}$

Then: decouple unknowns associated with bubble functions from system ⟶ *new problem* on original grid

The Streamline Diffusion Method Explained

Under appropriate assumptions: this new problem is

$$a_{sd}(u_{\mathrm{h}},v_{\mathrm{h}}) = l_{\mathrm{sd}}(v_{\mathrm{h}})$$

$$a_{sd}(u_h, v_h) = \varepsilon \int_\Omega \nabla u_h \cdot \nabla u_h + \int_\Omega (w \cdot \nabla u_h) v_h$$

$$+ \sum_{\Delta_k} \delta_k \int_{\Delta_k} (w \cdot \nabla u_h)(w \cdot \nabla v_h)$$

$$l(v_h) = \int_\Omega f v_h + \sum_k \delta_k \int_{\Delta_k} f (w \cdot \nabla v_h)$$

δ_k determined from elimination of bubble functions

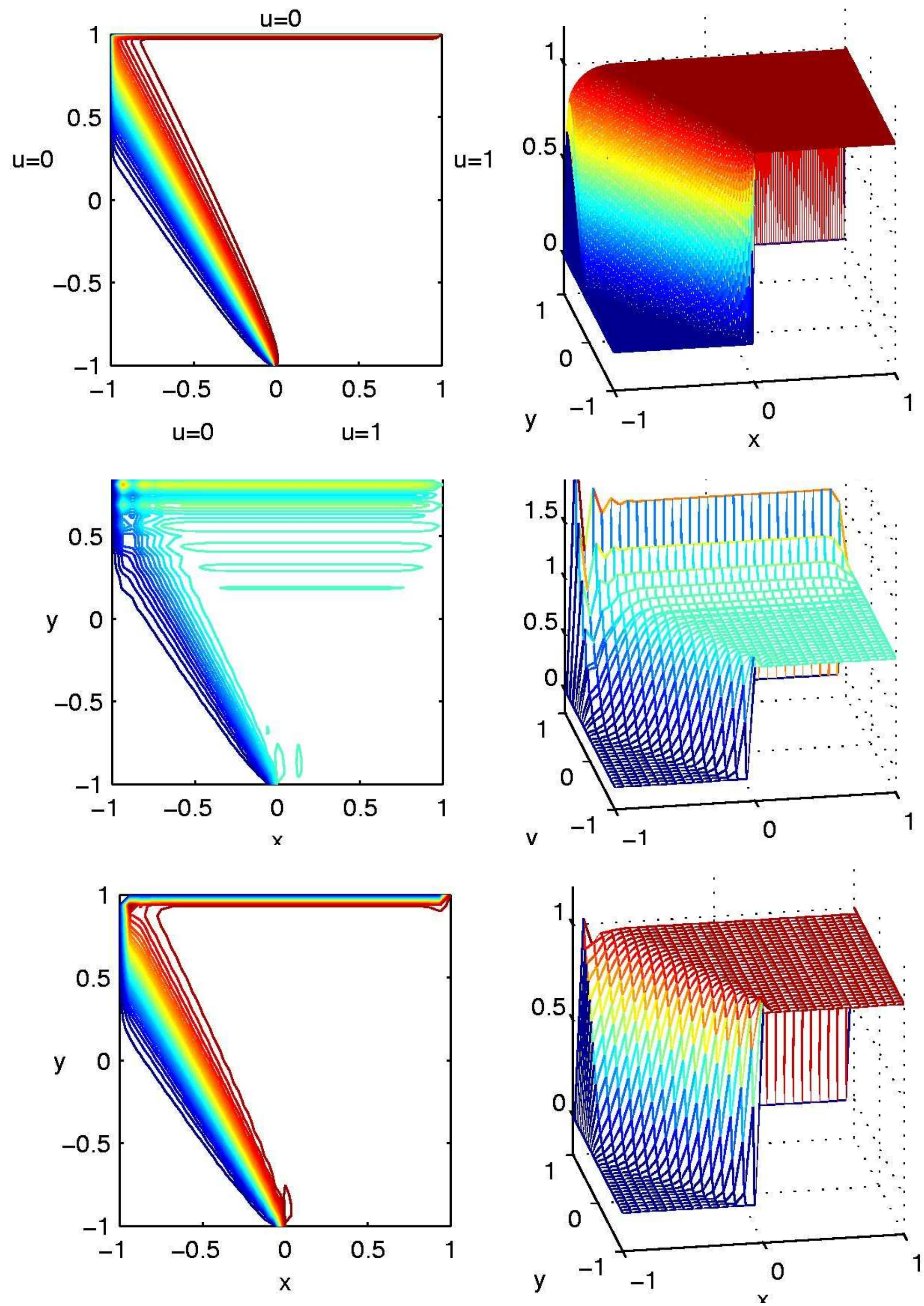
≡ ***Streamline diffusion***

Compare Galerkin and Streamline Diffusion

Top: accurate solution,
$\varepsilon = 1/200$

Middle: bilinear elements,
Galerkin,
32×32 grid

Bottom: bilinear elements,
streamline diffusion,
32×32 grid



Error Bounds

For Galerkin: as noted earlier, quasi-optimality:

$$\|\nabla(u-u_h)\| \le \frac{\Gamma_w}{\varepsilon} \inf_{v_h \in S_E^h} \|\nabla(u-v_h)\|$$

More careful analysis: for linear/bilinear elements,

$$\|\nabla(u-u_h)\| \le Ch \underbrace{\|D^2 u\|}$$

Large in exponential boundary layers for small ε

For streamline diffusion: use norm

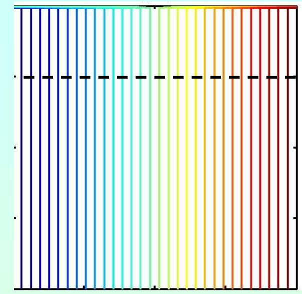
$$\|v\|_{sd} \equiv \left(\varepsilon \|\nabla v\|^2 + \delta \|w \cdot \nabla v\|^2\right)^{1/2}$$

Then

$$\|u-u_h\|_{sd} \le Ch^{3/2} \|D^2 u\|$$

These bounds do not tell the whole story

For one example (Problem 1, ε=1/64), compare errors $\|\nabla(u-u_h)\|$ on Ω and $\Omega_* = (-1,1)\times(-1,3/4)$ (to exclude boundary layer)



Grid (P_h)	Galerkin Ω	Str.Diff. Ω	Galerkin Ω_*	Str.Diff. Ω_*
8×8 (8)	5.62	4.34	3.25	8.16e-7
16×16 (4)	4.91	4.01	1.48	1.64e-5
32×32 (2)	3.81	3.23	5.30e-2	1.11e-5
64×64 (1)	2.39	2.39	4.98e-7	4.98e-7

Choice of parameter δ

Made element-wise: $$a_{sd}(u_h, v_h) = \varepsilon \int_\Omega \nabla u_h \cdot \nabla u_h + \int_\Omega (w \cdot \nabla u_h) v_h + \sum_{\Delta_k} \delta_k \int_{\Delta_k} (w \cdot \nabla u_h)(w \cdot \nabla v_h)$$

$$\delta_k = \begin{cases} \dfrac{h_k}{2|w_k|}\left(1 - 1/P_h^k\right) & \text{if } P_h^k > 1 \\ 0 & \text{if } P_h^k \le 1 \end{cases}$$

Matrix Properties

Given a basis $\{\varphi_j\}_{j=1}^{n}$ for S_0^h, extended by $\{\varphi_j\}_{n+1}^{n+n_\partial}$ for S_E^h

Finite element function is $u_h = \sum_j \mathrm{u}_j \varphi_j$, problem

becomes: find $\{\mathrm{u_j}\}$ such that

$$\sum_{\mathrm{j}} a(\varphi_j, \varphi_i)\mathrm{u}_j = l(\varphi_j), \quad i = 1,2,\ldots,n$$

or a_{sd}

Leads to matrix equation $F\boldsymbol{u}=\boldsymbol{f}$,

$$F=\varepsilon A + N \;\; (+ S)$$

Matrix Properties

Matrix equation $F\boldsymbol{u}=\boldsymbol{f}$, $F=\varepsilon A + N \ (+ S)$

$A=[a_{ij}]$, $a_{ij}=\int_\Omega \nabla\phi_j \cdot \nabla\phi_i$, *discrete Laplacian,*
symmetric positive-definite

$N=[n_{ij}]$, $n_{ij}=\int_\Omega (w\cdot\nabla\phi_j)\phi_i$, *discrete convection operator,*
skew-symmetric ($N=-N^T$)

$S=[s_{ij}]$, $s_{ij}=\int_\Omega (w\cdot\nabla\phi_j)(w\cdot\nabla\phi_i)$, *discrete streamline*
upwinding operator,
positive semi-definite

End of Part I

Next: how to solve $F\boldsymbol{u}=\boldsymbol{f}$?

Iterative Solution Algorithms: Krylov Subspace Methods

System $F\boldsymbol{u}=\boldsymbol{f}$

- F is a nonsymmetric matrix, so an appropriate Krylov subspace method is needed
- Examples:
  - GMRES
  - GMRES(k) restarted
  - BiCGSTAB
  - BiCGSTAB(l)
- Our choices:
  - Full GMRES for optimal algorithm, or
  - BiCGSTAB(2) for suboptimal

Properties of Krylov Subspace Methods

Drawback of GMRES: work & storage requirements at step *k* are proportional to kN

BiCGSTAB: Fixed cost per step, independent of *k*

Drawback: No convergence analysis

Variant: BiCGSTAB(*l*), more robust for complex eigenvalues, somewhat higher cost per step (*l*=2), but still fixed

Convergence of GMRES

GMRES: Starting with $\mathbf{u}_0$, with residual $\mathbf{r}_0=\mathbf{f}-F\mathbf{u}_0$, computes

$$\mathbf{u}_k \in span\{\mathbf{r}_0, F\mathbf{r}_0, \ldots, F^{k-1}\mathbf{r}_0\}$$

for which $r_k = \mathbf{f}-F\mathbf{u}_k$ satisfies

$$\|\mathbf{r}_k\| = \min_{p_k(0)=1} \| p_k(F)\mathbf{r}_0\|.$$

Consequence:

Theorem: For diagonalizable $F=V\Lambda V^{-1}$,

$$\|\mathbf{r}_k\| \leq \|V\|\,\|V^{-1}\| \min_{p_k(0)=1} \max_{\lambda\in\sigma(F)}|p_k(\lambda)|\,\|\mathbf{r}_0\|.$$

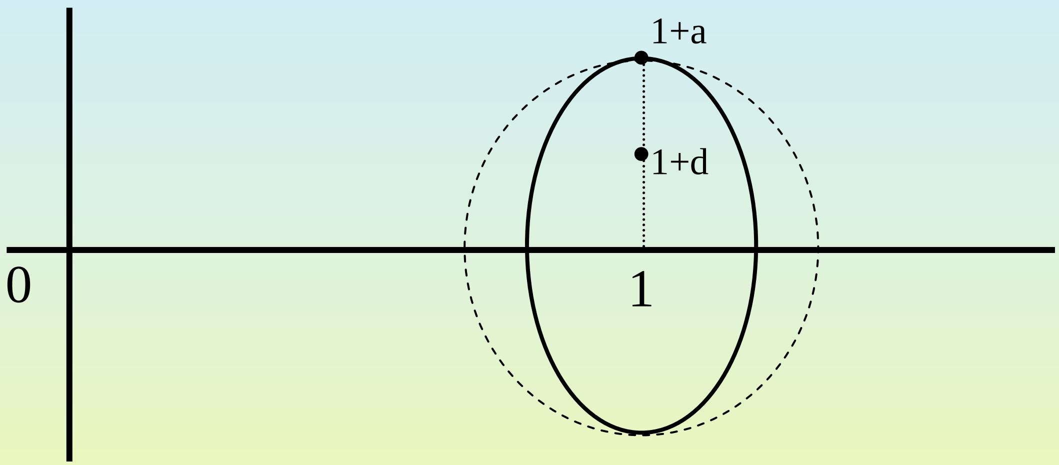
$$(\|\mathbf{r}_k\|/\|\mathbf{r}_0\|)^{1/k} \leq (\|V\|\,\|V^{-1}\|)^{1/k} \underbrace{(\min_{p_{k(0)=1}} \max_{\lambda\in\sigma(F)}|p_k(\lambda)|)^{1/k}}_{\hat{\rho}}$$

Loosely speaking: residual is reduced by factor of $\hat{\rho}$ *at each step*

Want eigenvalues to lie in compact set

Convergence of GMRES

Size of convergence factor $\hat{\rho}$



$$\hat{\rho} \approx \frac{a + \sqrt{a^2 - d^2}}{1 + \sqrt{1 - d^2}} \leq a = \rho(Q_F^{-1} R_F)$$

Key for Fast Convergence: Preconditioning Splitting operators

Seek $Q_F \approx F$ such that

- the approximation is good, and
- it is inexpensive to apply the action of Q^{-1} to a vector

Splitting: $F = Q_F - R_F$ ⟶ stationary iteration $u_{k+1} = Q_F^{-1}(R_F u_k + f)$

Error $\boldsymbol{e}_k = \boldsymbol{u}\text{-}\boldsymbol{u}_{\mathrm{k}}$ satisfies

$$u - u_{k+1} = Q_F^{-1} R_F (u - u_k) = (I - Q_F^{-1} F)(u - u_k) \Rightarrow$$

$$e_k = (I - Q_F^{-1} F)^k e_0$$

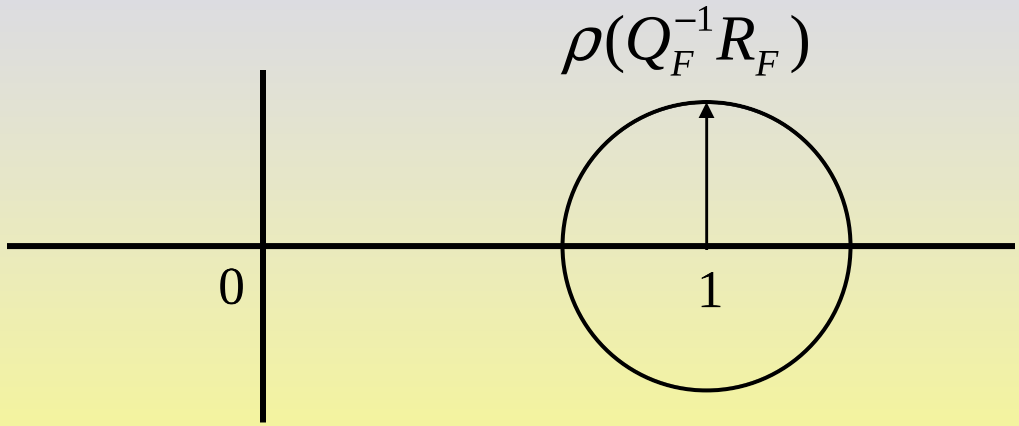
$$\|e_k\| \le \|(I - Q_F^{-1} F)^k\| \|e_0\|$$

$$\left(\|e_k\| / \|e_0\|\right)^{1/k} \le \|(I - Q_F^{-1} F)^k\|^{1/k} \approx \rho(I - Q_F^{-1} F)$$

Preconditioning / Splitting operators

Thus: want $\rho(I - Q_F^{-1}F)$ to be as small as possible

Equivalently: eigenvalues of $Q_F^{-1}F$ = eigenvalues of FQ_F^{-1} } as close to 1 as possible



This is similar to the requirement for rapid convergence of GMRES

Solve $Q_F^{-1}Fu = Q_F^{-1}f$ or $FQ_F^{-1}\hat{u} = f,\; u = Q_F^{-1}\hat{u}$

Examples of splitting operators

Gauss-Seidel	$Q_F = \text{lower triangle of } A$
Line Gauss-Seidel	$Q_F = \text{block lower triangle of } A$
Symmetric versions	$Q_F = L_F U_F$
Incomplete LU factorization	$F \approx LU = Q_F$

Comments:

- All depend on ordering of underlying grid
- Symmetric versions (symmetric GS, ILU) take some account of underlying flow
- Line/block versions can handle irregular grids

Convergence Analysis (Parter & Steuerwalt)

Seek maximal eigenvalue of $Q_F^{-1} R_F u = \lambda u$ or $\lambda\, Q_F u = R_F u$

Subtract $\lambda\, R_F u$ from both sides ⟶

$$(Q_F - R_F)u = \left(\frac{1-\lambda}{\lambda}\right) R_F u = \left(\frac{1-\lambda}{\lambda h^2}\right)(h^2 R_F)u$$

$$\boxed{Fu = \mu_h (h^2 R_F)u} \qquad (1)$$

Suggests relation to $\boxed{\mathcal{L}\, u = \mu \mathcal{R}\, u}$ (2)

$$\mathcal{L}\, u = -\varepsilon \nabla^2 u + w \cdot \nabla u$$

$\mathcal{R}$ to be determined

For many examples of splittings:

$h^2 (R_F u, v) \approx (r\, u_h, v_h)$, $r = r(x)$ (defines $\mathcal{R}$)

$h^2 R_F$ is a "weak multiplication operator"

and

$\mu_h \rightarrow \mu^{(0)} =$ minimal eigenvalue of (2)

Consequence: $$\rho(Q_F^{-1}R_F) = 1 - \mu^{(0)}h^2$$

For model problems:

(i) r is constant (will demonstrate in a moment)

(ii) on square domains, eigenvalues, eigenvectors of

$$\mathcal{L}u = \mu \mathcal{R}\, u$$

are known:

$$u^{(j,k)} = e^{w_1 x/2}\sin(j\pi x)\, e^{w_2 y/2}\sin(k\pi y)$$

$$\mu_{jk} = \frac{\varepsilon}{r}\left((j^2+k^2)\pi^2 + \left(\frac{w_x}{2\varepsilon}\right)^2 + \left(\frac{w_y}{2\varepsilon}\right)^2 \right)$$

$$\Rightarrow \mu^{(0)} = \frac{\varepsilon}{r}\left(2\pi^2 + \left(\frac{w_x}{2\varepsilon}\right)^2 + \left(\frac{w_y}{2\varepsilon}\right)^2 \right)$$

To find r: consider centered finite differences

$$-\left(\varepsilon - \frac{w_y h}{2}\right)$$

$$-\left(\varepsilon + \frac{w_x h}{2}\right) \quad — \quad 4\varepsilon \quad — \quad -\left(\varepsilon - \frac{w_x h}{2}\right)$$

$$-\left(\varepsilon + \frac{w_y h}{2}\right)$$

For horizontal line Jacobi splitting, Q_F = block tridiagonal:

$$[R_F u]_{ij} = \left(\varepsilon - \frac{w_y h}{2}\right) u_{i,j+1} + \left(\varepsilon + \frac{w_y h}{2}\right) u_{i,j-1} = 2\varepsilon\, u_{ij} + O(h^2)$$

$$\Rightarrow \quad r = 2\varepsilon, \qquad \rho = 1 - \frac{1}{2}\left(2\pi^2 + \left(\frac{w_x}{2\varepsilon}\right)^2 + \left(\frac{w_y}{2\varepsilon}\right)^2\right) h^2$$

Key point: convection terms lead to smaller convergence factors

Comments / Extensions

- Similar results obtained from matrix/Fourier analysis

- Young: $\rho(\text{line Gauss-Seidel operator}) = \rho(\text{line Jacobi operator})^2$

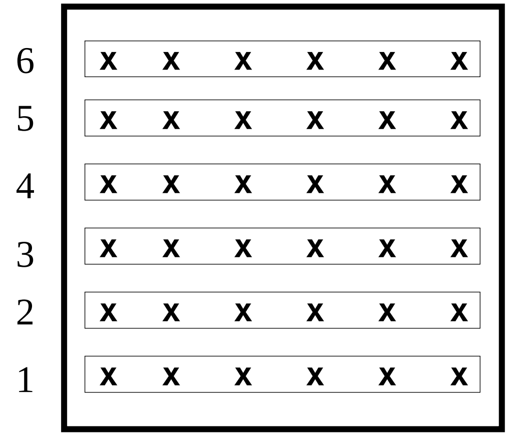
- "Multi-line" (k-line) splittings $\longrightarrow$ $r = 2\varepsilon / k$

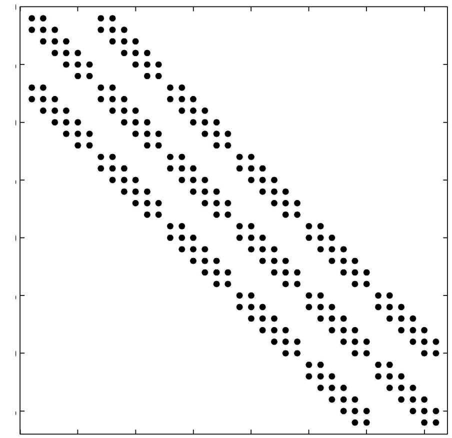
- Can extend to other splittings via *matrix comparison theorems* (Varga-Woźnicki): $Q_2^{-1} \geq Q_1^{-1} \Rightarrow \rho(Q_2^{-1}R_2) \leq \rho(Q_1^{-1}R_1)$

Limitations of analysis above:

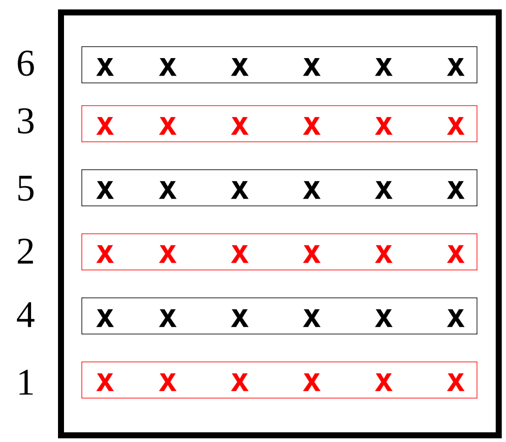
It does not discriminate among different orderings

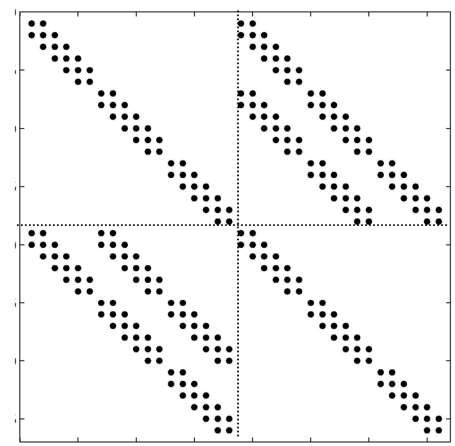
Natural ordering of grid
Left-to-right, bottom-to-top
Plus resulting matrix structure:





Horizontal line red-black
Ordering and matrix structure:





Young theory: spectral radii (Jacobi or Gauss-Seidel) independent of ordering

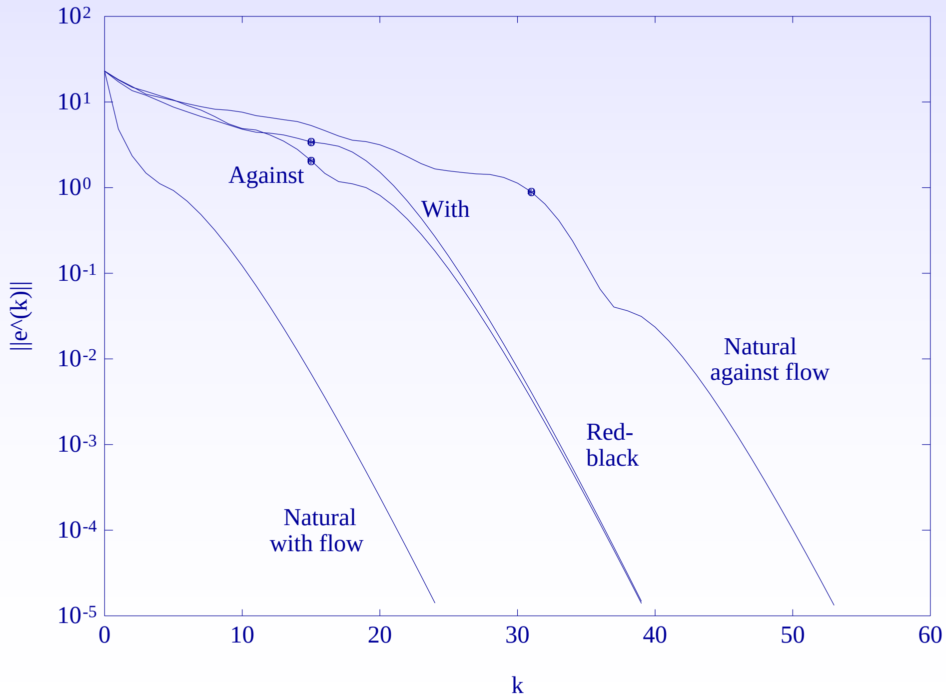
Performance of GS: depends on ordering

Example: Problem 1

$$-\varepsilon\nabla^2 u + u_y = f \quad (0,1)^2, \text{piecewise linear elements}, \quad P = 60$$

Four solution strategies: line Gauss-Seidel iteration with

- **natural line ordering, following the flow (bottom-to-top)**
- **natural line ordering, against the flow (top-to-bottom)**
- **red-black line ordering, with the flow**
- **red-black line ordering, against the flow**



Ordering effects

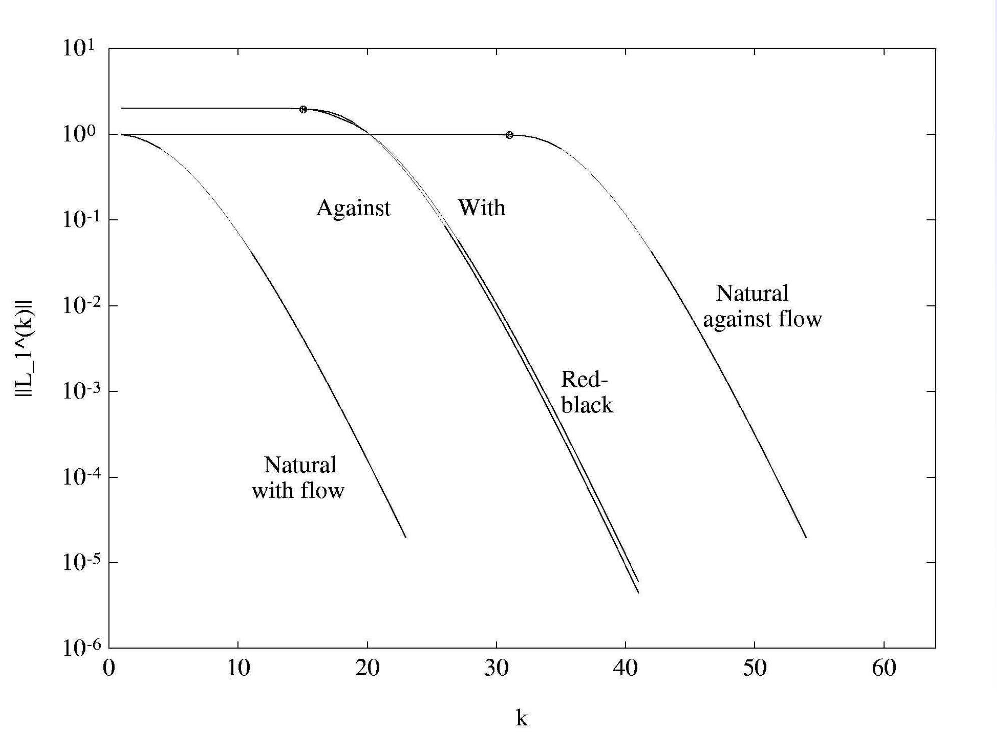
Error $e_k = u - u_k$ satisfies $e_k = (Q_F^{-1} R_F)^k e_0 \Rightarrow$

$$\Rightarrow \|e_k\| = \|(Q_F^{-1} R_F)^k e_0\| \le \|(Q_F^{-1} R_F)^k\| \|e_0\|$$

"Classical" analysis only provides insight in asymptotic sense:

$$\lim_{k\to\infty} \frac{\|(Q_F^{-1} R_F)^k\|^{1/k}}{\rho} = 1$$

E. & Chernesky:
bounds for $\|(Q_F^{-1} R_F)^k\|$
for 1D problems



Practical consequences

For *nonconstant* flows: inherent latencies if sweeps don't follow flow

Possible fixes:

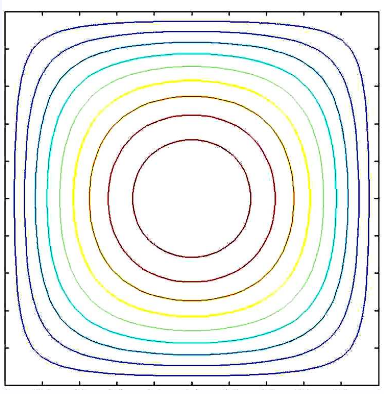
- flow-directed orderings (Bey & Wittum, Kellogg, Hackbusch, Xu)
- iterations based on multi-directional sweeps

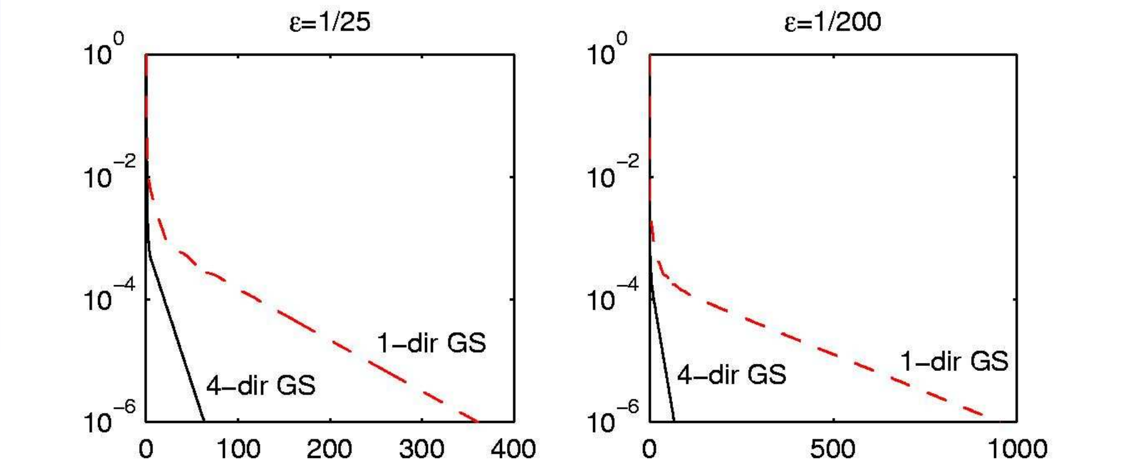
2D version:

$$u_{k+1/4} = u_k + Q_1^{-1}(f - Fu_k)$$
$$u_{k+1/2} = u_{k+1/4} + Q_2^{-1}(f - Fu_{k+1/4})$$
$$u_{k+3/4} = u_{k+1/2} + Q_3^{-1}(f - Fu_{k+1/2})$$
$$u_{k+1} = u_{k+3/4} + Q_4^{-1}(f - Fu_{k+3/4})$$

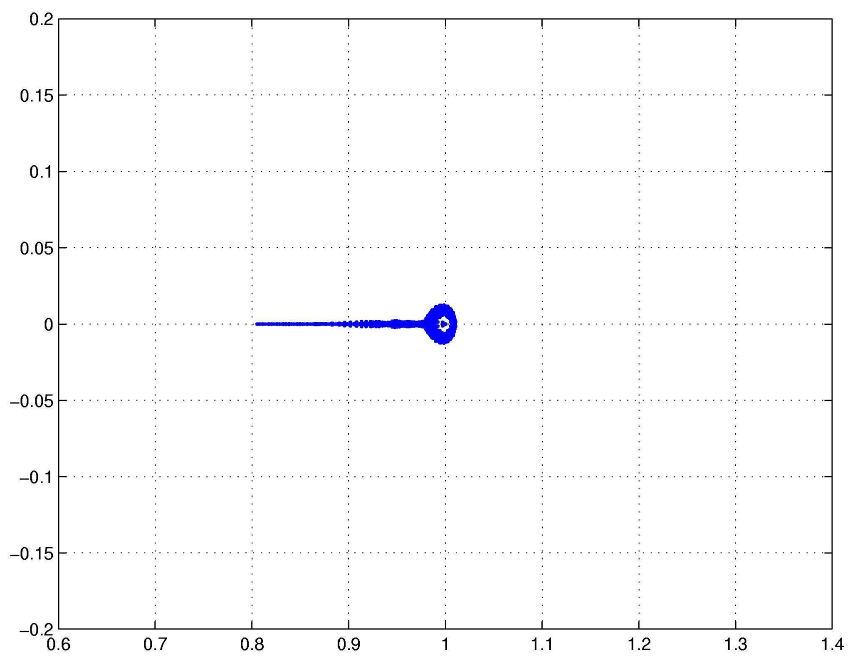
Speeds convergence when recirculations are present

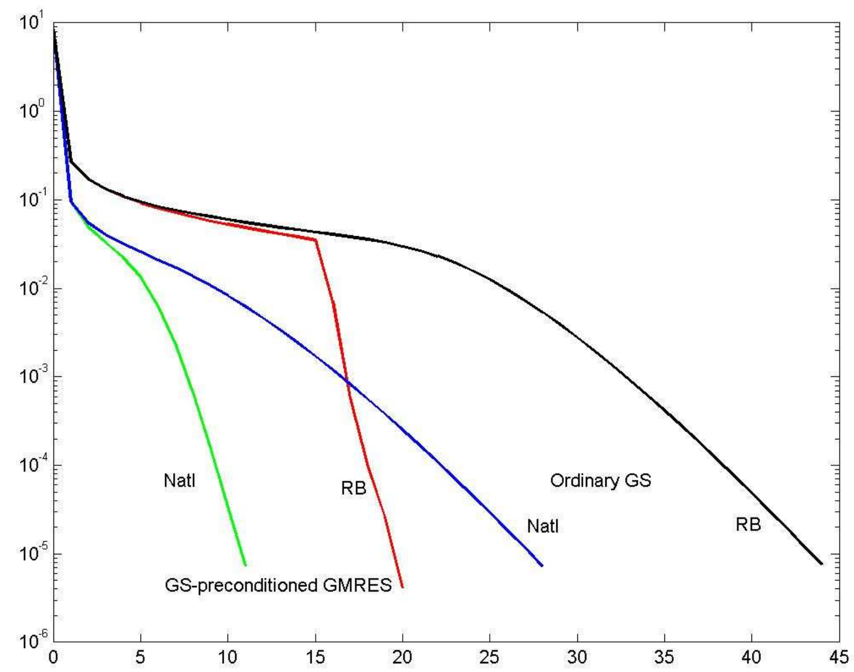
Contours of stream function





Summarizing with an experiment:





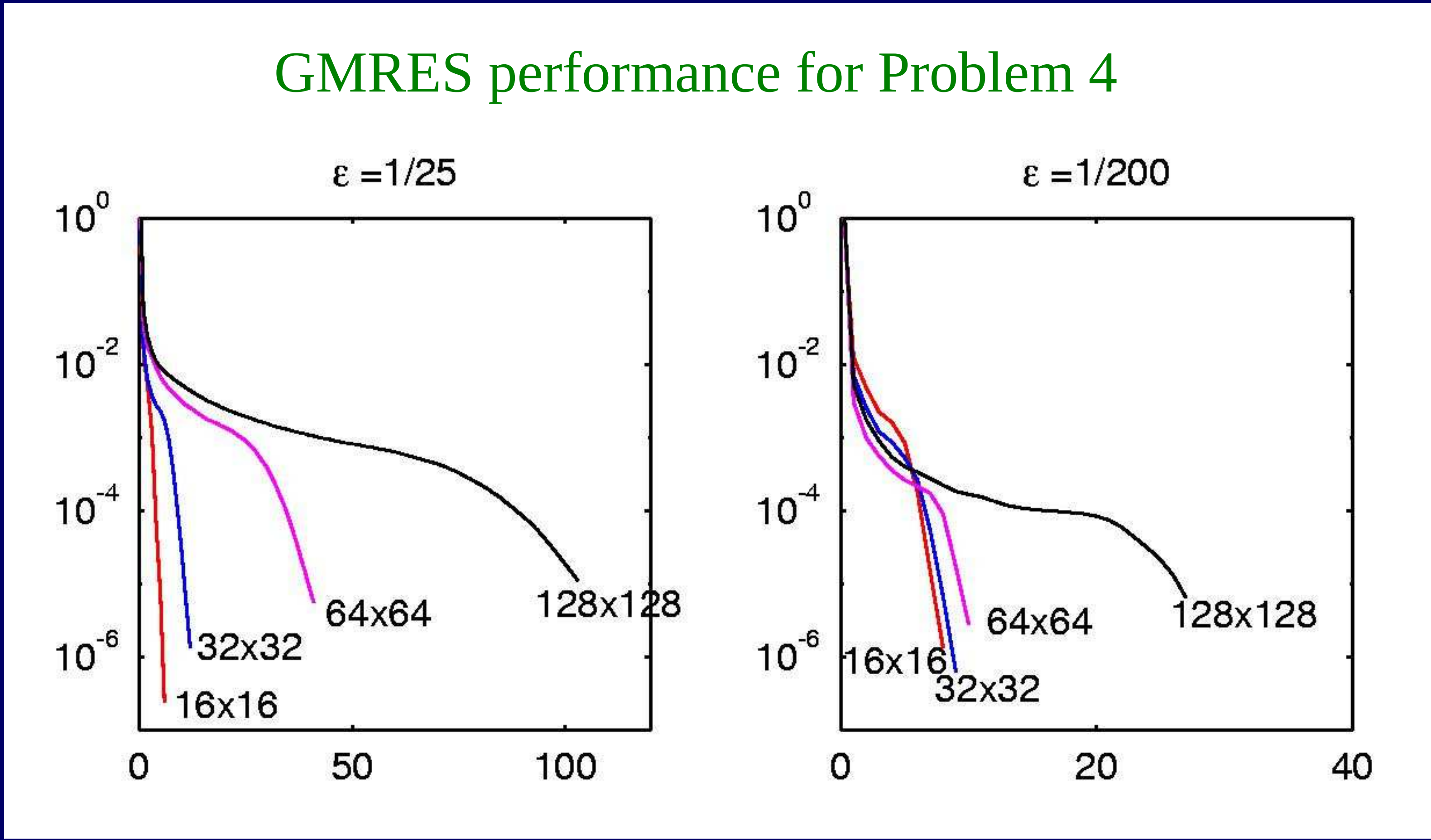
Eigenvalues of line-GS preconditioned operator, vertical flow, P=40, h=1/32

Asymptotic convergence rate is faster with Krylov acceleration

However: does not overcome latencies

Multigrid

Flow-following methods are effective for convection-dominated problems:



But: ultimately, solvers discussed above are mesh dependent

Multigrid

V-cycle multigrid:

Choose u_0

for $i = 0$ until convergence

for k steps, $u_i \leftarrow (I - Q_F^{-1} F) u_i + Q_F^{-1} f$ (presmooth)

$\hat{\mathrm{r}} = P^{\mathrm{T}} (f - F u_i)$ (restrict residual)

apply multigrid system to coarse problem $F^{2h} \hat{e} = \hat{r}$

$u_i \leftarrow u_i + P\hat{e}$ (prolong correction and update)

for m steps, $u_i \leftarrow (I - Q_F^{-1} F) u_i + Q_F^{-1} f$ (postsmooth)

$u_{i+1} \leftarrow u_i$ (update for next iteration)

end

Bottom Line: Performance

Multigrid iterations for $||r_k||/||r_0||<10^{-6}$

	ε=1/25				ε=1/200			
	Example				Example			
Grid	1	2	3	4	1	2	3	4
16× 16	4	3	3	3	3	3	6	6
32× 32	3	3	4	2	3	3	4	5
64×64	3	3	4	2	3	3	3	3
128× 128	3	3	5	2	3	3	3	2

For this to happen:

Two things have to be done correctly

1. Smoothing: $u_i \leftarrow (I - Q_F^{-1} F) u_i + Q_F^{-1} f$

Smoother must take underlying flow into account

For results above for Problem 4 (recirculating wind):
smoother is *4-directional*

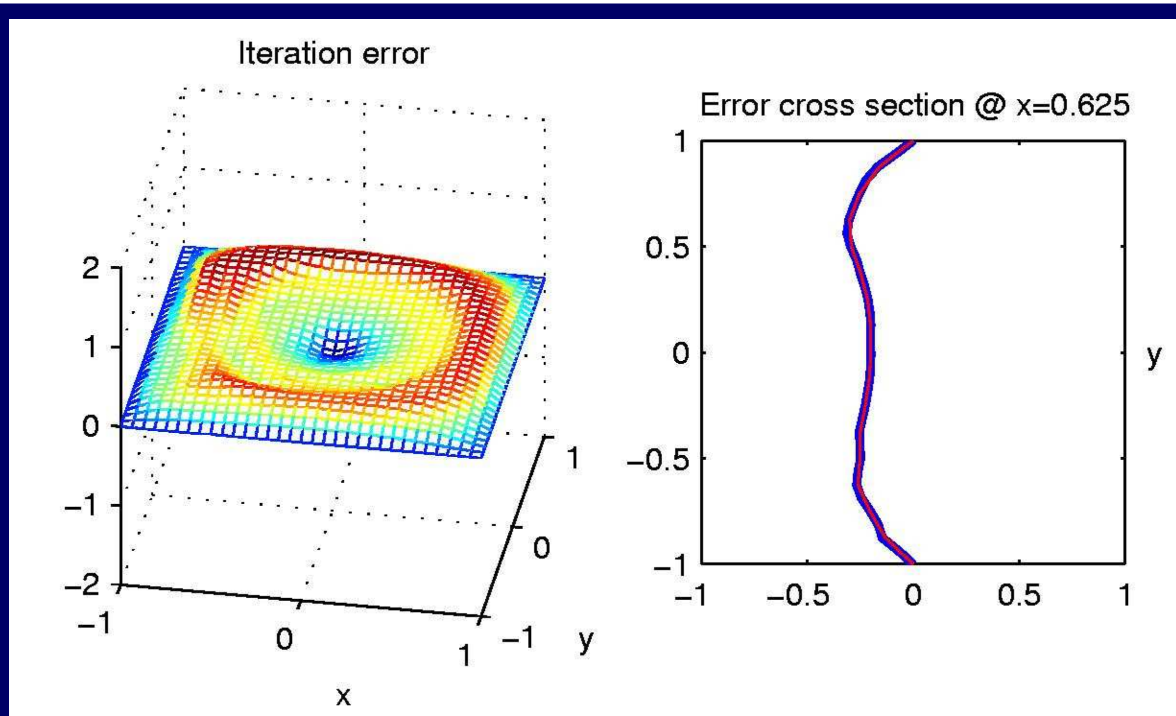
2. Coarse grid solve: $F^{2h} \hat{e} = \hat{r}$

Coarse grid operator must be stable

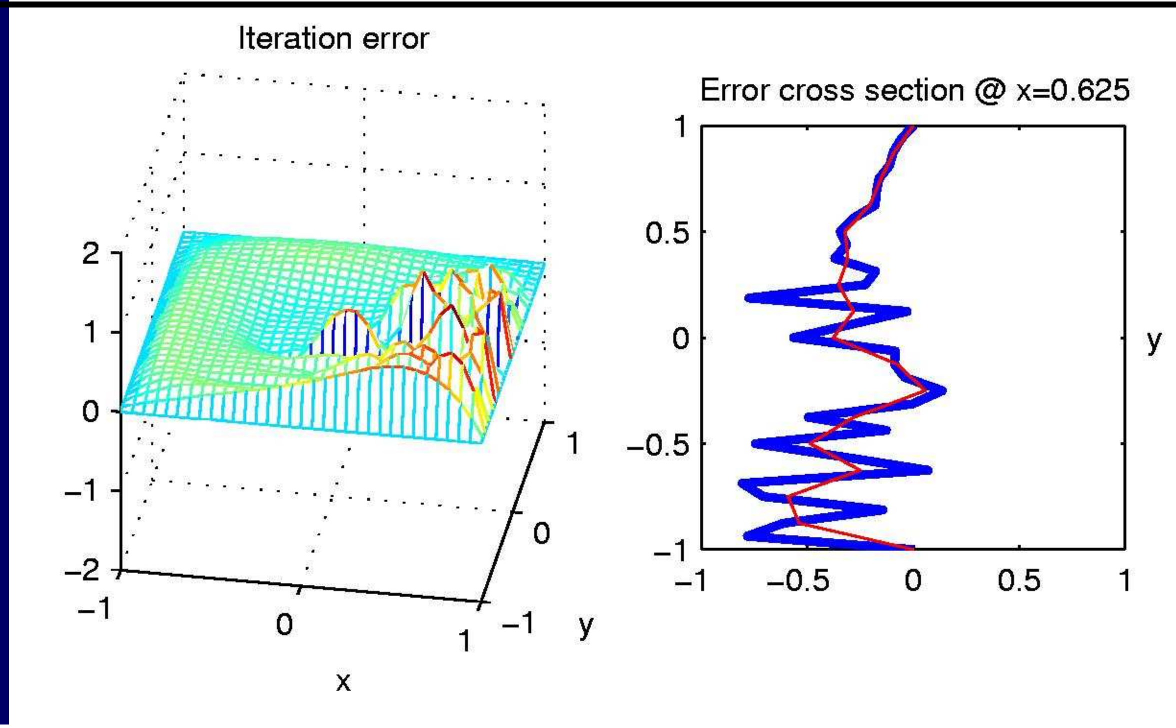
Even if fine grid is “fine enough,” coarse grid operators should include streamline diffusion

Example / effect of smoother

After one four-directional Gauss-Seidel step



After four one-directional Gauss-Seidel steps



Concluding Remarks

- Discretization requires stabilization for convection-dominated problems

- The best solution algorithms combine
  - general techniques of iterative methods
  - splitting strategies coupled to the underlying physics
  - stabilization when needed

References

- J. J. H. Miller, E. O'Riordan and G. I. Shishkin, *Fitted Numerical Methods for Singularly Perturbed Problems*, World Scientific, 1995.
- K. W. Morton, *Numerical Solution of Convection-Diffusion Problems*, Chapman & Hall, 1996.
- H.-G. Roos, M. Stynes and L. Tobiska, *Numerical Methods for Singularly Perturbed Differential Equations*, *Springer*, *1996.*

- H. C. Elman, D. J. Silvester and A. J. Wathen, *Finite Elements and Fast Iterative Solvers*, Oxford University Press, 2005.