Lecture 2, Part 1: Euler–Maruyama

- Definition of Euler–Maruyama Method
- Weak Convergence
- Strong Convergence
- Linear Stability
Recap: SDE

Given functions \( f \) and \( g \), the stochastic process \( X(t) \) is a solution of the SDE

\[
dX(t) = f(X(t)) \, dt + g(X(t)) \, dW(t)
\]

if \( X(t) \) solves the integral equation

\[
X(t) - X(0) = \int_0^t f(X(s)) \, ds + \int_0^t g(X(s)) \, dW(s)
\]

Discretize the interval \([0, T]\): let \( \Delta t = T/N \) and \( t_n = n \Delta t \)

Compute \( X_n \approx X(t_n) \)

Initial value \( X_0 \) is given
Euler–Maruyama

Exact solution:

\[ X(t_{n+1}) = X(t_n) + \int_{t_n}^{t_{n+1}} f(X(s)) \, ds + \int_{t_n}^{t_{n+1}} g(X(s)) \, dW(s) \]

Euler–Maruyama:

\[ X_{n+1} = X_n + \Delta t f(X_n) + \Delta W_n g(X_n) \]

where \( \Delta W_n = W(t_{n+1}) - W(t_n) \)

(Left endpoint Riemann sums)

In MATLAB, \( \Delta W_n \) becomes \( \text{sqrt}(Dt) \ast \text{randn} \)
\[ f(x) = \mu x \text { and } g(x) = \sigma x, \, \mu = 2, \, \sigma = 0.1, \, X(0) = 1 \]

**Solution:** \( X(t) = X(0) e^{(\mu - \frac{1}{2} \sigma^2)t} + \sigma W(t) \)

Disc. Brownian path with \( \delta t = 2^{-8} \), E-M with \( \Delta t = 4\delta t \):

\[
|X_N - X(T)| = 0.69
\]
Reducing to \( \Delta t = 2\delta t \) gives \( |X_N - X(T)| = 0.16 \)
Reducing to \( \Delta t = \delta t \) gives \( |X_N - X(T)| = 0.08 \)
Convergence?

$X_n$ and $X(t_n)$ are random variables at each $t_n$

In what sense does $|X_n - X(t_n)| \to 0$ as $\Delta t \to 0$?

There are many, non-equivalent, definitions of convergence for sequences of random variables.

The two most common and useful concepts in numerical SDEs are

- **Weak convergence:** error of the mean
- **Strong convergence:** mean of the error
Weak Convergence

Weak convergence: capture the average behaviour

Given a function $\Phi$, the **weak error** is

$$e_{\Delta t}^{\text{weak}} := \sup_{0 \leq t_n \leq T} |\mathbb{E}[\Phi(X_{t_n})] - \mathbb{E}[\Phi(X(t_n))]|$$

$\Phi$ from e.g. set of polynomials of degree at most $k$

**Converges weakly** if $e_{\Delta t}^{\text{weak}} \to 0$, as $\Delta t \to 0$

**Weak order** $p$ if $e_{\Delta t}^{\text{weak}} \leq K \Delta t^p$, for all $0 < \Delta t \leq \Delta t^*$

In practice we estimate $\mathbb{E}[\Phi(X_{t_n})]$ by Monte Carlo simulation over many paths $\Rightarrow "1/\sqrt{M}"$ sampling error
\( f(x) = \mu x \) and \( g(x) = \sigma x, \mu = 2, \sigma = 0.1, X(0) = 1 \)

**Solution** has \( \mathbb{E}[X(t)] = e^{\mu t} \)

Measure weak endpoint error \( |a_M - e^{\mu T}| \) over \( M = 10^5 \)

discretized Brownian paths. Try \( \Delta t = 2^{-5}, 2^{-6}, 2^{-7}, 2^{-8}, 2^{-9} \)

Least squares fit: power is 1.011

(Confidence intervals smaller than graphics symbols)

Suggests weak order \( p = 1 \)
Weak Euler–Maruyama

\[ X_{n+1} = X_n + \Delta t f(X_n) + \Delta W_n g(X_n) \]

where \( P(\Delta W_n = \sqrt{\Delta t}) = \frac{1}{2} = P(\Delta W_n = -\sqrt{\Delta t}) \)

E.g. use \( \sqrt{Dt} \cdot \text{sign(randn)} \)
or \( \sqrt{Dt} \cdot \text{sign(rand-0.5)} \)

Least squares fit: power is 1.03
Weak Euler–Maruyama

Generally, EM and weak EM have weak order $p = 1$ on appropriate SDEs for $\Phi(\cdot)$ with polynomial growth.

Can prove via **Feynman-Kac formula** that relates SDEs to PDEs.
Strong Convergence

Strong convergence: follow paths accurately

Strong error is

\[ e_{\Delta t}^{\text{strong}} := \sup_{0 \leq t_n \leq T} \mathbb{E} \left[ |X_n - X(t_n)| \right] \]

Converges strongly if \( e_{\Delta t}^{\text{strong}} \to 0 \), as \( \Delta t \to 0 \)

Strong order \( p \) if \( e_{\Delta t}^{\text{strong}} \leq K \Delta t^p \), for all \( 0 < \Delta t \leq \Delta t^* \)
\[ f(x) = \mu x \text{ and } g(x) = \sigma x, \; \mu = 2, \; \sigma = 1, \; X(0) = 1 \]

**Solution:** 
\[ X(t) = X(0)e^{(\mu - \frac{1}{2} \sigma^2)t + \sigma W(t)} \]

\( M = 5,000 \) disc. Brownian paths over \([0, 1]\) with \( \delta t = 2^{-11} \)

For each path apply EM with \( \Delta t = \delta t, 2\delta t, 4\delta t, 16\delta t, 32\delta t, 64\delta t \)

Record \( E[|X_N - X(1)|] \) for each \( \delta t \)

Least squares fit: power is 0.51
Strong Convergence

Generally EM has strong order $p = \frac{1}{2}$ on appropriate SDEs.

Can prove using Ito’s Lemma, Ito isometry and Gronwall.

Note: strong convergence $\Rightarrow$ weak convergence, but this doesn’t recover the optimal weak order.
Strong Convergence

Euler–Maruyama has

\[ \mathbb{E} [ |X_n - X(t_n)| ] \leq K \Delta t^{\frac{1}{2}} \]

Markov inequality says

\[ \mathbb{P} (|X| > a) \leq \frac{\mathbb{E}[|X|]}{a}, \quad \text{for any } a > 0 \]

Taking \( a = \Delta t^{\frac{1}{4}} \) gives

\[ \mathbb{P} \left( |X_n - X(t_n)| \geq \Delta t^{\frac{1}{4}} \right) \leq K \Delta t^{\frac{1}{4}}, \text{ i.e.} \]

\[ \mathbb{P} \left( |X_n - X(t_n)| < \Delta t^{\frac{1}{4}} \right) \geq 1 - K \Delta t^{\frac{1}{4}} \]

Along any path error is small with high prob.
Higher Strong Order

If \( g(x) \) is constant, then EM has strong order \( p = 1 \)

More generally, strong order \( p = 1 \) is achieved by the **Milstein** method

\[
X_{n+1} = X_n + \Delta t f(X_n) + \Delta W_n g(X_n) \\
+ \frac{1}{2} g(X_n) g'(X_n) (\Delta W_n^2 - \Delta t)
\]

(More complicated for SDE systems.)
Even Higher Strong Order: Warning!

Claims to derive arbitrarily high (strong?) order methods, with a Runge–Kutta approach. But using only Brownian increments, $\Delta W_n$, rather than more general integrals like

$$\int_{t_n}^{t_{n+1}} \int_{t_n}^{t_{n+1}} dW_1(s) dW_2(t)$$

there is an order barrier of $p = 1$ (Rümelin, 1982).
Beyond Convergence . . .

Numerical methods approximate the continuous by the discrete:

\[ X_n \approx X(t_n), \text{ with } t_{n+1} - t_n =: \Delta t \]

**Convergence:**
How small is \( X_n - X(t_n) \) at some finite \( t_n \)?

**Stability (Dynamics):**
Does \( \lim_{n \to \infty} X_n \) look like \( \lim_{t \to \infty} X(t) \)?

Study stability by applying the method to a **class of test problems**, where information about \( X(t) \) is known.

Hope to show good behavior either for all \( \Delta t > 0 \), or at least for sufficiently small \( \Delta t \).
Stochastic Theta Method

\[ X_{n+1} = X_n + (1 - \theta)\Delta t f(X_n) + \theta \Delta t f(X_{n+1}) + g(X_n) \Delta W_n \]

where we recall that \( \Delta W_n = W(t_{n+1}) - W(t_n) \), so \( \Delta W_n = \sqrt{\Delta t} V_n \), with \( V_n \sim \text{Normal}(0, 1) \) i.i.d.

\( X_n \approx X(t_n) \) in the SDE (Itô)

\[ dX(t) = f(X(t))dt + g(X(t))dW(t), \quad X(0) = X_0 \]
Stochastic Test Equation

\[ dX(t) = \mu X(t)dt + \sigma X(t)dW(t) \]

(Asset model in math-finance)

Mean-square stability

\[ \lim_{t \to \infty} \mathbb{E}(X(t)^2) = 0 \iff 2\mu + \sigma^2 < 0 \]

STM gives \( X_{n+1} = (a + bV_n)X_n \), with

\[
\begin{align*}
a &:= \frac{1 + (1 - \theta)\mu \Delta t}{1 - \theta \mu \Delta t}, \\
b &:= \frac{\sigma \sqrt{\Delta t}}{1 - \theta \mu \Delta t}
\end{align*}
\]
Mean-square stability

Saito & Mitsui, SIAM J Num Anal 1996

\[ 0 \leq \theta < \frac{1}{2}: \text{SDE stable } \Rightarrow \text{method stable} \iff \Delta t < \frac{|2\mu + \sigma^2|}{\mu^2(1 - 2\theta)} \]

\[ \theta = \frac{1}{2}: \text{SDE stable } \Leftrightarrow \text{method stable} \ \forall \Delta t > 0 \]

\[ \frac{1}{2} < \theta \leq 1: \text{SDE stable } \Rightarrow \text{method stable} \ \forall \Delta t > 0 \]
Stability Regions

Let \( x := \Delta t \mu \) and \( y := \Delta t \sigma^2 \)

SDE stable \( \iff y < -2x \)

Method stable \( \iff y < (2\theta - 1)x^2 - 2x \)
Stochastic Test Equation

\[ dX(t) = \mu X(t) dt + \sigma X(t) dW(t) \]

**Asymptotic stability**

\[ \lim_{t \to \infty} |X(t)| = 0, \text{ with prob. } 1 \iff 2\mu - \sigma^2 < 0 \]

Recall that STM gives \( X_{n+1} = (a + bV_n)X_n \), with

\[ a := \frac{1 + (1 - \theta)\mu \Delta t}{1 - \theta \mu \Delta t}, \quad b := \frac{\sigma \sqrt{\Delta t}}{1 - \theta \mu \Delta t} \]
Asymptotic Stability: \( \lim_{n \to \infty} |X_n| = 0 \), w.p. 1

\[
|X_n| = \left( \prod_{i=0}^{n-1} |a + bV_i| \right) |X_0|
\]

SLLN: \( \lim_{n \to \infty} |X_n| = 0 \iff E(\log |a + bV_i|) < 0 \)

Can be expressed in terms of **Meijer’s G-function**

Difficult to deal with analytically

No simple expression for stability region boundary
Asymptotic Stability for Backward Euler ($\theta = 1$)
Many open questions regarding asymptotic stability

E.g. is there an A-stable method?

Generalizations to nonlinear SDEs are also possible