Numerical Simulation of Stochastic Differential Equations: Lecture 1, Part 2

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Lecture 1, part 2: SDEs

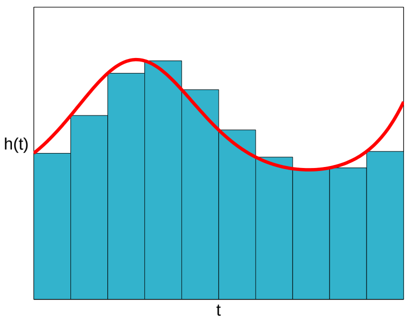
- Ito stochastic integrals
- Ito SDEs
- Examples of SDEs

Integration

For deterministic $h : \mathbb{R} \to \mathbb{R}$,
$t_i = i\delta t$, $\delta t = T/L$,
Riemann sum based on left endpoints

$$\sum_{i=0}^{L-1} h(t_i)(t_{i+1} - t_i)$$

Integral $\int_0^T h(t)\, dt$ defined by $\delta t \to 0$



Stochastic Integration

Integrate with respect to Brownian motion:
Riemann sum based on left endpoints

$$\sum_{i=0}^{L-1} h(t_i)(\mathbf{W}(t_{i+1}) - \mathbf{W}(t_i))$$

Integral $\int_0^T h(t)\, d\mathbf{W}(t)$ defined by $\delta t \to 0$

Digression: $\sum_{i=0}^{L-1} \delta \mathbf{W}_i^2$

$$\mathbb{E}\left[\sum_{i=0}^{L-1} \delta \mathbf{W}_i^2\right] = \sum_{i=0}^{L-1} \mathbb{E}\left[\delta \mathbf{W}_i^2\right] = L\delta t = T$$

$$\begin{aligned}
\mathbb{E}\left[\left(\sum_{i=0}^{L-1} \delta \mathbf{W}_i^2\right)^2\right] &= \sum_{i=0}^{L-1}\sum_{j=0}^{L-1} \mathbb{E}\left[\delta \mathbf{W}_i^2 \delta \mathbf{W}_j^2\right] \\
&= \sum_{i=0, i\neq j}^{L-1}\sum_{j=0}^{L-1} \mathbb{E}\left[\delta \mathbf{W}_i^2 \delta \mathbf{W}_j^2\right] + \sum_{i=0}^{L-1} \mathbb{E}\left[\delta \mathbf{W}_i^4\right] \\
&= \sum_{i=0, i\neq j}^{L-1}\sum_{j=0}^{L-1} \mathbb{E}\left[\delta \mathbf{W}_i^2\right] \mathbb{E}\left[\delta \mathbf{W}_j^2\right] + \sum_{i=0}^{L-1} \mathbb{E}\left[\delta \mathbf{W}_i^4\right] \\
&= L(L-1)\delta t^2 + 3L\delta t^2 \\
&= T^2 + O(\delta t)
\end{aligned}$$

Digression continued: $\sum_{i=0}^{L-1} \delta \mathbf{W}_i^2$

Hence,

$$\begin{aligned}
\text{var}\left[\sum_{i=0}^{L-1} \delta \mathbf{W}_i^2\right] &:= \mathbb{E}\left[\left(\sum_{i=0}^{L-1} \delta \mathbf{W}_i^2\right)^2\right] - \left(\mathbb{E}\left[\sum_{i=0}^{L-1} \delta \mathbf{W}_i^2\right]\right)^2 \\
&= T^2 + O(\delta t) - T^2 \\
&= O(\delta t)
\end{aligned}$$

The sum $\sum_{i=0}^{L-1} \delta \mathbf{W}_i^2$ has mean T and variance $O(\delta t)$.

Hence, as $\delta t \to 0$ it looks like the constant T.

Reminder: Stochastic Integration

Integrate with respect to Brownian motion:
Riemann sum based on left endpoints

$$\sum_{i=0}^{L-1} h(t_i)(\mathbf{W}(t_{i+1}) - \mathbf{W}(t_i))$$

Integral $\int_0^T h(t)\, d\mathbf{W}(t)$ defined by $\delta t \to 0$

Example: $h(t) = \mathbf{W}(t)$

$$\begin{aligned}
\sum_{i=0}^{L-1} \mathbf{W}(t_i)\left(\mathbf{W}(t_{i+1}) - \mathbf{W}(t_i)\right) &= \tfrac{1}{2}\sum_{i=0}^{L-1} \left(\mathbf{W}(t_{i+1})^2 - \mathbf{W}(t_i)^2\right. \\
&\quad \left. - \left(\mathbf{W}(t_{i+1}) - \mathbf{W}(t_i)\right)^2\right) \\
&= \tfrac{1}{2}\sum_{i=0}^{L-1} \left(\mathbf{W}(t_{i+1})^2 - \mathbf{W}(t_i)^2\right) \\
&\quad - \tfrac{1}{2}\sum_{i=0}^{L-1} \delta \mathbf{W}_i^2 \\
&\to \tfrac{1}{2}\mathbf{W}(T)^2 - \tfrac{1}{2}T
\end{aligned}$$

So

$$\int_0^T \mathbf{W}(t)\, d\mathbf{W}(t) = \tfrac{1}{2}\mathbf{W}(T)^2 - \tfrac{1}{2}T$$

Warning!

Similar analysis for the midpoint Riemann sum:

$$\sum_{i=0}^{L-1} \mathbf{W}(\tfrac{1}{2}(t_i + t_{i+1}))\,(\mathbf{W}(t_{i+1}) - \mathbf{W}(t_i)) \to \tfrac{1}{2}\mathbf{W}(T)^2$$

We will always use the left endpoint definition: Ito

Properties of the Ito Integral

We assume integrand is **non-anticipative**:
$h(t)$ independent of $\{W(s)\}_{s>t}$
Now

$$\mathbb{E}\left[\sum_{i=0}^{L-1} h(t_i)\delta\mathbf{W}_i\right] = \sum_{i=0}^{L-1} \mathbb{E}\left[h(t_i)\delta\mathbf{W}_i\right]$$

$$= \sum_{i=0}^{L-1} \mathbb{E}[h(t_i)]\mathbb{E}[\delta\mathbf{W}_i]$$

$$= 0$$

$\Rightarrow \mathbb{E}\left[\int_0^T h(t)\, d\mathbf{W}(t)\right] = 0$ martingale property

Properties of the Ito Integral

$$\mathbb{E}\left[\left(\sum_{i=0}^{L-1} h(t_i)\delta\mathbf{W}_i\right)^2\right] = 2\sum_{i<j}^{L-1} \mathbb{E}\left[h(t_i)h(t_j)\delta\mathbf{W}_i\delta\mathbf{W}_j\right]$$

$$+ \sum_{i=0}^{L-1} \mathbb{E}\left[h(t_i)^2\delta\mathbf{W}_i^2\right]$$

$$= 2A + B$$

$A = \sum_{i<j}^{L-1} \mathbb{E}\left[h(t_i)h(t_j)\delta\mathbf{W}_i\right]\mathbb{E}\left[\delta\mathbf{W}_j\right] = 0,$
$B = \sum_{i=0}^{L-1} \mathbb{E}\left[h(t_i)^2\right]\mathbb{E}\left[\delta\mathbf{W}_i^2\right] = \delta t \sum_{i=0}^{L-1} \mathbb{E}\left[h(t_i)^2\right]$

$\Rightarrow \mathbb{E}\left[\left(\int_0^T h(t)\, d\mathbf{W}(t)\right)^2\right] = \int_0^T \mathbb{E}\left[h(t)^2\right]\, dt$ Ito isometry

ODE

Given $f : \mathbb{R} \to \mathbb{R}$:

$$\frac{dx(t)}{dt} = f(x(t))$$

Typically, $x(0)$ given, solution required over $[0, T]$
Fundamental Theorem of Calculus $\Rightarrow$

$$x(t) - x(0) = \int_0^t f(x(s))\, ds$$

Can extend this to define an SDE . . .

SDE

Given functions f and g, the stochastic process $\mathbf{X}(t)$ is a soluton of the SDE

$$d\mathbf{X}(t) = f(\mathbf{X}(t))dt + g(\mathbf{X}(t))d\mathbf{W}(t)$$

if $\mathbf{X}(t)$ solves the integral equation

$$\mathbf{X}(t) - \mathbf{X}(0) = \int_0^t f(\mathbf{X}(s))\, ds + \int_0^t g(\mathbf{X}(s))\, d\mathbf{W}(s)$$

Note 1: $d\mathbf{X}(t)$ and $d\mathbf{W}(t)$ are just **shorthand**
Note 2: $d\mathbf{W}(t)$ not diff'ble, so we cannot write $d\mathbf{W}(t)/dt$

Repeat this:

If $\mathbf{X}(t)$ satisfies

$$\mathbf{X}(t) - \mathbf{X}(0) = \int_0^t f(\mathbf{X}(s))\, ds + \int_0^t g(\mathbf{X}(s))\, d\mathbf{W}(s)$$

then we say that $\mathbf{X}(t)$ solves the SDE

$$d\mathbf{X}(t) = f(\mathbf{X}(t))dt + g(\mathbf{X}(t))d\mathbf{W}(t)$$

Typically, $\mathbf{X}(0)$ given, solution required over $[0, T]$

We say $f(\cdot)$ is the **drift** and $g(\cdot)$ is the **diffusion**

Note: $\mathbf{X}(t)$ is a **random variable** at each time t

SDE Example: $f(x) = \mu x$ and $g(x) = \sigma x$

$$d\mathbf{X}(t) = \mu\mathbf{X}(t)dt + \sigma\mathbf{X}(t)d\mathbf{W}(t)$$

Here μ and σ are real constants
Used to model **asset prices** in finance
Arises in Black–Scholes theory for option valuation
Solution:

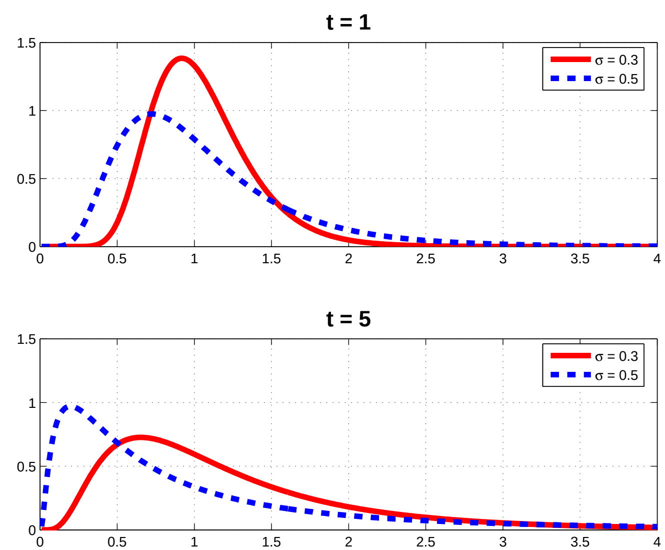
$$\mathbf{X}(t) = \mathbf{X}(0)e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma\mathbf{W}(t)}$$

Satisfies

$$\mathbb{E}\left[\mathbf{X}(t)\right] = \mathbb{E}\left[\mathbf{X}(0)\right] e^{\mu t}$$

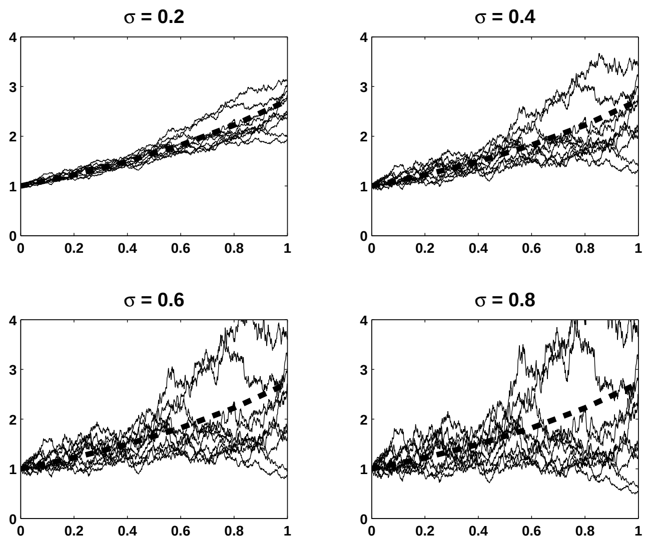
$$\text{var}\left[\mathbf{X}(t)^2\right] = \mathbb{E}\left[\mathbf{X}(0)^2\right] e^{(2\mu+\sigma^2)t}$$

$f(x) = \mu x$ and $g(x) = \sigma x$: Density



$f(x) = \mu x$ and $g(x) = \sigma x$: paths

10 discretized Brownian paths, $\mu = 1$



SDE Example: Interest Rates

Mean-reverting square root process

$$d\mathbf{X}(t) = \lambda\,(\mu - \mathbf{X}(t))\,dt + \sigma\sqrt{\mathbf{X}(t)}d\mathbf{W}(t)$$

Also named after **Cox, Ingersoll and Ross**, 1985

We assume that $\mathbf{X}(0) > 0$

If $\sigma^2 > 2\lambda\mu$ then $\mathbf{X}(t)$ can attain the value 0

SDE Example: Stochastic Volatility

Heston (1993)

$$d\mathbf{S}(t) = \lambda_1\,(\mu_1 - \mathbf{S}(t))\,dt + \sigma_1\mathbf{S}(t)\sqrt{\mathbf{X}(t)}d\mathbf{W}_1(t)$$
$$d\mathbf{X}(t) = \lambda_2\,(\mu_2 - \mathbf{X}(t))\,dt + \sigma_2\sqrt{\mathbf{X}(t)}d\mathbf{W}_2(t)$$

- $\mathbf{S}(t)$ is asset price
- $\mathbf{X}(t)$ is squared volatility

SDE Example: Population Dynamics

$$d\mathbf{X}(t) = r\,(K - \mathbf{X}(t))\,dt + \beta\mathbf{X}(t)d\mathbf{W}(t)$$

- $\mathbf{X}(t)$ is population size
- $1/r$ is a characteristic timescale
- K is carrying capacity
- β is environmental noise strength

Solution:

$$\mathbf{X}(t) = \frac{\mathbf{X}(0)\,e^{(rK-\frac{1}{2}\beta^2)t+\beta\mathbf{W}(t)}}{1+\mathbf{X}(0)\,r\int_0^t e^{(rK-\frac{1}{2}\beta^2)s+\beta\mathbf{W}(s)}\,ds}$$

SDE Example: Political Opinions, Cobb (1981)

$$d\mathbf{X}(t) = r\,(G - \mathbf{X}(t))\,dt + \sqrt{\epsilon \mathbf{X}(t)(1-\mathbf{X}(t))}\,d\mathbf{W}(t)$$

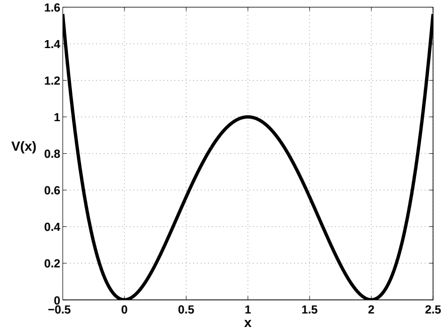
$\mathbf{X}(t)$ is political opinion of an individual at time t

$\mathbf{X}(t) = 0$ means ultra-liberal
$\mathbf{X}(t) = 1$ means ultra-conservative

- G is the long term average ($\mathbb{E}[\mathbf{X}(t)] \to G$)
- r is the rate at which $\mathbb{E}[\mathbf{X}(t)]$ approaches G
- ϵ is a noise strength

Idea: Extreme views $\Rightarrow$ less likely to change mind

Double-Well Potential: $V(x) = x^2(x-2)^2$



ODE version: $\frac{dx(t)}{dt} = -V'(x(t))$ satisfies

$$\frac{d}{dt}V(x(t)) = V'(x(t))\frac{dx(t)}{dt} = -\left(V'(x(t))\right)^2$$

SDE version: $d\mathbf{X}(t) = -V'(\mathbf{X}(t))dt + \sigma d\mathbf{W}(t)$

stint.m

```
%STINT   Approximate   stochastic   integral
%
% Ito  integral   of  W  dW

randn('state',100)                        % set  the  state  of  randn
T  =  1;  N  =  500;  dt  =  T/N;

dW  =  sqrt(dt)*randn(1,N);               % increments
W  =  cumsum(dW);                         % cumulative   sum

ito  =  sum([0,W(1:end-1)].*dW)

itoerr  =  abs(ito  -  0.5*(W(end)^2-T))
```