# Numerical Simulation of Stochastic Differential Equations: Lecture 2, Part 1

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## Lecture 2, Part 1: Euler–Maruyama

- Definition of Euler–Maruyama Method
- Weak Convergence
- Strong Convergence
- Linear Stability



## Recap: SDE

Given functions $f$ and $g$, the stochastic process $\mathbf{X}(t)$ is a soluton of the SDE

$$d\mathbf{X}(t) = f(\mathbf{X}(t))dt + g(\mathbf{X}(t))d\mathbf{W}(t)$$

if $\mathbf{X}(t)$ solves the integral equation

$$\mathbf{X}(t) - \mathbf{X}(0) = \int_0^t f(\mathbf{X}(s))\,ds + \int_0^t g(\mathbf{X}(s))\,d\mathbf{W}(s)$$

Discretize the interval $[0, T]$: let $\Delta t = T/N$ and $t_n = n\Delta t$
Compute $\mathbf{X}_n \approx \mathbf{X}(t_n)$
Initial value $\mathbf{X}_0$ is given



## Euler–Maruyama

**Exact solution**:

$$\mathbf{X}(t_{n+1}) = \mathbf{X}(t_n) + \int_{t_n}^{t_{n+1}} f(\mathbf{X}(s))\,ds + \int_{t_n}^{t_{n+1}} g(\mathbf{X}(s))\,d\mathbf{W}(s$$

**Euler–Maruyama**:

$$\mathbf{X}_{n+1} = \mathbf{X}_n + \Delta t f(\mathbf{X}_n) + \Delta\mathbf{W}_n\, g(\mathbf{X}_n)$$

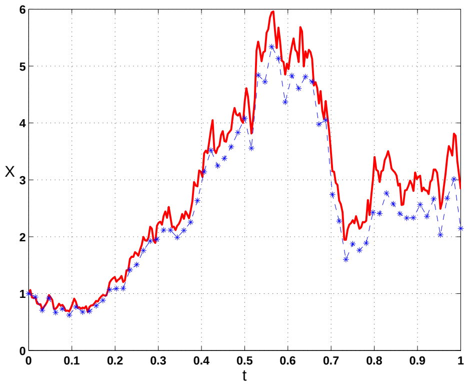
where $\Delta\mathbf{W}_n = \mathbf{W}(t_{n+1}) - \mathbf{W}(t_n)$
(Left endpoint Riemann sums)

In MATLAB, $\Delta\mathbf{W}_n$ becomes `sqrt(Dt)*randn`

## $x) = \mu x$ and $g(x) = \sigma x$, $\mu = 2$, $\sigma = 0.1$, $X(0) = 1$

**Solution**: $\mathbf{X}(t) = \mathbf{X}(0)e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma \mathbf{W}(t)}$

Disc. Brownian path with $\delta t = 2^{-8}$, E-M with $\Delta t = 4\delta t$:




$|\mathbf{X}_N - \mathbf{X}(T)| = 0.69$

Reducing to $\Delta t = 2\delta t$ gives $|\mathbf{X}_N - \mathbf{X}(T)| = 0.16$

Reducing to $\Delta t = \delta t$ gives $|\mathbf{X}_N - \mathbf{X}(T)| = 0.08$



# Convergence?

$\mathbf{X}_n$ and $\mathbf{X}(t_n)$ are **random variables** at each $t_n$

In what sense does $|\mathbf{X}_n - \mathbf{X}(t_n)| \to 0$ as $\Delta t \to 0$?

There are many, non-equivalent, definitions of convergenc for sequences of random variables

The two most common and useful concepts in numerical SDEs are

- **Weak convergence:** error of the mean
- **Strong convergence:** mean of the error



# Weak Convergence

**Weak convergence:** capture the average behaviour

Given a function $\Phi$, the **weak error** is

$$e_{\Delta t}^{\text{weak}} := \sup_{0 \le t_n \le T} \left| \mathbb{E}\left[\Phi(\mathbf{X}_n)\right] - \mathbb{E}\left[\Phi(\mathbf{X}(t_n))\right] \right|$$

$\Phi$ from e.g. set of polynomials of degree at most $k$

**Converges weakly** if $e_{\Delta t}^{\text{weak}} \to 0, \quad$ as $\Delta t \to 0$

**Weak order** $p$ if $e_{\Delta t}^{\text{weak}} \le K \Delta t^p, \quad$ for all $0 < \Delta t \le \Delta t^\star$

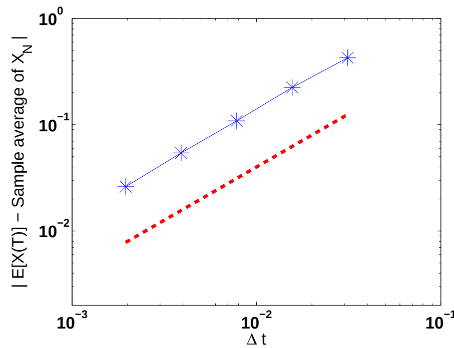
In practice we estimate $\mathbb{E}[\Phi(\mathbf{X}_n)]$ by Monte Carlo simulation over many paths $\Rightarrow$ "$1/\sqrt{M}$" sampling error



## $f(x) = \mu x$ and $g(x) = \sigma x$, $\mu = 2$, $\sigma = 0.1$, $X(0)$

**Solution** has $\mathbb{E}[X(t)] = e^{\mu t}$

Measure weak endpoint error $|a_M - e^{\mu T}|$ over $M = 10^5$ discretized Brownian paths. Try $\Delta t = 2^{-5}, 2^{-6}, 2^{-7}, 2^{-8}, 2^{-9}$




Least squares fit: power is $1.011$

(Confidence intervals smaller than graphics symbols)

Suggests **weak order $p = 1$**

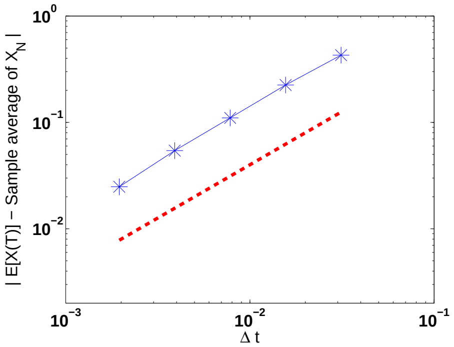
# Weak Euler–Maruyama

$$\mathbf{X}_{n+1} = \mathbf{X}_n + \Delta t\, f(\mathbf{X}_n) + \widehat{\Delta \mathbf{W}}_n\, g(\mathbf{X}_n)$$

where $\mathbb{P}\left(\widehat{\Delta \mathbf{W}}_n = \sqrt{\Delta t}\right) = \frac{1}{2} = \mathbb{P}\left(\widehat{\Delta \mathbf{W}}_n = -\sqrt{\Delta t}\right)$

E.g. use `sqrt(Dt)*sign(randn)`

or `sqrt(Dt)*sign(rand-0.5)`




Least squares fit: power is $1.03$



# Weak Euler–Maruyama

Generally, EM and weak EM have weak order $p = 1$ on appropriate SDEs for $\Phi(\cdot)$ with polynomial growth

Can prove via **Feynman-Kac formula** that relates SDEs t
PDEs



# Strong Convergence

**Strong convergence:** follow paths accurately

**Strong error** is

$$e^{\text{strong}}_{\Delta t} := \sup_{0 \le t_n \le T} \mathbb{E}\left[|\mathbf{X}_n - \mathbf{X}(t_n)|\right]$$

**Converges strongly** if $e^{\text{strong}}_{\Delta t} \to 0$, as $\Delta t \to 0$

**Strong order** $p$ if $e^{\text{strong}}_{\Delta t} \le K \Delta t^p$, for all $0 < \Delta t \le \Delta t^\star$



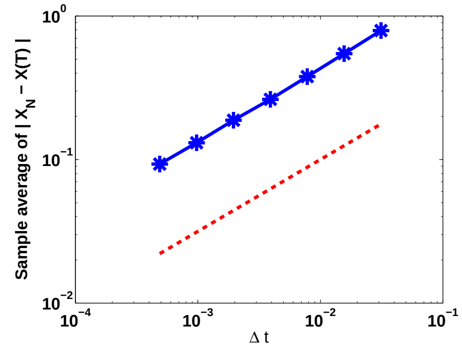
# $f(x) = \mu x$ and $g(x) = \sigma x$, $\mu = 2$, $\sigma = 1$, $X(0) =$

**Solution**: $\mathbf{X}(t) = \mathbf{X}(0)e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma \mathbf{W}(t)}$

$M = 5{,}000$ disc. Brownian paths over $[0, 1]$ with $\delta t = 2^{-11}$

For each path apply EM with $\Delta t = \delta t, 2\delta t, 4\delta t, 16\delta t, 32\delta t, 64\delta$

Record $\mathbb{E}\left[|\mathbf{X}_N - \mathbf{X}(1)|\right]$ for each $\delta t$




Least squares fit: power is $0.51$

## Strong Convergence

Generally EM has strong order $p = \frac{1}{2}$ on appropriate SDEs

Can prove using Ito's Lemma, Ito isometry and Gronwall

Note: **strong convergence** $\Rightarrow$ **weak convergence**,
but this doesn't recover the optimal weak order



## Strong Convergence

Euler–Maruyama has

$$\mathbb{E}\left[|\mathbf{X}_n - \mathbf{X}(t_n)|\right] \le K\Delta t^{\frac{1}{2}}$$

**Markov inequality** says

$$\mathbb{P}\left(|\mathbf{X}| > a\right) \le \frac{\mathbb{E}[|\mathbf{X}|]}{a}, \qquad \text{for any } a > 0$$

Taking $a = \Delta t^{\frac{1}{4}}$ gives $\mathbb{P}\left(|\mathbf{X}_n - \mathbf{X}(t_n)| \ge \Delta t^{\frac{1}{4}}\right) \le K\Delta t^{\frac{1}{4}}$, i.e.

$$\mathbb{P}\left(|\mathbf{X}_n - \mathbf{X}(t_n)| < \Delta t^{\frac{1}{4}}\right) \ge 1 - K\Delta t^{\frac{1}{4}}$$

Along any path error is small with high prob.



## Higher Strong Order

If $g(x)$ is constant, then EM has strong order $p = 1$

More generally, strong order $p = 1$ is achieved by the **Milstein** method

$$\begin{aligned}\mathbf{X}_{n+1} = \mathbf{X}_n &+ \Delta t\, f(\mathbf{X}_n) + \Delta\mathbf{W}_n\, g(\mathbf{X}_n)\\ &+ \tfrac{1}{2} g(\mathbf{X}_n) g'(\mathbf{X}_n)\left(\Delta\mathbf{W}_n^2 - \Delta t\right)\end{aligned}$$

(More complicated for SDE systems.)



## Even Higher Strong Order: Warning!

**Numerical methods for stochastic differential equations**
**Joshua Wilkie**
**Physical Review E, 2004**

Claims to derive arbitrarily high (strong?) order methods, with a Runge–Kutta approach.
But using only Brownian increments, $\Delta\mathbf{W}_n$, rather than more general integrals like

$$\int_{t_n}^{t_{n+1}} \int_{t_n}^{t_{n+1}} d\mathbf{W}_1(s) d\mathbf{W}_2(t)$$

there is an order barrier of $p = 1$ (Rümelin, 1982).

# Beyond Convergence ...

Numerical methods approximate the continuous by the discrete:
$X_n \approx X(t_n)$, with $t_{n+1} - t_n =: \Delta t$

**Convergence**:
How small is $X_n - X(t_n)$ at some finite $t_n$?

**Stability (Dynamics)**:
Does $\lim_{n\to\infty} X_n$ look like $\lim_{t\to\infty} X(t)$?

Study stability by applying the method to a **class of test problems**, where information about $X(t)$ is known.
Hope to show good behavior either for all $\Delta t > 0$, or at least for sufficiently small $\Delta t$.



# Stochastic Theta Method

$$\mathbf{X}_{n+1} = \mathbf{X}_n + (1-\theta)\Delta t f(\mathbf{X}_n) + \theta \Delta t f(\mathbf{X}_{n+1}) + g(\mathbf{X}_n)\Delta\mathbf{W}_n$$

where we recall that $\Delta\mathbf{W}_n = \mathbf{W}(t_{n+1}) - \mathbf{W}(t_n)$,
so $\Delta\mathbf{W}_n = \sqrt{\Delta t}\mathbf{V}_n$, with $\mathbf{V}_n \sim$ Normal$(0,1)$ i.i.d.

$\mathbf{X}_n \approx \mathbf{X}(t_n)$ in the SDE (Itô)

$$d\mathbf{X}(t) = f(\mathbf{X}(t))dt + g(\mathbf{X}(t))d\mathbf{W}(t), \quad \mathbf{X}(0) = \mathbf{X}_0$$



# Stochastic Test Equation

$$d\mathbf{X}(t) = \mu\mathbf{X}(t)dt + \sigma\mathbf{X}(t)d\mathbf{W}(t)$$

(Asset model in math-finance)

**Mean-square stability**

$$\lim_{t\to\infty} \mathbb{E}(\mathbf{X}(t)^2) = 0 \Leftrightarrow 2\mu + \sigma^2 < 0$$

STM gives $\mathbf{X}_{n+1} = (a + b\mathbf{V}_n)\mathbf{X}_n$, with

$$a := \frac{1 + (1-\theta)\mu\Delta t}{1 - \theta\mu\Delta t}, \qquad b := \frac{\sigma\sqrt{\Delta t}}{1 - \theta\mu\Delta t}$$



# Mean-square stability

**Saito & Mitsui**, SIAM J Num Anal 1996

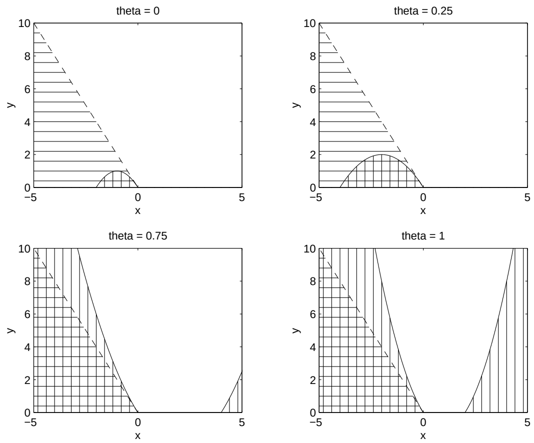
$0 \le \theta < \frac{1}{2}$: SDE **stable** $\Rightarrow$ method **stable** iff

$$\Delta t < \frac{|2\mu + \sigma^2|}{\mu^2(1-2\theta)}$$

$\theta = \frac{1}{2}$: SDE **stable** $\Leftrightarrow$ method **stable** $\forall \Delta t > 0$
$\frac{1}{2} < \theta \le 1$: SDE **stable** $\Rightarrow$ method **stable** $\forall \Delta t > 0$

# Stability Regions

Let $x := \Delta t\mu$ and $y := \Delta t\sigma^2$
SDE **stable** $\Leftrightarrow y < -2x$
Method **stable** $\Leftrightarrow y < (2\theta - 1)x^2 - 2x$






# Stochastic Test Equation

$$d\mathbf{X}(t) = \mu\mathbf{X}(t)dt + \sigma\mathbf{X}(t)d\mathbf{W}(t)$$

**Asymptotic stability**

$$\lim_{t\to\infty} |\mathbf{X}(t)| = 0, \text{ with prob. } 1 \Leftrightarrow 2\mu - \sigma^2 < 0$$

Recall that STM gives $\mathbf{X}_{n+1} = (a + b\mathbf{V}_n)\mathbf{X}_n$, with

$$a := \frac{1 + (1-\theta)\mu\Delta t}{1 - \theta\mu\Delta t}, \qquad b := \frac{\sigma\sqrt{\Delta t}}{1 - \theta\mu\Delta t}$$



# Asymptotic Stability: $\lim_{n\to\infty} |X_n| = 0, \text{w.p. } 1$

$$|\mathbf{X}_n| = \left(\prod_{i=0}^{n-1} |a + b\mathbf{V}_i|\right) |\mathbf{X}_0|$$

SLLN: $\lim_{n\to\infty} |\mathbf{X}_n| = 0 \Leftrightarrow \mathbb{E}(\log |a + b\mathbf{V}_i|) < 0$

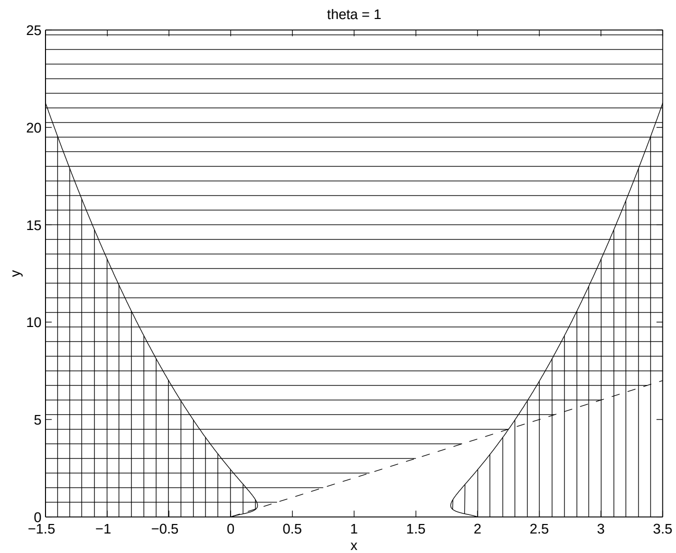
Can be expressed in terms of **Meijer's G-function**

Difficult to deal with analytically

No simple expression for stability region boundary



# Asymptotic Stability for Backward Euler ($\theta = $



Many open equestions regarding asymptotic stability

E.g. is there an A-stable method?

Generalizations to nonlinear SDEs are also possible