Numerical Simulation of Stochastic Differential Equations: Lecture 2, Part 2

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Lecture 2, Part 2: Mean Exit Times

- Statement of Problem
- Differential Equation Formulation
- SDE/Monte Carlo Algorithm
- Convergence Property

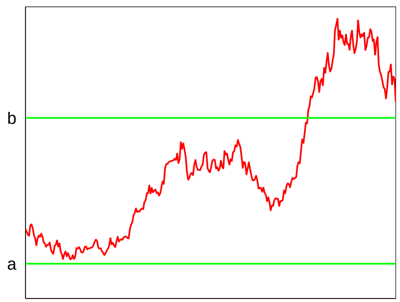
$$d\mathbf{X}(t) = f(\mathbf{X}(t))dt + g(\mathbf{X}(t))d\mathbf{W}(t), \quad \mathbf{X}(0) = \mathbf{X}_0$$

Suppose $\mathbf{X}_0$ is a constant in (a, b)

Define the random variable $\mathbf{T}_{\mathrm{exit}}$ to be

$$\mathbf{T}_{\mathrm{exit}} := \inf\{t : \mathbf{X}(t) = a \text{ or } \mathbf{X}(t) = b\}$$

In words: $\mathbf{T}_{\mathrm{exit}}$ is first time solution leaves (a, b)



Problem: find the **mean exit time** $T_{\mathrm{exit}}^{\mathrm{mean}} := \mathbb{E}\left[\mathbf{T}_{\mathrm{exit}}\right]$

Monte Carlo for Mean Exit Time

Choose a stepsize, Δt
Choose a number of paths, M
for s = 1 to M
  Set $t_n = 0$ and $X_n = X_0$
  While $X_n > a$ and $X_n < b$
    Compute a N(0,1) sample ξ_n
    Replace X_n by $X_n + \Delta t f(X_n) + \sqrt{\Delta t}\, \xi_n g(X_n$
    Replace t_n by $t_n + \Delta t$
  end
  set $T_{\mathrm{exit}}^s = t_n - \frac{1}{2}\Delta t$
end
set $a_M = \frac{1}{M}\sum_{s=1}^{M} T_{\mathrm{exit}}^s$
set $b_M^2 = \frac{1}{M-1}\sum_{s=1}^{M}(T_{\mathrm{exit}}^s - a_M)^2$

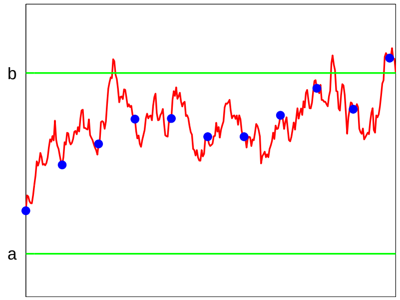
Errors

Two sources of error

- **Sampling error**: sample mean $\approx$ expected value
- **Numerical SDE error**: EM $\approx$ SDE paths

Third source of error

- **Discrete time**: $\{t_i\}$ is monitored, not continuous time



Analysis

Let $X_0 = x$. Then $T^{\text{mean}}_{\text{exit}} = u(x)$, where

$$\tfrac{1}{2} g(x)^2 \frac{d^2u}{dx^2} + f(x)\frac{du}{dx} = -1, \qquad \text{for } a < x < b$$

with b.c.'s $u(a) = u(b) = 0$

We can solve (analytically or numerically) to get "exact" solution. Then investigate accuracy of SDE/Monte Carlo.

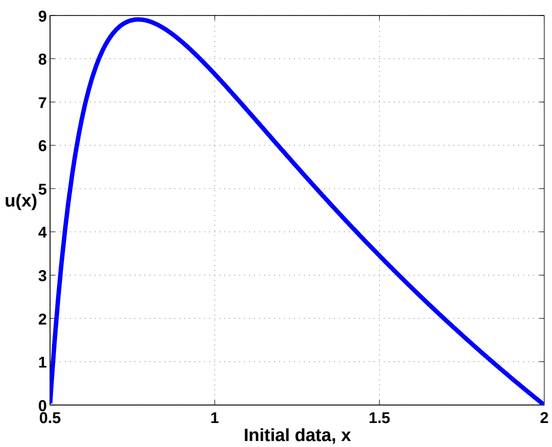
SDE/Monte Carlo approach is attractive for

- **high-dimensional** problems, e.g. $\mathbf{X} \in \mathbb{R}^{32}$
- **complicated geometries**,
- computing $u(x)$ at a **single point**

$f(x) = \mu x$ and $g(x) = \sigma x$

$$u(x) = \frac{1}{\frac{1}{2}\sigma^2 - \mu}\left(\log(x/a) - \frac{1-(x/a)^{1-2\mu/\sigma^2}}{1-(b/a)^{1-2\mu/\sigma^2}}\log(b/a)\right)$$

E.g. $\mu = 0.1$, $\sigma = 0.2$, $a = 0.5$ and $b = 2$:



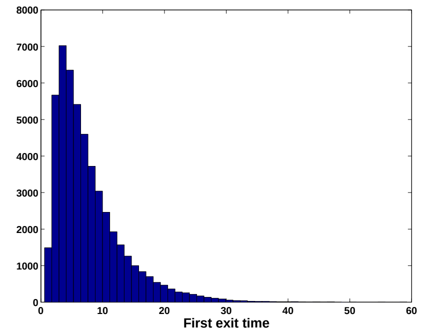
$f(x) = \mu x$ and $g(x) = \sigma x$

$$\text{Replace } X_n \text{ by } X_n + \Delta t \mu X_n + \sqrt{\Delta t}\, \xi_n \sigma X_n$$

can be improved:

$$\text{Replace } X_n \text{ by } X_n \exp\left((\mu - \tfrac{1}{2}\sigma^2)\Delta t + \sqrt{\Delta t}\, \xi_n \sigma\right)$$

E.g. $\mu = 0.1$, $\sigma = 0.2$, $a = 0.5$, $b = 2$ and $X_0 = 1$ with $M = 5 \times 10^4$ and $\Delta t = 10^{-2}$:



Accuracy? $T_{\mathrm{exit}}^{\mathrm{mean}} = 7.6450$

With $M = 5\times 10^4$ and $\Delta t = 10^{-2}$ we get
$a_M = 7.8056$ with a 95% conf. int. of $[7.7561, 7.8552]$

- exact answer **well outside** conf. int.
- Monte Carlo method **overestimates** mean exit time

Explanation: error from checking paths only at discrete time points $\{t_i\}$ is dominating the statistical sampling error

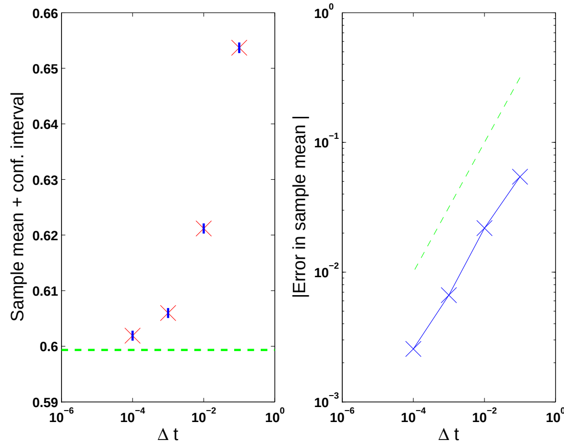
$\Rightarrow$ decrease Δt rather than increase M

$\Delta t = 10^{-3}$:	$a_M = 7.7137$	conf. int. $[7.6641, 7.7634]$
$\Delta t = 10^{-4}$:	$a_M = 7.6688$	conf. int. $[7.6200, 7.7177]$

Convergence rate . . .

$M = 5\times 10^5$ and try $\Delta t = 10^{-1}, 10^{-2}, 10^{-3}, 10$

$\mu = 0.5,\ \sigma = 0.2,\ a = 0.5,\ b = 2,\ X_0 = 1.5$:



Least-squares fit: power = 0.45, (resid = 0.12)

Convergence Rate

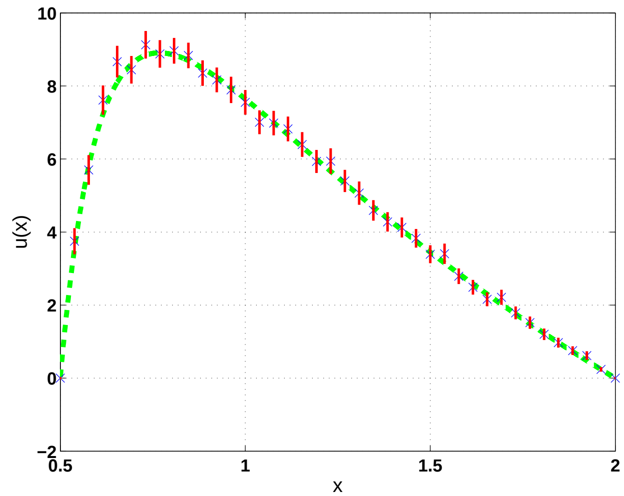
The rate $O(\Delta t^{\frac{1}{2}})$ has been widely reported

Overall error is then $O(\Delta t^{\frac{1}{2}} + 1/\sqrt{M})$

$\Rightarrow$ take $M \propto \Delta t^{-1}$

$M = 10^4$, $\Delta t = 10^{-4}$, 40 X_0 values

$\mu = 0.1,\ \sigma = 0.2,\ a = 0.5,\ b = 2$:

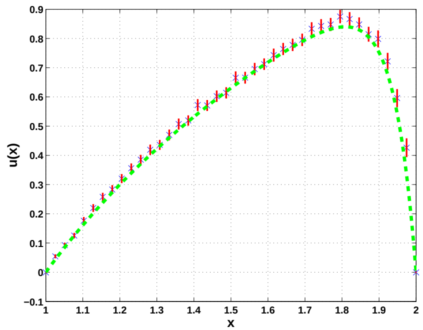


$f(x) = \lambda(\mu - x)$ and $g(x) = \sigma\sqrt{x}$

Safe EM step is

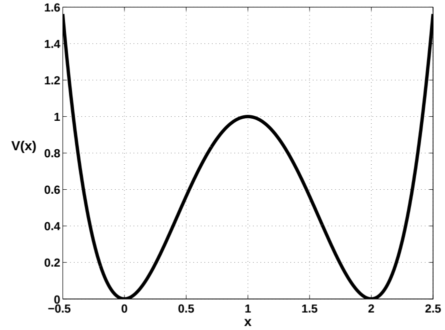
$$\text{Replace } X_n \text{ by } X_n + \Delta t \lambda(\mu - X_n) + \sqrt{\Delta t}\, \xi_n \sigma \sqrt{|X_n|}$$

Take $\lambda = 1$, $\mu = 0.5$, $\sigma = 0.3$, $a = 1$ and $b = 2$,
$M = 10^3$ and $\Delta t = 10^{-3}$:



"Exact" solution from bvp4c.m

Double-Well Potential: $V(x) = x^2(x-2)^2$

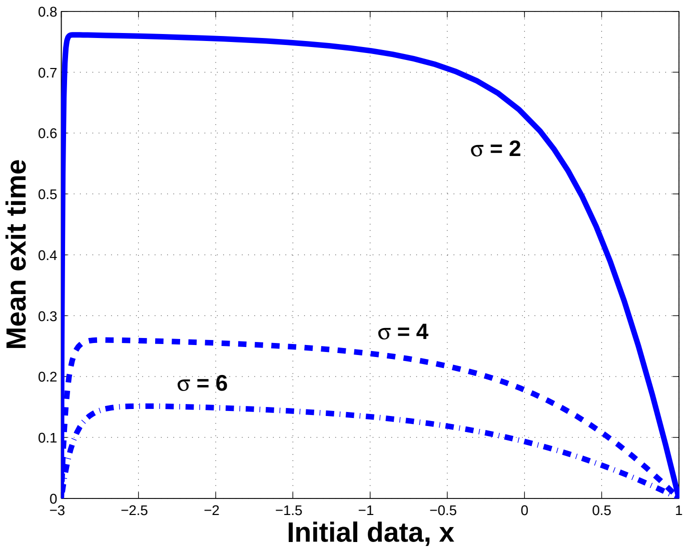


$$dX(t) = -V'(X(t))dt + \sigma dW(t)$$

Take $a = -3$ and $b = 1$

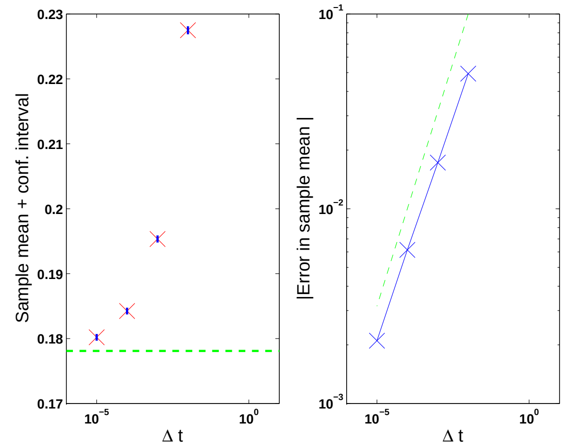
Measuring expected time to **climb over the central hump**

$$\tfrac{1}{2}\sigma^2 \frac{d^2u}{dx^2} - V'(x)\frac{du}{dx} = -1$$



Convergence

Fix $X_0 = 0$ and $\sigma = 4$
$M = 5 \times 10^5$ and $\Delta t = 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}$:



Least-squares fit: power = 0.46, (resid = 0.017)

Monte Carlo/SDE for Mean Exit Time

Research problems:

- Develop a provably $O(\Delta t)$ algorithm. Ideas:
  - **Adaptively reduce** Δt near boundary.
  - After each step, **calculate probability that exit was missed**. Then draw from a uniform $(0, 1)$ random number generator in order to decide whether to record an exit.
  - Use **random** Δt_n from a suitable exponential distribution.
- Develop an efficient method for high-dimension/complicated region