MATH 589 001 Advanced Probability Theory II
Winter Term 2006
Homework 1/2
Due: Thursday, January 12, 2006, 5pm, in Room 1123.

This homework is worth half the weight of the other homeworks in the course.

Cite all books you consult other than the course books and list all people you discuss your work with.

1. (Rosenthal, 3.6.6) Determine the following probabilities where $\{r_i\}$ are i.i.d. (independent, identically distributed) random variables with $\mathbb{P}(r_i = 0) = \mathbb{P}(r_i = 1) = 1/2$. Here *i.o.* means infinitely often. (Hint: Use the Borel-Cantelli Lemma.)

   (a) $\mathbb{P}(r_n = r_{n+1} = 1 \ i.o.)$.

   (b) $\mathbb{P}(r_{n+1} = r_{n+2} = \ldots = r_{2n} = 1 \ i.o.)$.

   (c) $\mathbb{P}(r_{n+1} = r_{n+2} = \ldots = r_{n+\lfloor 2\log_2 n \rfloor} = 1 \ i.o.)$.

   (d) $\mathbb{P}(r_{n+1} = r_{n+2} = \ldots = r_{n+\lfloor \log_2 n \rfloor} = 1 \ i.o.)$.

2. (Billingsley, 20.22) Suppose that $X_1 \leq X_2 \leq \cdots$ and that $X_n \to_P X$ (convergence in probability). Show that $X_n \to X$ with probability 1.

3. (Billingsley, 21.21) Let $X_1, X_2, \ldots$ be identically distributed random variables with finite second moment.

   (a) Show that $n\mathbb{P}[|X_1| \geq \epsilon\sqrt{n}] \to 0$ for all $\epsilon > 0$.

   (b) Show that $n^{-1/2} \max_{k \leq n} |X_k| \to_P 0$.

4. Let $X_1, X_2, \ldots$ be i.i.d. random variables that are positive with probability one. Let $S_n = X_1 + \cdots + X_n$ for $n > 0$ and $S_0 = 0$. Define $U(T)$ to be the largest integer n such that $S_n \leq T$. Show that $\mathbb{E}U(T) < \infty$ for all $T > 0$.

   You may want to do this by proving and using the following lemma. If U is a random variable that takes nonnegative integer values then $\mathbb{E}U = \sum_{n \geq 1} \mathbb{P}(U \geq n)$.

   Interpretation: Imagine you install a new lightbulb at time zero. Every time the lightbulb burns out you instantly install a new one. Suppose that the lightbulbs always last for some random (nonzero) period of time. Then over any finite time period the expected number of lightbulbs you use will be finite.