MATH 589 001 Advanced Probability Theory II
Winter Term 2006
Homework 2
Due: Friday, February 3, 2006, 5pm, in Room 1123.

Cite all books you consult other than the course books and list all people you discuss your work with.

1. (Durrett)
   (a) Suppose that $X_n \Rightarrow X$ and X_n has a normal distribution with mean 0 and variance σ_n^2. Prove that $\sigma_n^2 \to \sigma^2 \in [0, \infty)$.
   (b) Show that if X_n and Y_n are independent for $1 \le n$, $X_n \Rightarrow X$ and $Y_n \Rightarrow Y$ then $X_n + Y_n \Rightarrow X + Y$ where X and Y are independent.
2. (Billingsley 26.1) A random variable has a *lattice distribution* if for some a and b, $b > 0$, the lattice $[a + nb : n = 0, \pm 1, \ldots]$ supports the probability measure of X. Let X have characteristic functions ϕ. Show that if X is a lattice variable then $|\phi(t)| = 1$ for some $t \neq 0$. (The converse is also true.)
3. Let X_n be a sequence of positive i.i.d. random variables with distribution function F that is differentiable at 0 and satisfies $F'(0) > 0$ (a one-sided derivative). Define
$$Y_n = \min_{i=1,\ldots,n} nX_i.$$
Show that as $n \to \infty$, Y_n converges weakly to an exponential random variable with parameter $\lambda = F'(0)$.
4. (Billingsley 14.5) The *Lévy distance* between two distribution functions F and G is defined to be
$$\rho(F, G) := \inf\{\epsilon : F(x - \epsilon) - \epsilon \le G(x) \le F(x + \epsilon) + \epsilon \text{ for all } x\}.$$
Show that the Lévy distance is a metric on the space of distribution functions. That is, for any distribution functions F, G, H, show that
   (a) $\rho(F, G) \ge 0$,
   (b) $\rho(F, G) = 0$ if and only if $F = G$,
   (c) $\rho(F, G) = \rho(G, F)$,
   (d) $\rho(F, H) \le \rho(F, G) + \rho(G, H)$.
5. (Billingsley 14.5 continued) Show that $\rho(F_n, F) \to 0$ if and only if $F_n \Rightarrow F$.
6. (Durrett) Let X_n be an i.i.d. sequence with $\mathbb{E}X_i = 0$, and let $S_n = X_1 + \cdots + X_n$. Use the central limit theorem and Kolmogorov's zero-one law to show that $\limsup S_n/\sqrt{n} = \infty$ a.s.