MATH 589 001 Advanced Probability Theory II
Winter Term 2006
Homework 3
Due: Monday, February 27, 2006, 5pm, in Room 1123.

Cite all books you consult other than the course books and list all people you discuss your work with.

1. (Billingsley 27.12) There can be assymptotic normality even if there are no moments at all. Construct a simple example.

2. (Durrett)

   (a) Let ϕ be the ch.f. of a distribution F on $\mathbb{R}$. What is the distribution on $\mathbb{R}^k$ that corresponds to the ch.f. $\psi(t_1, \dots, t_k) = \phi(t_1 + \cdots + t_k)$?

   (b) Show that the random variables $X_1, \dots, X_k$ are independent if and only if

$$\phi_{X_1,\dots,X_k}(t) = \prod_{j=1}^{k} \phi_{X_j}(t_j).$$

3. Let Y_n, $n \geq 1$ be a sequence of random variables with $\mathbb{E}Y_n = 0$, $\mathbb{E}Y_n^2 = 1$. Let $X_{n,m}$, $n \geq 1$, $1 \leq m \leq n$, be a triangular array of independent random variables where $X_{n,m}$ is distributed as $Y_n/\sqrt{n}$. Consider applying the Lindeberg-Feller Central Limit Theorem to this array.

   (a) What does the Lindeberg-Feller condition reduce to in this case?

   (b) What does the Lyapunov condition reduce to in this case?

   (c) Give an example of a sequence Y_n for which the Lindeberg-Feller condition does not hold.

   (d) Give an example of a sequence Y_n for which the Lyapunov condition does not hold but the Lindeberg-Feller condition does.

   Hint: I was able to construct Y_n for each of (c) and (d) that consisted of three atoms for each n.

4. Prove the following multivariate version of the Lindeberg-Feller Theorem using the one-dimensional LFT and the Cramér-Wold device. For each n, let $X_{n,m}, 1 \leq m \leq n$, be independent random vectors with $\mathbb{E}X_{n,m} = 0$. Suppose
   (i) $\sum_{m=1}^{n} \mathbb{E}X_{n,m}X_{n,m}^T \to \Gamma \in \mathbb{R}^{k\times k}$
   (ii) For all $\epsilon > 0$, $\lim_{n\to\infty} \sum_{m=1}^{n} \mathbb{E}(\|X_{n,m}\|^2 1_{\|X_{n,m}\|>\epsilon}) = 0$
   Here $\|\cdot\|$ denotes the usual 2-norm on $\mathbb{R}^k$.

5. Let $U_n, n \geq 1$ be an i.i.d. sequence of random variables uniformly distributed on $[-1, 1]$. What is the assymptotic distribution of

$$M_n = n^{1/2} \text{median}_{i=1,\ldots,2n+1} U_i.$$

Use the usual Central Limit Theorem and the following identity for medians:

$$\{M_n \leq x\} = \{\sum_{i=1}^{2n+1} \mathbf{1}_{n^{1/2} X_i \leq x} \geq n+1\}.$$

6. (Rosenthal)

   (a) Suppose X and Y are discrete random variables. Let $p(x, y) = P(X = x, Y = y)$. Show that we can set

   $$\mathbb{E}(Y|X) = \frac{\sum_y y p(X, y)}{\sum_y p(X, y)}.$$

   (b) Let $\mathcal{G}$ be a sub-σ-algebra, and let X and Y be two *independent* random variables. Prove by example that $\mathbb{E}(X|\mathcal{G})$ and $\mathbb{E}(Y|\mathcal{G})$ need not be independent.