

The Standard Deviation is Sub-Additive

A sub-additive risk measure $\rho[X]$ satisfies the following:

$$\rho[X + Y] \leq \rho[X] + \rho[Y]$$

A super-additive risk measure satisfies:

$$\rho[X + Y] \geq \rho[X] + \rho[Y]$$

When only equality holds, the risk measure is additive.

The variance ($\text{var}[X]$) of two random variables with **positive covariance** ($\text{cov}[X, Y] > 0$) is a super-additive risk measure:

$$\text{var}[X+Y] = \text{var}[X] + \text{var}[Y] + 2 \text{cov}[X, Y] \geq \text{var}[X] + \text{var}[Y]$$

However the standard deviation ($\text{sd}[X]$) is subadditive:

$$\text{sd}[X + Y] \leq \text{sd}[X] + \text{sd}[Y]$$

To see this, consider the case where $\text{sd}[X] = \text{sd}[Y] = 1$ and $\text{cov}[X, Y] = .5$. In this case:

$$\sqrt{1 + 1 + 2(.5)} = \sqrt{3} \leq 1 + 1$$

Only in the perfectly, positively correlated case will equality hold.