The problem of determining the cost of risk

Basic idea: *The cost of risk is the difference between the expected value of a risky prospect, such as terminal wealth, and the certainty-equivalent value of that prospect.*

The solution to this problem would be useful in analyzing whether to buy insurance or to invest in a risky capital project. While there are a number of possible methods to extract the cost of risk, consider the following solution. Let the expected value of terminal wealth be: $E[W(T)] = \Omega$. Observe that Ω is a parameter which permits the *certainty equivalent* income of a risky prospect to be defined as $\Omega - C$, where C is the cost of risk. It follows from the expected utility axioms that the cost of risk, C, can be calculated as the difference between the expected value of the risky prospect and the associated certainty equivalent income:

$$U[\Omega - C] = \sum_{i=1}^{S} \theta_i \ U[W_i] = EU[W(T)]$$

It is now possible to expand $U[\Omega - C]$ in a Taylor series and estimate the cost of risk by manipulating the first and second order approximations.

More precisely, expanding the function $U[\Omega - C]$ around Ω the first order approximation is:

$$U[\Omega - C] = U[\Omega] + U'[\Omega] \ (\Omega - C - \Omega) = U[\Omega] - U'[\Omega] \ C$$

Similarly, a second order approximation for the function $U[W_{t+1}]$ can provide:

$$U[W_{t+1}] = U[\Omega] - U'[\Omega] \ (W_{t+1} - \Omega) + \frac{1}{2} U''[\Omega] \ (W(T) - \Omega)^2$$

$$\rightarrow \quad EU[W(T)] = U[\Omega] + \frac{1}{2} \ U''[\Omega] \ var[W(T)]$$

Using $U[\Omega\text{-}C]=EU[W_{t+1}]$ and manipulating gives:

$$C = -\frac{U''[\Omega]}{2 \ U'[\Omega]} \ var[W(T)] \quad \rightarrow \quad \frac{C}{W_t} = -\frac{U''[\Omega] \ W(0)}{2 \ U'[\Omega]} \ var[1 + R]$$

This demonstrates theoretically that the cost of risk will vary across utility functions. This result also provides theoretical measures of the cost of risk. The measures of absolute risk aversion, $-\{U''/U'\}$, and relative risk aversion, $-\{U'' \ W_t\}/U'$ are now textbook concepts, e.g., Elton and Gruber (1995).

Absolute Risk Aversion –> The cost of risk is proportional to the variance of income –> initial wealth does not affect the assessment of risky projects –> constant absolute risk aversion requires the ranking of risky prospects to be the same, regardless of wealth level

Relative Risk Aversion –> the fraction of initial wealth invested in risky projects –> constant relative risk aversion requires the ranking of risky prospects to improve as wealth increases, i.e., wealthy individuals are less averse to risky prospects