Lecture 8
Review of Option Properties

- Definitions and Expiration Date Profit Diagram
- Put-Call Parity and Replication
- Risk Management with Options
- Introduction to Black-Scholes

Basics of Expiration Date Profit Diagrams

- These graphs (diagrams) plot the relationship between the ***expiration date*** stock price, *S(T),* and the profit function, $\pi(T)$, from positions that involve holding an option or combination of options.

- Because the option is purchased prior to the expiration date, the profit function will involve an option premium payment

Basics of Expiration Date Diagrams (cont'd)

- **Warnings about the Diagrams**
  - The cash flows in the positions are not accurately represented, e.g., the funds used to long the stock are not incorporated.
  - Interest on the premium over the life of the option is not analyzed.
  - The diagram only takes a valuation on the expiration date into account – this ignores the option time value on the purchase date.

Expiration Date Profit Diagrams

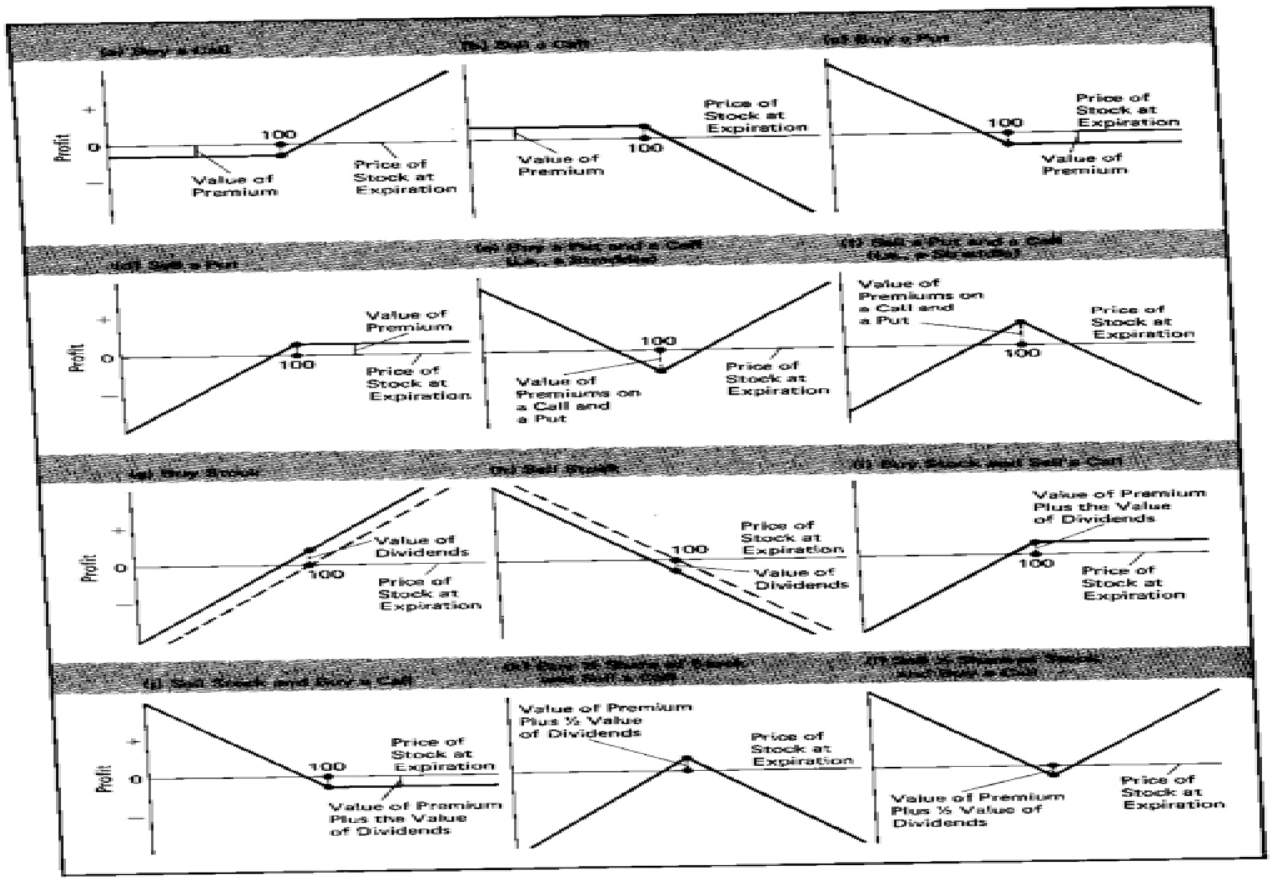
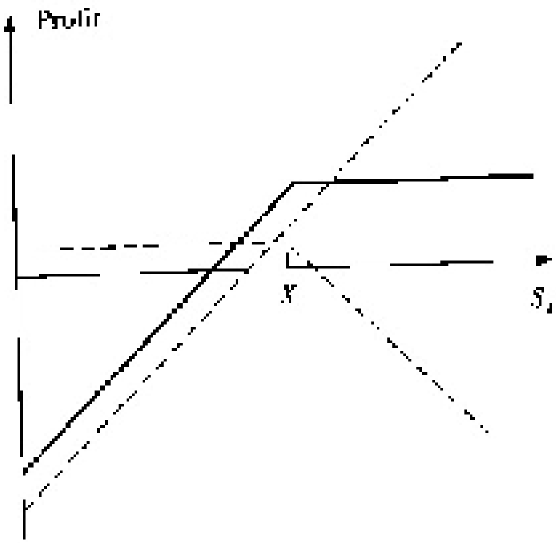


FIGURE 16-7
Profits and Losses from Various Strategies.

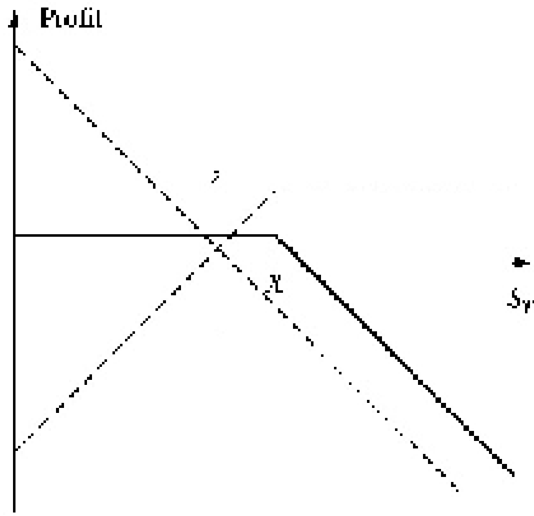
Replication Trades:

Short the Put



(c)
Long Position in a Stock
Combined with Short Position in a Call

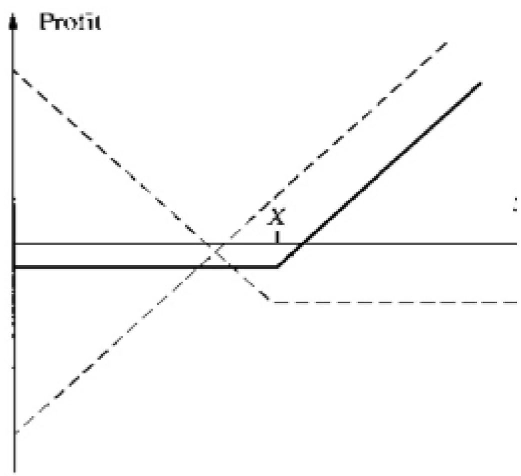
Short the Call



(d)
Short Position in a Stock
Combined with Short Position in a Put

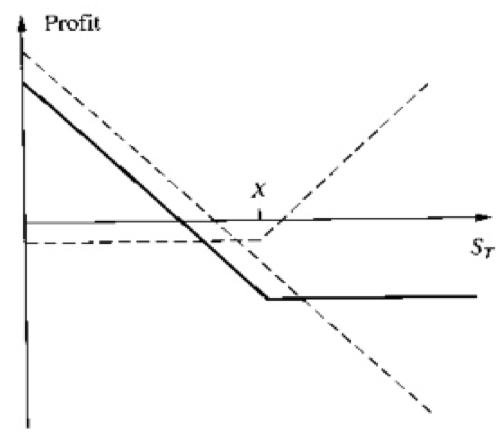
Replication Trades:

Long the Call



(c)
Long Position in a Stock
Combined with Long Position in a Put

Long the Put



(b)
Short Position in a Stock
Combined with Long Position in a Call

Algebra of Expiration Date Diagrams

- The profit function for a long position in a call option can be written as:

$$\pi = C(T,X) - \{call\ premium\}$$

- where the expiration date payoff for a call option $C(T,X) = max[0, S(T) - X]$

Algebra (cont'd)

- Consider the replication trade profit function for a long call (long the stock + long a put)

$$\pi = S(T) - S(0) + P(T,X) - \{put\ prem.\}$$

$$= S(T) - S(0) + max[\ 0, X - S(T)] - \{put\ prem.\}$$

$$= max[\ 0, S(T) - X] - \{put\ prem. + S(0) - X\}$$

Readings and Exercises

- Reading: RSD, p.382-9
- Exercises: Develop the expiration date profit diagrams for:
  - Long the stock
  - Short the stock
  - Long the Vertical Spread with Puts
  - Long the Straddle
  - Short the Strangle

Distribution Free Properties of Options

- Reading: RSD, p.375-82.
- Formulas like Black-Scholes are obtained by making a distributional assumption about the random variable determining the option value, e.g., stock prices.
- Distribution free properties apply to all options – irrespective of assumptions about the distribution.

Distribution Free Properties of Options (cont'd)

- Some of the distribution free properties are obvious or well known:
  - Property 1 requires prices to be non-negative
  - Property 2 is the expiration date values of the call and put options
  - Property 3 requires the prices of American options to be greater than European options (with the same *X* and *T*)

Distribution Free Properties of Options (cont'd)

- Key Distribution Free Property:

  Put-Call Parity

  For a non-dividend paying stock and options with the same *X* and *T*:

  $$S(t) + P(t; X,T) = C(t; X,T) + X\, e^{-rt^*}$$

  Like CIP, put-call parity is an arbitrage relationship.

Variations of Put-Call Parity

- The put-call parity relationship can be re-expressed in terms of replication

- Replicate the stock:

$$S(t) = C(t; X, T) + X\, e^{-rt^*} - P(t; X, T)$$

- Replicate a call:

$$C(t; X, T) = S(t) + P(t) - X\, e^{-rt^*}$$

Variations of Put-Call Parity (cont'd)

- Replicate the bond:

$$X\, e^{-rt^*} = S(t) - C(t; X, T) + P(t; X, T)$$

- Replicate the put:

$$P(t; X, T) = C(t; X, T) + X\, e^{-rt^*} - S(t)$$

The security on the lhs is being replicated by the portfolio on the rhs.

Reading: RSD, p.393-403, esp. p393-400.

Risk Management with Options

- Options can introduce or offset non-linear elements in firm cash flows
  - Put Options as insurance
    - Traditional Risk Management (Fire, Health, etc.)
  - Replication Strategies
    - Path Independent and Dynamic Portfolio Insurance
- Real Options and Natural Hedging

Figure 6.7 **Comparison of a Different Methods of Hedging a Contingent Foreign Currency Position**

Scenario: A U.S. telecommunications firm is planning a hostile takeover of a British firm. If the takeover is successful, payment of a fixed number of British pounds is required. If the takeover is unsuccessful, no pound payments are required. The U.S. firm is concerned that the pound will strengthen during the uncertain time period during which the takeover bid is in effect. The table below indicates various possible combinations of forward, futures or option contracts which could be used to hedge the pound exposure.

Outcome of Takeover	**No Hedge**	**Long Futures Hedge**	**Long Forward Hedge**	**Option Hedge (Buy Call)**	**Option Hedge (Buy Call Write Put)**
Successful:					
Pound increases	Cost of takeover in US$ increases	Increased US$ payment offset by hedge gain;	Increased US$ payment offset by hedge gain;	Call exercised; Increased US$ payment offset by gain on option	Call exercised; put expires; Increased US$ payment offset.
Pound decreases	Cost of takeover in US$ decreases	Lower US$ payment increased by hedge loss;	Lower US$ payment increased by hedge loss; payout	Call expires; Lower US$ payment after option premium	Call expires, put is exercised; lower US$ payment offset by put
Unsuccessful:					
Pound increases	No effect	Cash inflow from gain on hedge	Cash inflow from gain on hedge	Call exercised	Call exercised; Put expires Cash inflow from call.
Pound decreases	No effect	Cash outflow from loss on hedge	Cash outflow from loss on hedge	Call expires; Premium lost	Call expires, put exercised; Cash outflow from put exercise
Advantages	No hedging cost	Easy to Unwind	No basis risk	Contingent Payoff	Ability to set different Put and Call Exercise Prices

The Black-Scholes Formula

THE BLACK-SCHOLES CALL OPTION FORMULA (the put solution is solved using the put-call parity condition).

$$C[S,t^*;X,r,\sigma] = C(t) = S(t)\,N[d_1] - X\,e^{-rt^*}\,N[d_2]$$

- where *N[d]* is the **cumulative** standard normal distribution evaluated at the value *d* and where:

$$d_1 = \frac{\ln[\frac{S}{X}] + (r + \frac{1}{2}\sigma^2)t^*}{\sigma\sqrt{t^*}}$$

$$d_2 = d_1 - \sigma\sqrt{t^*} = \frac{\ln[\frac{S}{X}] + (r - \frac{1}{2}\sigma^2)t^*}{\sigma\sqrt{t^*}}$$

Solving the Formula

Assume: $S(t)$ = \$36, X = \$40, τ = 3 months $\Rightarrow$ t^* = .25, r = .05, σ = .5. Both the interest rate and standard deviation are expressed in annualized form. For sigma, estimating the **historical** standard deviation over the relevant sampling frequency (e.g., weekly) requires annualizing as appropriate. Given this:

d_1 = {ln[36/40] + (.05 + .5(.5)2) .25} / {.5 ($\sqrt{.25}$)} = -.25
d_2 = {ln[36/40] + (.05 - .5(.5)2) .25} / {.5 ($\sqrt{.25}$)} = -.50

Evaluating the $N[d]$ values: $N[-.25]$ = .4013 and $N[-.50]$ = .3085 and exp{(.05)(.25)} = .9877, it is possible to solve for the Black-Scholes call option price:

C = S [.4013] - X [.9877] [.3085] = 14.4468 - 12.1882 = \$2.26

Evaluating the Cumulative Normal d.f.

- The **cumulative normal distribution function** *N[x]* provides the area under the normal density function *n[x]* at a given point *x* --- integrate the area under the density function from $-\infty$ to *x* to get *N[x]* value.

- The normal density function is sometimes referred to as the bell curve.

- $N[0] = .5 \quad N[+\infty] = 1 \quad N[-\infty] = 0$

- The value of *N[x]* is the delta for a Black-Scholes call option.

Deriving Black-Scholes: Basic Elements

- The Black-Scholes (1973) derivation requires some basic concepts (readings in brackets):

  The Black-Scholes Assumptions (RSD, p.439)

  The riskless hedge portfolio (RSD, p.439-40)

  Ito’s Lemma (RSD, p.434-6)

  The Fundamental Partial Differential Equation

The Black-Scholes Assumptions

a) Non-dividend paying stock.

b) European option.

c) The instantaneously riskless continuous interest rate *r* is constant over time (with a flat term structure).

d) The model has only **one** source of randomness, the single state variable, the price of the stock which follows a log-normal diffusion process. This log normal process is defined only over S ε [0, ∞].

e) No transactions costs or taxes.

f) No penalties on short selling.

g) Riskless lending and borrowing at r.

h) Continuous trading.

e)-h) are conventional perfect markets assumptions.

Riskless Hedge Portfolio and Arbitrage

- Discussion of pricing for futures and forward contracts used the cash-and-carry arbitrage
- The **riskless hedge portfolio** used in Black-Scholes (1973) predates arbitrage arguments – the riskless hedge portfolio has a net investment of funds.
- The riskless hedge portfolio argument can be reconstructed as an arbitrage portfolio argument where the funds invested in the position are borrowed.

The Riskless Hedge Portfolio

- There is both a long and a short hedge portfolio, the long portfolio combines a long stock position with β written call options on the stock.
- The value of this portfolio (V) can be expressed as $V(t) = S(t) - \beta\, C(t)$
- The hedge portfolio condition can be specified as:

$$\frac{\partial V}{\partial S} = 1 - \beta \frac{\partial C}{\partial S} = 0 \quad \rightarrow \quad \beta = \frac{1}{\frac{\partial C}{\partial S}} \quad \rightarrow \quad \frac{\partial C}{\partial S} = \frac{1}{\beta}$$

Comments on the Hedge Portfolio

- The hedge portfolio condition $(\partial V / \partial S) = 0$ involves the partial derivative with respect to S. Do not confuse this with $dV = 0$.

- This is a riskless hedge portfolio condition because the amount of funds required to purchase S is less than the funds received from selling β call options, i.e., there is a net investment of funds in the position. (This violates the requirements for an arbitrage).

- The number of written calls needed to hedge the portfolio value will change as C and S change over time. The **delta** of the call $(\partial C / \partial S) = \Delta_C$ provides this information.

Basic Calculus and Stochastic Calculus

- Conventional calculus deals with deterministic functions, *f[x,y]*, where *x* and *y* variables are non-random.
- The total derivative rule:

$$df[x,y] = (\partial f / \partial x)\, dx + (\partial f / \partial y)\, dy$$

- The call price function ***C[S,t*;X,r,σ]*** contains the random variable *S* (Semantics: random = stochastic).

- Calculus for functions with stochastic variables is more complicated because the process of taking the slope at a point is not possible because the function is not well defined at a point if there is a random component.

Ito’s Lemma and Diffusion Processes

- Ito’s Lemma provides the rule for totally differentiating functions of random variables where the randomness is associated with a diffusion process.

Ito's Lemma: Let $y(x,t)$ be a continuous random function with continuous partial derivatives y_t, y_x, and y_{xx}. If $x(t)$ is a random process obeying a diffusion of the form:

$$dx(t) = a(t)\, dt + v(t)\, dW(t)$$

where $W(t)$ is a standard Wiener process and $a(t)$ and $v(t)$ are the drift and volatility of the diffusion, then the function $y(t) = y(x(t),t)$ also has a differential on $[0,T]$ given by:

$$dy(t) = \{y_t + y_x\, a(t) + 1/2\, y_{xx}\, v(t)2\}\, dt + y_x\, v(t)\, dW(t)$$

Ito's Lemma for the Call Price Function

The non-dividend paying stock price is assumed to follow a log-normal diffusion:

$$dS = \alpha\, S\, dt + \sigma\, S\, dW$$

In this case, ***a(t) = αS* and *v(t) = σS*** where α and σ are constants.

- The functional relationship between the call option price *(C)* and the stock price takes the form: $C = C[S,t; X]$. Application of Ito's lemma gives:

$$dC = \{C_t + C_S\, \alpha\, S + 1/2\, C_{SS}\, \sigma^2\, S^2\}\, dt + \{C_S\, \sigma\, S\}\, dW$$

Solving for the Fundamental PDE

- The Black-Scholes derivation uses two conditions:

  $dV = dS - \beta\, dC$ (Total Derivative)

  $dV = (S - \beta\, C)\, r\, dt$ (Riskless Return Condition)

The first condition can be solved by substituting in: the diffusion equation for dS, $\beta = C_S$ (from the riskless hedge solution) and dC from Ito's Lemma.

The two conditions are equated and solved to get the **fundamental partial differential equation** (PDE) for a European call on a non-dividend paying stock (which is subject to the boundary and terminal conditions of $C[T] = max[0, S-X]$ and $C[S=0] = 0$)

$$C_t = rC - rS\,C_S - \frac{1}{2}\sigma^2 S^2 C_{SS}$$

Solving the PDE

- The Black-Scholes formula is the function that solves the fundamental PDE problem.
- There are a number of possible methods that can be used to derive this solution – solving PDE's is an active research area in mathematics.
- In this class the solution will be verified when the Greeks are being discussed (see RSD, p.497-8).

(Notice that the fundamental PDE provides a relationship among delta, gamma and theta).