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Partial Immunization Bounds and Non-Parallel Term Structure Shifts*

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ABSTRACT

A variety of approaches have been proposed to extend classical fixed income portfolio immunization theory to cases where shifts in the term structure are not parallel. Following Reitano (1991a, 1991b, 1992, 1996) and Poitras (2007), this paper uses partial durations and convexities to specify benchmark partial immunization bounds for non-parallel term structure shifts. Theoretical results are obtained by exploiting properties of the multivariate Taylor series expansion of the spot interest rate pricing function. It is demonstrated that the partial immunization bounds can be effectively manipulated by adequate selection of the securities being used to immunize the portfolio. The inclusion of time values permits the results obtained to be related to previous studies by Christiansen and Sorensen (1994), Chance and Jordan (1996), Barber and Copper (1997) and Poitras (2005, ch.5) on the time value-convexity tradeoff.

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The classical theory of fixed income portfolio immunization, e.g., Redington (1952), Shiu (1987, 1990), employs a univariate Taylor series expansion to derive two rules for immunizing a life insurance company balance sheet against a change in the level of interest rates. The associated classical rules are: match the duration of cash inflows and outflows; and, set the asset cash flows to have more convexity (dispersion) than the liability cash flows around that duration. From that beginning, a number of improvements to classical immunization rules have been proposed aimed at correcting limitations in the classical formulation. Particular attention has been given to generalizing the classical model to allow for non-parallel shifts in the yield curve, e.g., Nawalkha and Soto (2009); Soto (2004, 2001); Nawalkha et al. (2003); Navarro and Nave (2001); Crack and Nawalka (2000); Balbas and Ibanez (1998); and, Bowden (1997).[1] While most studies aim to identify a 'best' rule for specifying fixed income portfolios that are immunized against instantaneous non-parallel shifts, it is gradually being recognized that the optimal solution depends on "the nature of the term structure shifts expected" (Nawalka and Soto 2009, p.10). Instead of seeking a 'best rule', Reitano (1991a, 1991b,1992, 1996) and Poitras (2007) explore the properties of partial immunization bounds applicable for bench marking a range of level, slope and shape changes associated with 'non-parallel shifts'.

The objective of this paper is to outline and further extend the partial immunization framework by clarifying the role that time value changes have in the immunization bounds approach. In particular, partial durations, convexities and time values are exploited to identify bounds on gains and losses for an instantaneous unit shift in the term structure of spot interest rates. The partial immunization bounds are derived from fundamental properties of vector spaces: the Cauchy-Schwarz inequality restrictions for the duration bounds; and, the largest and smallest characteristic roots for the matrix of partial convexities for the convexity bounds. Comparison of the bounds for different *ex ante* scenarios involving changes in slope, shape and level of the same 'length' requires specification of an appropriate 'norm' to measure distance. The inherent advantages of the partial immunization bound approach in practical fixed income portfolio management is illustrated by comparison with studies on the time value-convexity tradeoff that employ one factor interest rate models, especially Christensen and Sorensen (1994), Chance and Jordan (1996), Barber and Copper (1997) and Poitras (2005).

1. Background Literature

The seminal work that initiated classical fixed income portfolio immunization theory (Redington 1952) also recognized that the classical immunization conditions may be sub-optimal when shifts in the yield curve are not parallel. Starting with Fisher and Weil (1971), the development of techniques to address non-parallel yield curve shifts led to the recognition of a connection between immunization strategy specification and the type of assumed shocks, e.g., Bierwag and Khang (1979), Fong and Vasicek (1984), Chambers et al. (1988), Barber and Copper (1998). Sophisticated risk measures, such as M^2 and *M-absolute*, were developed to select the best duration matching portfolio from the set of potential portfolios, e.g., Nawalkha and Chambers (1996). Being derived using a specific assumption about the stochastic process generating the term structure, these theoretically attractive models encounter difficulties in practice when actual term structure behaviour differs significantly from the assumed stochastic model. For example, Bierwag et al. (1993, p.1165) find that "minimum M^2 portfolios *fail* to hedge as effectively as portfolios including a bond maturing on the horizon date", a problematic empirical results referred to as the 'duration puzzle' where portfolios containing a maturity-matching bond have smaller deviations from the promised target return than portfolios not containing a maturity-matching bond (Bierwag et al. 1993; Soto 2001; Poitras 2007). Such problematic results beg the question: are such empirical limitations due to failings of the stochastic process assumption underlying the theoretically derived immunization measures or is there some deeper property of the immunization process not being accurately modelled?

Instead of assuming a specific stochastic process and deriving the optimal immunization conditions, it is possible to leave the process unspecified and work directly with the properties of an expansion of the spot rate pricing function. Immunization then proceeds by making assumptions based on the empirical behaviour of the term structure or yield curve. Soto (2001) divides these "empirical multiple factor duration models" into three categories. *Polynomial duration models* fit yield curve movements using a polynomial function of the terms to maturity, e.g., Crack and Nawalka (2000), Soto (2001), or the distance between the terms to maturity and the planning horizon, e.g., Nawalka et al. (2003). *Directional duration models* identify general risk factors using data reduction techniques such as principal components to capture the empirical yield curve behaviour, e.g., Elton et al. (2000), Barber and Copper (1996), Hill and Vaysman (1998), Navarro and Nave (2001). *Key rate duration models* decompose the yield curve into number of linear segments based on the selection of key rates, e.g., Ho (1992), Dattareya and Fabozzi (1995), Phoa and Shearer (1997). Whereas the dimension of polynomial duration models is restricted by the degree of the polynomial and the directional duration models are restricted by the number of empirical components or factors that are identified, e.g., Soto (2004), the number of key rates used is exogenously determined by the desired fit of immunization procedure.

Similar to the approach used in the key rate duration models, Reitano (1991a,b) and Poitras (2007) demonstrate that fixed income portfolio immunization can be approached using partial durations and convexities to determine 'partial immunization bounds' for the specified fixed income portfolio. Instead of seeking the 'optimal' immunizing portfolio, the partial immunization approach examines the exposure of pre-specified portfolios to benchmark shifts in the term structure. The immunization bounds for partial durations and convexities are determined by evaluating a multivariate Taylor series for the asset and liability price functions specified using spot interest rates. More precisely, defining a norm applicable to a unit parallel term structure shift allows Cauchy-Schwarz and quadratic form inequality restrictions to identify: extreme bounds on the possible deviations from classical immunization conditions; and, the specific types of shifts that represent the greatest loss or gain. In turn, these extreme bounds can be compared with bounds for term structure shifts that are consistent with absence of arbitrage conditions on the evolution of spot rates. In other words, even though classical immunization rules are violated for non-parallel yield curve shifts, with appropriate selection of a norm it is still possible to benchmark theoretical bounds on deviations from the classical outcome.

In assessing the implications of instantaneous non-parallel term structure shifts, the partial immunization bounds approach lacks the precise, if potentially inaccurate, portfolio composition recommendations of alternative 'immunization' models. When the analysis is extended to include the time-value convexity tradeoff identified by Christiansen and Sorensen (1994), Barber and Copper (1997) and Poitras (2005, ch.5), the partial immunization approach has the desirable feature of providing a direct relationship between the convexity and time value elements of the immunization problem. This follows because convexity has a time value cost associated with the initial yield curve shape and the expected future path of interest rates for reinvestment of coupons and rollover of short-dated principal. Despite the essential character of the time value decision in actual fixed income portfolio management, available results on the time value-convexity tradeoff have only been developed in the classical Fisher-Weil framework involving parallel yield curve shifts, e.g., Christensen and Sorensen (1994); Chance and Jordan (1996). The partial immunization approach has the desirable property of allowing attention to focus on the implications of specific term structure shifts that are of practical *ex ante* interest.

2. Classical and Partial Immunization Analytics

Classical fixed income portfolio immunization theory involves explicit recognition of the balance sheet relationship for a fixed income portfolio, $A = L + S \rightarrow S[y] = A[y] - L[y]$ where: A, L and S are assets, liabilities and surplus (equity) of the portfolio (fund); and, y is the yield to maturity. In the classical case, a univariate Taylor series expansion is applied to the surplus function ($S = A - L$) to give:

$$S[y] = S[y_0] + \frac{dS}{dy}(y - y_0) + \frac{1}{2}\frac{d^2S}{dy^2}(y - y_0)^2 + \ldots \; H.O.T.$$

$$\frac{S[y] - S[y_0]}{S[y_0]} \cong -DUR_S\,(y - y_0) + \frac{1}{2}\,CON_S\,(y - y_0)^2$$

where: DUR_S and CON_S are the modified duration and convexity of surplus; y_0 is the (fixed point) yield around which $S[y]$ is expanded; and, $H.O.T.$ indicates higher order terms in the expansion. Given this, the classical zero duration of surplus and positive convexity of surplus conditions can be determined as:

$$S = A - L \quad \rightarrow \quad \frac{1}{S}\frac{dS}{dy} = 0 = \frac{A}{S}\,DUR_A - \frac{L}{S}\,DUR_L$$

$$\rightarrow \quad DUR_A = \frac{L}{A}\,DUR_L \quad \rightarrow \quad CON_A > \frac{L}{A}\,CON_L$$

where the subscripts A and L indicate assets and liabilities, respectively. In most presentations, the classical immunization conditions assume a zero surplus which requires setting the duration of assets equal to the duration of liabilities. In other words, $S = 0$ in the classical case implies that the surplus is managed separately. In turn, immunization with a positive surplus requires the duration of assets to be equal to the duration of liabilities, multiplied by the loan-to-value ratio for the market values of the assets held in the fixed income portfolio and the lender liabilities used to fund the portfolio, e.g., Messmore (1990).

In order to incorporate term structure shifts, the partial immunization approach specifies the fund surplus using a multivariate function of spot interest rates. For ease of notation, this function is specified in vector notation as $S[z]$ where $z = [z_1, z_2, \ldots, z_T]'$ is the $Tx1$ (column) vector of spot interest rates. Observing that spot interest rates are bootstrapped from observed yields to maturity (e.g., Poitras 2011, ch.4) provides a basic starting point for presenting the model:

$$S[y] = \sum_{t=1}^{T} \frac{C_t}{(1 + y)^t} + \frac{M}{(1 + y)^T} = S[z] = \sum_{t=1}^{T} \frac{C_t}{(1 + z_t)^t} + \frac{M}{(1 + z_T)^T} = \sum_{t=1}^{T} \frac{CF_t}{(1 + z_t)}$$

where: $S[z]$ is the market value of the fixed income portfolio surplus calculated using spot interest rates; C_t is the fund net cash flow payment at time t; M is the terminating (time T) value of the fund; $CF_t = C_t$ for $t = \{1,2, \ldots ,T\text{-}1\}$ and $= C_t + M$ at $t = T$; z_t is the spot interest rate (implied zero coupon interest rate) applicable to cash flows at time t; and, T is the termination date of the fund in years. Because C_t will be negative when the liability cash outflow exceeds the asset cash inflow at a given time t, it is possible for either the duration of surplus or convexity of surplus to take negative values, depending on the selected portfolio composition and term structure shape. Recognizing that $S[z]$ is a function of the T spot interest rates contained in z, it is possible to apply a multivariate Taylor series expansion to the surplus value function, that leads immediately to the concepts of partial duration and partial convexity:

$$S[z] = S[z_0] + \sum_{t=1}^{T} \frac{\partial S[z_{t,0}]}{\partial z_t}(z_t - z_{t,0}) + \frac{1}{2!} \sum_{i=1}^{T}\sum_{j=1}^{T} \frac{\partial^2 S[\,\cdot\,]}{\partial z_i \, \partial z_j}(z_i - z_{i,0})(z_j - z_{j,0}) + H.O.T.$$

$$\rightarrow \quad \frac{S[z] - S[z_0]}{S[z_0]} \cong -\sum_{t=1}^{T} D_t \, (z_t - z_{t,0}) + \frac{1}{2!} \sum_{i=1}^{T}\sum_{j=1}^{T} CON_{i,j} \, (z_i - z_{i,0})(z_j - z_{j,0}) \qquad (1)$$

where: $z_0 = [z_{1,0}, z_{2,0}, \ldots, z_{T,0}]'$ is the $Tx1$ vector of initial spot interest rates at $t = 0$; D_t is partial duration associated with z_t, the spot interest rate for time t; and, $CON_{i,j}$ is the partial convexity associated with the spot interest rates z_i and z_j for i,j defined over $\{1,2, \ldots, T\}$.

Observing that the partial durations can be identified with a $Tx1$ vector $D_{\underline{T}} = [D_1 , D_2 , \ldots , D_T]'$ and the partial convexities with a TxT matrix Γ_T with elements $CON_{i,j}$, the model proceeds by applying results from the theory of normed linear vector spaces to identify theoretical extreme bounds on $D_{\underline{T}}$ and Γ_T. In the case of $D_{\underline{T}}$, the Cauchy-Schwarz inequality is used. For Γ_T the extreme bounds are based on restrictions on the eigenvalues of Γ_T derived from the theory of quadratic forms. To access these results, it is necessary to specify a norm so that term structure shifts of the same 'length' can be compared. In the partial immunization approach, the norm selected is required to specify a ***direction vector***. More precisely, taken as a group, the $(z_t - z_{0,t})$ changes in the individual spot interest rates represent shifts in term structure shape. These individual changes can be reexpressed as the product of a direction shift N and a magnitude Δi:

$$(z_t - z_{t,0}) = n_t \, \Delta i \qquad where: \quad N = [n_1, n_2, \ldots, n_T]'$$

This leads to the vector space representation of the second order multivariate Taylor series expansion:

$$\frac{S[z] - S[z_0]}{S[z_0]} \cong -\Delta i \, (N' \, D_{\underline{T}}) + \Delta i^2 \, (N' \, \Gamma_T \, N) \qquad (2)$$

From this the spot rate curve, i.e., the term structure of interest rates, is shocked according to some rule and the associated immunization bounds are derived. Using the spot rate approach, it follows that $n_1 = n_2 = n_3 = \ldots = n_T$ represents a parallel shift in the spot rate curve.[2]

The partial duration immunization bound provided by the partial durations is a consequence of the Cauchy-Schwarz inequality, a fundamental property of distance measurement in vector spaces. In order to measure length of a given shift in the term structure of interest rates, the **norm** of a column vector $x = [x_1, x_2, x_3, \ldots, x_T]'$ is defined as: $\| x \| = \sqrt{x'x} = \sqrt{\sum_{i=1}^{T} x_i^2}$. The Cauchy-Schwarz inequality requires that:

$$-\| N \| \, \| D_T \| \leq N' D_T \leq \| N \| \, \| D_T \|$$

Without loss of generality, the norm of the shift vector N can be selected as: $\| N \| = 1$. This allows the length of the vector of partial durations, $\| D_T \|$ to determine the partial duration immunization bounds for different unit shifts in the spot rate curve. In turn, the individual partial durations are given by the cash flow pattern of the assets and liabilities in the fixed income portfolio. Once the fixed income portfolio has been specified, it is possible to solve for $\| N^* \| = 1$, the term structure shift of unit length that achieves the partial duration immunization bound.

3. Partial Duration Immunization Bound Examples

For comparability with previous partial immunization studies, the various fixed income portfolio scenarios are examined using the yield curve and spot rates from Fabozzi (1993) as the baseline.[3] This steep yield curve has a 450 basis point slope between the ½ year and 10 year yields. The associated spot rate term structure has a slope of 558 basis points. Tables 1-3 report: the partial durations; the extreme bound shift weights n_t*; and, extreme duration bounds calculated from the Cauchy-Schwarz inequality (see Appendix). Table 1 examines a high surplus ($50.75) and a low surplus portfolio ($4.67) where the liability is a five year zero coupon obligation with market value of $87.51. Consistent with examples in Reitano (1996) and Poitras (2007), the assets are a ½ year zero coupon money market security and a ten year 12% semi-annual coupon bond. In Table 2, the surpluses are approximately equal ($10.32 and $10.91) with the same liability (5 year zero coupon with market value of $87.51) is used as in Table 1. In addition to the assets used in Table 1, the assets also include a 5 year par bond (same maturity as the liability) in one case and 3 year and 7 year par bonds in the other case. Table 3 changes the liability to a ten year semi-annual coupon annuity with the same market value ($87.51, semi-annual coupon of $14.96) as in Tables 1-2. The assets used are the 'maturity bond' portfolio in Table 2 and the 'low surplus' portfolio in Table 1.

Examining Table 1, comparison of the bounds between the low and high surplus cases depends crucially on the observation that the bounds relate to the percentage change in the surplus. Due to the smaller position in the 6 month asset, the larger bounds for the low surplus case also translate to a larger relative change when compared to the high surplus case. This result is calculated by multiplying the reported bound by the size of the surplus.[4] As expected, because all cash flow dates are used the extreme shift vector for duration, *N**, exhibits a sawtooth change, with about 80% of the worst shift concentrated on a fall in the 5 year yield and 17-20% on an increase in the 10 year rate.[5] This is an immediate implication of the limited exposure to cash flows in other time periods. However, even in this relatively simple portfolio management problem, the n_t* provide useful information about the worst case non-parallel term structure shift. Consistent with basic intuition, the worst type of non-parallel shift has a sizeable fall in midterm rates combined with smaller, but still significant, rise in long term rates. In practice, actual unit shifts cannot be determined arbitrarily but must satisfy basic absence of arbitrage conditions. When this is done, the extreme duration bounds can be compared with absence of arbitrage consistent N_i where the unit shift is maintained, an issue explored in Section 6.

INSERT TABLE 1

Following Poitras (2007), Table 1 is constructed to be roughly comparable to the example in Reitano (1991a, 1992, 1996). Table 2 extends this example by increasing the number of assets to include a maturity matching bond. As such, including such bonds addresses the source of the 'duration puzzle'. Table 2 provides results for two cases with similar surplus levels but with somewhat different asset compositions. The ***maturity bond*** portfolio involves a par bond with a term to maturity that matches that of the zero coupon liability (T=5). The ***split maturity*** portfolio does not include the maturity matching bond but, instead, uses 3 and 7 year par bonds. For both portfolios the 1/2 year zero coupon and 10 year coupon bond of Table 1 are included, with the position in the 10 year bond being the same in both portfolios. The 1/2 year bond position is permitted to vary, with the maturity matching portfolio holding a slightly higher market value of the 1/2 year asset. *A priori*, the split maturity portfolio would seem to have an advantage as four fixed income assets are being used to immunize instead of the three assets in the maturity matching portfolio.

INSERT TABLE 2

Given this, the results in Table 2 reveal that the portfolio with the maturity matching bond has much smaller extreme bounds even though more bonds are being selected in the split maturity portfolio. The partial durations reveal that, as expected, the presence of a maturity matching bond reduces the partial duration at T=5 compared to the split maturity case. The partial durations at T=3 and T=7 are proportionately higher in the split maturity case to account for the difference at T=5. The small difference in the partial duration at T=10 is due solely to the small difference in the size of the surplus. Examining the n_t* reveals that there is not a substantial difference in the sensitivity to changes in five year rates, as might be expected. Rather, the split maturity portfolio redistributes the interest rate sensitivity along the term structure. In contrast, the maturity bond portfolio is more heavily exposed to changes in 10 year rates. This greater exposure along the term structure by the split maturity portfolio results in wider extreme duration bounds because the norming restriction dampens the allowable movement in any individual interest rate. In other words, spreading interest rate exposure along the term structure by picking assets across a greater number of maturities acts to increase the exposure to extreme non-parallel shifts of unit length.

INSERT TABLE 3

Table 3 considers the implications of immunizing a liability with a decidedly different cash flow pattern. In particular, the liability being immunized is an annuity over T=10. The immunizing asset portfolios are a ‘maturity matching' portfolio similar to that in Table 2 combining the 6 month zero coupon with 5 year and 10 year bonds. The other case considered is a ‘low surplus’ portfolio, similar to that of Table 1, containing the 6 month zero and 10 year bond as assets. Table 3 reveals a significant relative difference between the extreme bounds for the two portfolios compared with the similar portfolios in Tables 1 and 2. The extreme bound for the low surplus portfolio has been reduced to about one third the value of the bound in Table 1 with the N* vector being dominated by the n_t* value for T=10. The extreme bound for the maturity bond portfolio has been reduced by just over one half compared to the extreme bound for the Table 2 portfolio with the N* vector being dominated by the n_t*

values at T=5 and T=10. When the extreme bounds for the two portfolios in Table 2 are multiplied by the size of the surplus, there is not much difference in the potential extreme change in the value of the surpluses between the two portfolios in Table 3. This happens because, unlike the zero coupon 5 year liability of Table 2, the liability cash flow of the annuity is spread across the term structure and the addition of the five year asset provides greater coverage of the cash flow pattern. In the annuity liability case, the dramatic exposure to the T=10 rate indicated by the n_t* of the low surplus portfolio is a disadvantage compared to the maturity bond portfolio which distributes the rate exposure between the T=5 and T=10 year maturities.

5. Partial Convexity Immunization Bounds and Time Value

While the partial duration immunization bounds are based on the Cauchy-Schwarz inequality, the partial convexity immunization bounds follow from the theory of quadratic forms. Recognizing that Γ_T is a real symmetric matrix permits a number of results from Bellman (1960, Sec.4.4, Sec. 7.2) to be accessed. In particular, if A is real symmetric then the characteristic roots will be real and have characteristic vectors (for distinct roots) that are orthogonal. Ordering the characteristic roots from smallest to largest, the following bounds apply to the quadratic form $N' \Gamma_T N$:

$$\lambda_1 \geq \lambda_2 \geq \ldots\ldots \geq \lambda_t \quad \rightarrow \quad \lambda_1 \geq N' \, \Gamma_T \, N \geq \lambda_T$$

Morrison (1976, p.73) develops these results further by recognizing that the solution to the optimization problem corresponds to the defining equation for characteristic vectors: $[\Gamma_T - \lambda I]\, N = 0$, where pre-multiplication by N' and use of the constraint gives: $\lambda = N' \, \Gamma_T \, N$. Hence, for the maximum (and minimum) characteristic roots of Γ_T, the extreme shift vector is the characteristic vector associated with that characteristic root.

One final point arising from the implementation of the partial immunization model concerns the associated convexity calculation. Consider the direct calculation of the partial convexity of surplus,

$CON_{i,j}^{S}$, where $i \neq j$:

$$S(z) = \sum_{t=1}^{T} \frac{CF_t}{(1 + z_t)^t} \quad \rightarrow \quad \frac{\partial S(z)}{\partial z_i} = \frac{i\ CF_i}{(1 + z_i)^{i+1}} \quad \rightarrow \quad \frac{\partial^2 S(z)}{\partial z_i\ \partial z_j} = 0 \quad for \quad i \neq j$$

where CF_t is the cash flow at time t for $t = \{1,2, ..., T\}$. From (1) and (2), it follows that the quadratic form $N' \Gamma_T^S N$ reduces to:

$$N' \Gamma_T^S N = \sum_{t=1}^{T} n_t^2\ CON_{t,t}^{S} = \sum_{t=1}^{T} n_t^2 \frac{1}{S(z_0)} \frac{\partial^2 S(z_0)}{\partial z_t^2}$$

In terms of the extreme bounds on convexity (see Appendix), this is a significant simplification. Because the *TxT* convexity matrix is diagonal, the extreme bounds are now given by the maximum and minimum diagonal ($CON_{i,i}$) elements. If the *i*th element is a maximal element, the associated extreme N^* vector for the convexity bounds is a *Tx1* with a one in the *i*th position and zeroes elsewhere.

In the classical model, for an instantaneous shift in the yield curve, higher convexity portfolios will outperform lower convexity portfolios of equal duration and yield. In practice, this result is problematic because the positive slope of the yield curve implies that higher convexity comes at the cost of lower time value. To this end, consider the multivariate Taylor series expansion of the function $S[y,t]$:

$$\frac{S[y,t] - S[y_0,\ t_0]}{S[y_0,\ t_0]} = \frac{1}{S}\frac{\partial S}{\partial y} (y - y_0) + \frac{1}{S}\frac{\partial S}{\partial t} (t - t_0)$$

$$+ \frac{1}{2} \{ \frac{1}{S}\frac{\partial^2 S}{\partial y^2} (y - y_0)^2 + \frac{1}{S}\frac{\partial^2 S}{\partial t^2} (t - t_0)^2 \} + \frac{1}{S}\frac{\partial^2 S}{\partial y\ \partial t} (y - y_0)(t - t_0) + \ldots\ H.O.T.$$

This solution can be manipulated to produce the result:

$$\% \Delta S \cong -DUR\ (y - y_0) + \theta\ (t - t_0)$$

$$+ \frac{1}{2} [\ CON\ (y - y_0)^2 + \Theta_2\ (t - t_0)^2\ ] + CROSS\ (y - y_0)(t - t_0)$$

where: θ is the theta or time value of the portfoli; Θ_2 relates to the second derivative with respect to time; and, *CROSS* relates to the cross product terms. Evaluating the first order term for the time value in the expansion for the surplus value function $S[z,t]$ produces:

$$\frac{1}{S}\frac{\partial S}{\partial t} = -\left(\sum_{t=1}^{T} \frac{CF_t \ln(1 + z_t)}{(1 + z_t)^T} \right) \frac{1}{S} = \theta \qquad (3)$$

The sign on the time value can be ignored by adjusting time to count backwards, e.g, changing time from t=20 to t=19 produces Δt = -1. Taking the Δt to be positive permits the negative sign to be ignored. Much like an option, time counts backwards for a fixed income portfolio: as time increases, the time to maturity goes down and the value of the positive surplus position increases, i.e., the time derivative is positive. Unlike options where the premium value falls as time increases, for θ to be taken as a positive value it is not necessary to define θ by multiplying by minus one.

Following Redington (1952), classical immunization requires the satisfaction of both duration and convexity conditions: duration matching of asset and liability cash flows is required, accompanied with higher asset convexity. The convexity requirement ensures that, for an instantaneous change in yields, the market price of assets will outperform the market price of liabilities. Yet, higher convexity does have a cost. In particular, when the yield curve is upward sloping, there is a tradeoff between higher convexity and lower time value (Christensen and Sorensen 1994, Poitras 2005, ch.5). To see this, evaluate the multivariate Taylor series using $y_0 = E[y] = \Omega$. Assuming $S = 0 = H.O.T. = \Theta_2$ produces:

$$E[\ \%\Delta S \mid y_0 = \Omega] = (\ CON_A - CON_L\ )\ \frac{\sigma_y^2}{2} - (\theta_A - \theta_L) = 0$$

$$\rightarrow \quad (\ CON_A - CON_L\ )\frac{\sigma_y^2}{2} = (\theta_L - \theta_A)$$

where: $\sigma_y^2 = E[\ (y - E[y])^2\ ]$; and, the subscripts A and L refer to assets and liabilities, respectively. This extension to the classical immunization condition that $CON_A > CON_L$ highlights a limitation of the univariate Taylor series expansion in (1): the portfolio surplus value function depends on time as well as the vector of spot interest rates. If yields do not change, higher convexity will typically result in a lower portfolio return due to the impact of the time value cost of convexity.

Though some progress has been made in exploring the relationship between convexity and time value in a one-factor setting (Chance and Jordan 1996, Barber and Copper 1997), it is not difficult to construct examples where non-parallel term structure shifts produce results contrary to the classical predictions, e.g., a pivot shift. The precise relationship between time value and convexity when non-parallel shifts are incorporated into the analysis is complicated by the different possible shifts and portfolio compositions. Given this, Table 4 provides incrementally more information on convexity and time value for the portfolios examined in Tables 1-3. Certain pieces of relevant information are repeated from Tables 1-3, i.e., the surplus and the extreme bounds for duration. Table 4 also provides the annualized time value, the sum of the partial convexities ($N_0' C_T N_0$), the maximum and minimum partial convexities and the quadratic form defined by the extreme-duration-shift convexities, $N^{*\prime} \Gamma_T N^*$, where $N^* = [n_1^*,....,n_T^*]'$ is the vector containing the n_t^*'s from Tables 1-3 and Γ_T is a diagonal matrix with the $CON_{t,t}$ elements along the diagonal.[6] The quadratic form calculated using the N^* for the extreme duration bound is of interest because it provides information about whether convexity will be improving or deteriorating when the extreme duration shift occurs.

Table 4 illustrates the relevance of examining the convexity and time value information, in conjunction with the partial duration results. In particular, consider a comparison with the maturity matching and the split maturity portfolios of Table 2. The primary result in Table 2 was that the split maturity portfolio had greater potential exposure to term structure shifts, as reflected in the wider extreme bounds associated with a term structure shift of length one. Whether this was a positive or negative situation was unclear, as the extreme bounds permitted both larger potential gains, as well larger potential losses, for the split maturity portfolio. Regarding the duration puzzle, Table 4 reveals that the maturity bond portfolio has a marginally higher time value. This happens because, despite having a higher surplus and a smaller holding of the 1/2 year bond, the split maturity portfolio has to hold a disproportionately larger amount of the three year bond relative to the higher yielding seven year bond to achieve duration matching of asset and liability cash flows. With an upward sloping term structure, this lower time value is combined with a ***higher*** convexity, as measured by $N_0' \Gamma_T N_0$. This supports the results from the one factor model where a tradeoff between convexity and time value is proposed, albeit for the classical stochastic model using a single interest rate process to capture the evolution of the term structure.[7]

INSERT TABLE 4

Table 4 also provides a number of other useful results. For example, comparison of the high and low Surplus portfolios from Table 1 adds the conclusions that, all other things equal, the time value will depend on the size of the surplus. However, as illustrated in Table 4, the relationship is far from linear. Even though the surpluses of the two portfolios from Table 1 differ by a factor of 10.9, the time values only differ by a factor of 2.5. High surplus portfolios permit a proportionately smaller amount of the longer term security to be held with corresponding impact on all the various measures for duration, convexity and time value. In addition, unlike the classical interpretation of convexity which is derived from the case where all net cash flows are positive, partial convexity of the surplus can, in general, take negative values and, in the extreme case, these negative values can be larger than the extreme positive values. However, this is not always the case, as evidenced in the Table 3 portfolios where the liability is an annuity. The absence of a future liability cash flow concentrated in a particular period produces a decided asymmetry in the Max *CON* and Min *CON* measures, with the Max values being much larger than the absolute value of the Min values. This is a consequence of the large market value of the 10 year bond relative to the individual annuity payments for the liability.

6. Immunization Benchmarks for Arbitrage Free Shifts

The appropriate procedure for immunizing a fixed income portfolio against arbitrage free term structure shifts is difficult to identify, e.g., Reitano (1996). Previous efforts that have approached this problem, e.g,, Fong and Vasicek (1984), have typically developed duration measures with weights on future cash flows depending on an arbitrage free stochastic process assumed to drive term structure movements. This introduces 'stochastic process risk' into the immunization problem. If the assumed stochastic process is empirically incorrect the immunization strategy will not perform as anticipated and may even underperform portfolios constructed using classical immunization conditions. In general, short of cash flow matching, it is not possible to theoretically solve the problem of designing a ***practical*** immunization strategy that can provide "optimal" protection against non-parallel term structure shifts. In the spirit of Hill and Vaysman (1998), it is possible to evaluate a specific portfolio's sensitivity to predetermined types of yield curve shifts. In practice, this will be sufficient for many purposes. For example, faced with a gently sloped yield curve and government policy aimed at keeping the short term rate near zero, a fixed income portfolio manager is likely to be more concerned about the impact of the yield curve steepening than with a further flattening. If there is some prior information about the expected change in location and shape of the yield curve, it is possible to determine the properties of portfolios that satisfy a surplus immunizing condition at the initial yield curve location.

The basic procedure for evaluating the impact of specific yield curve shifts requires a spot rate shift vector $(\Delta i)\ N_i = [(z_{1,1} - z_{1,0}), (z_{2,1} - z_{2,0}), \ldots, (z_{T,1} - z_{T,0})]'$ to be specified that reflects the anticipated shift from the initial location at $z_0 = [z_{1,0}, z_{2,0}, \ldots, z_{T,0}]'$ to the target location $z_1 = [z_{1,1}, z_{2,1}, \ldots, z_{T,1}]'$. This step begs an obvious question: what is the correct method for adequately specifying N_i? It is well known that, in order to avoid arbitrage opportunities, shifts in the term structure cannot be set arbitrarily, e.g., Boyle (1978); Barber and Copper (2006). If a stochastic model is used to generate shifts, it is required that N_i

be consistent with absence of arbitrage restrictions on the assumed stochastic model. These restrictions, which apply to the set of all possible *ex ante* paths generated from the stochastic model, are not needed when the set of assumed future shifts is restricted to *ex post* term structure movements based on historical experience. Where such empirical shift scenarios are notional, relevant restrictions for maintaining consistency between changes in individual spot rates are required. In terms of implied forward rates, necessary restrictions for absence of arbitrage take the form:

$$(1 + z_j)^{j} = (1 + z_i)^{i}(1 + f_{i,j})^{j-i} = (1 + z_1)(1 + f_{1,2})(1 + f_{2,3})....(1 + f_{j-1,j})$$

where the implied forward rates are defined as $(1 + f_{1,2}) = (1 + z_2)^2 / (1 + z_1)$ and $f_{j-1,j} = (1 + z_j)^j / (1 + z_{j-1})^{j-i}$ with other forward rates defined appropriately. This imposes a smoothness requirement on spot rates restricting the admissible deviation of adjacent spot rates. Considerable effort has been given to identifying stochastic models for generating appropriate forward rates that are consistent with absence of arbitrage and the observed empirical behaviour of interest rates, e.g., the Black, Derman and Toy model; the Heath-Jarrow-Morton model.

In contrast to such models of forward rates, the partial immunization bounds approach is considerably less ambitious. In addition to smoothness restrictions on adjacent spot rates, the partial immunization approach only requires that admissible N_i shifts satisfy the norming condition $\| N_i \| = 1$. In the set of unit length spot rate curve shifts, there are numerous shifts which do not satisfy the spot rate smoothness requirement. Because smoothing will allocate a substantial portion of the unit shift to spot rates that have small partial durations, restricting the possible non-parallel shifts by using smoothness restrictions tightens the convexity and duration bounds compared to the extreme bounds reported in Tables 1-4. To illustrate this, three scenarios for shifting the term structure of interest rates are considered: flattening with an upward move in level, holding the T=10 spot rate constant; flattening with a downward move in level, holding the T=6 month rate constant; and, flattening with a pivoting around the T=5 rate, where the $T > 5$ year rates fall and the $T < 5$ year rates rise. Given these scenarios, what remains is to specify the elements of N_i for shifts of unit length. The absence of arbitrage smoothness restrictions require that changes in term structure shape will distribute the shift proportionately along the term structure. For example, when flattening with an upward move in level, the change in the T=6 month rate would be largest, with the size of the shift getting proportionately smaller as T increases, reaching zero at T=10.

Solving for a factor of proportionality in the geometric progression, subject to satisfaction of the norming condition, produces a number of possible solutions, depending on the size of the spot rate increase at the first step. The following three unit length N_i shift vectors were identified:

Time	**Flatten Up (YC1)**	**Flatten Down (YC2)**	**Pivot (YC3)**
1	0.400	0.000	0.371
2	0.368	-0.090	0.180
3	0.339	-0.097	0.164
4	0.312	-0.106	0.146
5	0.287	-0.115	0.126
6	0.264	-0.125	0.105
7	0.243	-0.136	0.082
8	0.224	-0.147	0.057
9	0.206	-0.160	0.030
10	0.189	-0.174	0.000
11	0.174	-0.189	-0.032
12	0.160	-0.206	-0.067
13	0.147	-0.224	-0.105
14	0.136	-0.243	-0.146
15	0.125	-0.264	-0.191
16	0.115	-0.287	-0.240
17	0.106	-0.312	-0.293
18	0.097	-0.339	-0.351
19	0.090	-0.368	-0.413
20	0.000	-0.400	-0.481

As in (1) and (2), the actual change in a specific spot rate requires the magnitude of the shift to be given. Observing that each of these three scenario N_i vectors is constructed to satisfy the norming condition ‖

$N\| = 1$, the empirical implications of this restriction are apparent. More precisely, unit length shifts do not make distinction between the considerably higher volatility for empirical changes in short term rates compared to long term rates. Imposing both unit length shift and smoothness restrictions on spot rates is not enough to restrict the set of theoretically admissible shifts to capture all aspects of empirical consistency. Specific practical applications impose further empirically-based restrictions on the set of admissible shifts. These three empirically plausible term structure shift scenarios represent plain vanilla benchmarks for more complicated practical applications.

Given the three unit length spot rate curve shifts, Table 5 provides the calculated values associated with (2) and (3) for the six portfolios of Tables 1-3. Because the initial duration of surplus is approximately zero and the portfolio convexity ($N_0' \Gamma_T N_0$) is positive in all cases, classical immunization theory predicts that the portfolio surplus will not be reduced by yield changes. The information in Table 5 illustrates how the partial duration approach generalizes this classical immunization result to assumed non-parallel spot rate curve shifts. Comparison of the size of $N_i'D_{\underline{T}}$ with the Cauchy bound reveals the dramatic reduction that combining smoothness with unit length imposes on the potential change in surplus value. In particular, from (2) it follows that a negative value for the partial duration measure $N_i'D_{\underline{T}}$ is associated with an ***increase*** in the value of the fund surplus projected by the duration componentat $t = 0$. All such values in Table 5 are negative, consistent with $N_i'D_{\underline{T}}$ indicating all three curve shifts produce an additional increase in the value of surplus. As in Tables 1-3, the change in the surplus from the duration component can be calculated by multiplying the surplus by the $N_i'D_{\underline{T}}$ value and the assumed shift magnitude Δi. For every portfolio, the YC3 shift produced a larger surplus increase from the duration component than YC1 and YC2. Given that the YC3 shift decreases the interest rate for the high duration 10 year asset and increases the interest rate for the low duration 6 month asset, this result is not surprising. In contrast, while the YC2 shift increased surplus more than YC1 in most cases, the reverse result for the maturity bond portfolio with the annuity liability indicates how portfolio composition can matter when the term structure shift is non-parallel.

INSERT TABLE 5

Following Chance and Jordan (1996) and Poitras (2005, ch.5), interpreting the contribution from convexity depends on the assumed shift magnitude Δi^2. A positive value for the convexity component ($N_i' \Gamma_T N_i$) indicates an improvement in the surplus change in addition to the increase from $N_i' D_{\underline{T}}$. For all portfolios, there was small negative contribution from convexity for the term structure flattening up (YC1). When multiplied by empirically plausible values for Δi^2 the negative values are small relative to $N_i' D_{\underline{T}}$, indicating that the additional contribution from convexity does not have much additional impact. Results for the term structure flattening down (YC2) and the pivot (YC3) produced positive convexity values for all portfolios. While for empirically plausible shift magnitudes the convexity values for the YC2 shift were also not large enough to have a substantial impact on the calculated change in surplus, the YC3 values could have a marginal impact if the shift magnitudes were large enough. For example, assuming $\Delta i = .01$, the 23.7% surplus increase predicted by $N_i' D_{\underline{T}}$ in the low surplus portfolio in Table 1 is increased by 1.12% from the convexity contribution. Also of interest is the magnitude of $N_i' \Gamma_T N_i$ relative to $N_0' \Gamma_T N_0$. While the calculated $N_i' D_{\underline{T}}$ term is small in comparison to the Cauchy bound even for YC3, the calculated $N_i' \Gamma_T N_i$ is over half as large as $N_0' C_T N_0$ for YC3 and more than one third the value for YC2.

Results for partial durations and partial convexities are relevant to the determination of the change in surplus associated with various non-parallel term structure shifts. Table 5 also reports results for the change in time value for the six portfolios and three term structure shift scenarios. Following Christensen and Sorensen (1994) and Poitras (2005, ch.5), time value measures the rate of change in the surplus if the term structure remains unchanged over a time interval. For portfolios with equal duration and different convexities, such as those in Table 2, differences in time value reflect the cost of convexity. For a steep term structure, the cost of convexity is high and for a flat term structure the cost of convexity is approximately zero. From (3), it is apparent that term structure shifts will also impact the time value. While YC1-YC3 all reflect a flattening of the term structure, the level of the curve after the shift is different. In addition, because the calculation of time value involves discounting of future cash flows, it is not certain that an upward flattening in the level of the term structure (YC1) will necessarily produce a superior increase in time value compared to a flattening pivot of the term structure (YC3).

Significantly, the YC3 shift produced the largest increases in surplus for all portfolios except the high surplus portfolio of Table 1 where the YC1 shift produced the largest increase in time value. In all cases, the YC2 shift produced the smallest increase in time value. These results are not apparent from a visual inspection of the different shifts, which appear to favour YC1 where spot rates increase the most at all maturity dates. To see how this occurs, consider the partial time values from the low surplus portfolio in Table 1. For the initial term structure, θ_t associated with the largest cash flows are $\theta_{½} = .18698$, $\theta_5 = -.97942$ and $\theta_{10} = .30677$. For YC3, these values become $\theta_{½} = .19513$, $\theta_5 = -.97942$ and $\theta_{10} = .30990$ while for YC1 the values are $\theta_{½} = .19576$, $\theta_5 = -.98683$ and $\theta_{10} = .30677$. While the pivot leaves the time value of the liability unchanged and increases the time value of the principal associated with the10 year bond, the overall upward shift in rates associated with YC1 is insufficient to compensate for the negative impact of the rate increase for the liability. Comparing this to the high surplus case where YC1 had a superior increase in time value compared to YC3, the initial term structure values for the high cash flow points are $\theta_{½} = .056045$, $\theta_5 = -.090227$ and $\theta_{10} = .026552$ which change to $\theta_{½} = .058678$, $\theta_5 = -.09091$ and $\theta_{10} = .034144$ for YC1 and $\theta_{½} = .058489$, $\theta_5 = -.090227$ and $\theta_{10} = .034493$. Because the higher surplus portfolio has more relative asset value, the higher overall level of rates associated with YC1 is reflected in the time value change

Finally, Table 5 provides further evidence on the duration puzzle. Based on the results in Table 4, comparison of the maturity matching portfolio with the split maturity portfolio reveals the time value-convexity tradeoff identified by Christensen and Sorensen (1994). The higher reported time value measure for the maturity bond portfolio was offset by the higher convexity values for the split maturity portfolio. Under classical conditions, this implies a superior value change for the split maturity portfolio if interest rates change sufficiently. Yet, in Table 5 the maturity matching portfolio has a larger change in surplus than the split maturity portfolio for the flattening up shift (YC1) while retaining the time value advantage across all three scenarios. However, for both the flattening down and the pivot shifts, the surplus increase for the split maturity portfolio does outperform the maturity matching portfolio as expected. This is another variant of the duration puzzle. Because the result does not apply to all three scenarios, this implies that the duration puzzle is not a general result but, rather, is associated with specific types of term structure shifts. If this result extends to real time data, the presence of the duration puzzle can be attributed to the prevalence of certain types of term structure shifts compared to other types. The presence of the duration puzzle in the selected scenarios is due to a complicated interaction between the partial durations, partial convexities and time values. As such, the partial immunization approach is well suited to further investigation of this puzzle.

7. Conclusions

In the spirit of Redington (1952), there is a considerable distance to travel from the simple portfolio illustrations used in academic discussions of fixed income portfolio immunization to the complex risk management problems arising in financial institutions such as pension funds, life insurance companies, securities firms and depository institutions. Cash flow patterns in financial institutions are decidedly more complicated. Not only are the cash flows more numerous, there is also an element of randomness that is not easy to model. This paper develops a theoretical method for assessing the impact that non-parallel term structure shifts have on fixed income portfolio immunization. This is done by changing the type of data that is needed to implement the risk management strategy. Whereas classical duration can make use of the simplification that the portfolio duration is the value weighted sum of the durations of the individual assets and liabilities, the approach used here requires the net cash flows at each payment date to be determined. In simple portfolio illustrations this requirement is not too demanding. However, this requirement could present problems to a financial institution faced with large numbers of cash flows, a significant fraction of which may be relatively uncertain. Extending the partial immunization approach to incorporate randomness in cash flows would provide further insight into practical immunization problems.

The simplicity of the classical approach to fixed income portfolio immunization is appealing. Immunizing portfolios can be readily constructed even where there are large numbers of assets and liabilities involved. Unfortunately, classical immunization breaks down when the term structure of interest rates is not flat and shifts are not parallel. By developing a methodology that directly incorporates non-parallel term structure shifts into the immunization problem, it is possible to analyse the properties of classical immunization strategies by comparing the performance of classical portfolios with those constructed using other methodologies. In particular, classical immunizing portfolios can be analysed using time value, partial duration and partial convexity measures. For different types of initial term structure shape and different specifications of the shift direction vector, the performance of classical portfolios can be compared with portfolios constructed to satisfy conditions derived from the partial immunization approach. This can provide useful information about the types of situations in which classical immunization does poorly relative to other types of portfolios. This paper provides a framework for structuring such comparisons, suggesting a number of additional directions for future research.

Appendix

Solution for the Extreme Duration Bound Weights $n_1^*, n_2^*, \ldots, n_T^*$:

The optimization problem is to set the *N* weighted sum of the partial durations of surplus equal to zero, subject to the constraint that the *N'N* equals the norming value. Without loss of generality assume that the norming value is one, which leads to:

$$\underset{\{n_i\}}{opt}\ L = \sum_{i=1}^{k} n_i\, D_i - \lambda(\{\sum_{i=1}^{k} n_i^2\} - 1)$$

This optimization leads to $k + 1$ first order conditions in the k n_i and λ. Observing that the first order condition for the n_i can be set equal to $\lambda/2$, appropriate substitutions can be made into the first order condition for λ that provides the solution for the individual n_t^* weights.

Solution for the Extreme Convexity Bound Weights $n_1, n_2, \ldots, n_T$:

The optimization problem for convexity involves the objective function:

$$\underset{\{n_i\}}{opt}\quad L_c = N^{\prime}\, \Gamma_T\, N - \lambda\ (N^{\prime}N - 1)$$

Recognizing that Γ_T is a real symmetric matrix permits a number of results from Bellman (1960, Sec.4.4, Sec. 7.2) to be accessed. In particular, if *A* is a real symmetric matrix then the characteristic roots will be real and have characteristic vectors (for distinct roots) that are orthogonal. Ordering the characteristic roots from smallest to largest, the following bounds apply to the quadratic form $N'\,\Gamma_T\,N$:

$$\lambda_1 \geq \lambda_2 \geq \ldots\ldots \geq \lambda_t \qquad \rightarrow \qquad \lambda_1 \geq N^{\prime}\, C_T\, N \geq \lambda_T$$

Morrison (1976, p.73) develops these results further by recognizing that the solution to the optimization problem corresponds to the defining equation for characteristic vectors: $[\Gamma_T - \lambda I]N = 0$, where premultiplication by N' and use of the constraint gives: $\lambda = N'\,\Gamma_T\,N$. Hence, for the maximum (and minimum) characteristic roots of Γ_T, the optimum shift vector is the characteristic vector associated with that characteristic root.

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Table 1

Partial durations, extreme bounds and shift weights {n_t*} for the High and Low Surplus Examples*

	High Surplus		Low Surplus	
Date	D_t	n_t*	D_t	n_t*
0.5	0.687	0.0775	2.292	0.0236
1.0	0.075	0.0085	0.870	0.0089
1.5	0.107	0.0121	1.238	0.0128
2.0	0.136	0.0153	1.569	0.0162
2.5	0.161	0.0182	1.863	0.0192
3.0	0.183	0.0206	2.111	0.0217
3.5	0.201	0.0226	2.318	0.0239
4.0	0.214	0.0241	2.472	0.0254
4.5	0.226	0.0255	2.611	0.0269
5.0	−7.933	−0.89545	−86.115	−0.88658
5.5	0.244	0.0275	2.815	0.0290
6.0	0.246	0.0277	2.837	0.0292
6.5	0.247	0.0279	2.854	0.0294
7.0	0.246	0.0278	2.845	0.0293
7.5	0.243	0.0275	2.810	0.0289
8.0	0.242	0.0273	2.792	0.0287
8.5	0.239	0.0269	2.758	0.0284
9.0	0.231	0.0260	2.664	0.0274
9.5	0.221	0.0249	2.551	0.0263
10.0	3.786	0.42733	43.742	0.45033
Extreme Duration Bounds:	Cauchy = ‖D‖ ± 8.860%		Cauchy = ‖D‖ ± 97.131%	
Surplus:	50.75		4.66735	

* The High Surplus Portfolio is composed of ($68.3715) 1/2 year and ($69.89445) 10 year bonds. The Low Surplus Portfolio is composed of ($17.8382) 1/2 year and ($74.343) 10 year bonds. The liability for the High Surplus Portfolio is a 5 year zero coupon bond with $100 par value and market value of $58.3427. The liability for the Low Surplus Portfolio is a 5 year zero coupon bond with $150 par value and market value of $87.51. The extreme Cauchy bounds are derived using ‖ N ‖ = 1.

Table 2

Partial durations, extreme bounds and shift weights {n_t*} for the Maturity Bond and Split Maturity Examples*

	Maturity Bond		Split Maturity	
Date	D_t	n_t*	D_t	n_t*
0.5	0.476	0.0183	0.251	0.0063
1.0	0.454	0.0174	0.446	0.0112
1.5	0.646	0.0248	0.635	0.0159
2.0	0.818	0.0314	0.804	0.0201
2.5	0.971	0.0373	0.955	0.0239
3.0	1.101	0.0422	7.737	0.1939
3.5	1.208	0.0464	0.834	0.0209
4.0	1.288	0.0494	0.889	0.0223
4.5	1.361	0.0522	0.939	0.0235
5.0	−23.866	−0.91558	−36.998	−0.92716
5.5	0.636	0.0244	1.012	0.0254
6.0	0.641	0.0246	1.020	0.0256
6.5	0.645	0.0247	1.026	0.0257
7.0	0.643	0.0247	8.177	0.2049
7.5	0.635	0.0244	0.601	0.0151
8.0	0.631	0.0242	0.597	0.0150
8.5	0.623	0.0239	0.590	0.0148
9.0	0.602	0.0231	0.569	0.0143
9.5	0.577	0.0221	0.545	0.0137
10.0	9.884	0.37921	9.350	0.23430
Extreme Duration Bounds:	Cauchy = ‖D‖ ± 26.07%		Cauchy = ‖D‖ ± 39.905%	
Surplus:	10.32685		10.91804	

* The Maturity Bond Portfolio is composed of ($5.13) 1/2 year, ($55.54) 5 year and ($37.172) 10 year bonds. The Split Maturity Portfolio is composed of ($0.4105) 1/2 year, ($33.8435) 3 year, (27.0) 7 year and ($37.172) 10 year bonds. The liability is a 5 year zero coupon bond with $150 par value and market value of $87.51. The extreme Cauchy bounds are derived using ‖ N ‖ = 1.

Table 3

Partial durations, extreme bounds and shift weights $\{n_t*\}$ for the 10 Year Annuity Liability Immunized with the Maturity Bond and Low Surplus Examples*

	Maturity Bond		Low Surplus	
Date	D_t	n_t*	D_t	n_t*
0.5	1.034	0.0841	2.856	0.0869
1.0	−0.275	−0.0224	−0.705	−0.0214
1.5	−0.392	−0.0319	−1.003	−0.0305
2.0	−0.497	−0.0404	−1.271	−0.0387
2.5	−0.590	−0.0480	−1.509	−0.0459
3.0	−0.668	−0.0543	−1.710	−0.0520
3.5	−0.734	−0.0597	−1.877	−0.0571
4.0	−0.782	−0.0636	−2.001	−0.0609
4.5	−0.826	−0.0672	−2.115	−0.0643
5.0	8.257	0.67159	−2.201	−0.0670
5.5	−1.401	−0.1139	−2.280	−0.0694
6.0	−1.412	−0.1148	−2.298	−0.0699
6.5	−1.420	−0.1155	−2.311	−0.0703
7.0	−1.415	−0.1151	−2.304	−0.0701
7.5	−1.398	−0.1137	−2.276	−0.0693
8.0	−1.389	−0.1130	−2.262	−0.0688
8.5	−1.372	−0.1116	−2.234	−0.0680
9.0	−1.325	−0.1078	−2.157	−0.0656
9.5	−1.269	−0.1032	−2.066	−0.0629
10.0	7.855	0.63888	31.647	0.9629
Extreme Duration Bounds:	Cauchy = ‖D‖ ± 12.29%		Cauchy = ‖D‖ ± 32.87%	
Surplus:	10.5979		4.68	

* The Maturity Bond Portfolio is composed of ($25.97) 1/2 year, ($34.956) 5 year and ($37.172) 10 year bonds. The Low Surplus Portfolio is composed of ($31.388) 1/2 year and ($60.791) 10 year bonds. The liability has market value of $87.500 with annual coupon, paid semi-annually, of $14.96. The extreme bounds are derived using ‖ N ‖ = 1.

Table 4

Time Value, Convexity and Other Measures for the Immunizing Portfolios*

TABLE 1	High Surplus	Low Surplus
Surplus	50.75	4.667
Time Value = $2N_0'\Theta$	0.07118	0.0278
$N_0'\ C_T\ N_0$	17.18	221.94
$N^{*}{}'\ C_T\ N^{*}$	-25.34	-259.59
Max CON_t	37.23	430.09
Min CON_t	-41.34	-448.79
Cauchy Duration Bound	± 8.86%	± 97.13%
$N_0'D_{\underline{T}}$	-0.000	-0.102

TABLE 2	Maturity Bond	Split Maturity
Surplus	10.327	10.918
Time Value = $2N_0'\Theta$	0.0664	0.0648
$N_0'\ C_T\ N_0$	42.37	44.21
$N^{*}{}'\ C_T\ N^{*}$	-85.39	-142.06
Max CON_t	97.19	91.93
Min CON_t	-124.38	-192.81
Cauchy Duration Bound	± 26.07%	± 39.90%
$N_0'D_{\underline{T}}$	-0.025	-0.022

TABLE 3	Maturity Bond	Low Surplus
Surplus	10.598	4.68
Time Value = $2N_0'\Theta$	0.0724	0.0510
$N_0'\ C_T\ N_0$	11.96	109.34
$N^{*}{}'\ C_T\ N^{*}$	21.61	287.65
Max CON_t	77.23	311.17
Min CON_t	-11.89	-19.36
Cauchy Duration Bound	±12.29%	±32.86%
$N_0'D_{\underline{T}}$	-0.019	-0.077

* See Notes to Tables 1-3. Making appropriate adjustment for semiannual payments, the annualized time value $2N_0'\Theta$ is defined in (3). The sum of the partial convexities is $N_0'C_T\ N_0$ The quadratic form, $N^{*}{}'\ C_T\ N^{*}$, is the sum of squares for the relevant N* from Tables 1-3 multiplied term-by-term with the appropriate partial convexities. Max CON and Min CON are the maximum and minimum individual partial convexities.

Table 5

Partial Durations and Convexities for the Immunizing Portfolios under Different Term Structure Shift Assumptions*

TABLE 1

	High Surplus			Low Surplus		
Surplus		50.75			4.667	
	YC1	YC2	YC3	YC1	YC2	YC3
$N_i'D_T$	−0.606	−.879	−1.875	−8.205	−11.127	−23.75
$N_i'C_T\ N_i$	−0.828	6.082	9.778	−9.398	71.151	112.23
$2N_0'\Theta_i$	.09144	.0878	.08922	.0436	.0408	.0554
Cauchy Bound ($\lVert D_T\rVert$)		±8.86%			±97.13%	
$N^{*\prime}\ C_T\ N^*$		−25.34			−259.59	
$N_0'\ C_T\ N_0$		17.18			221.94	
Time Value = $2N_0'\Theta$		.07118			.0274	

TABLE 2

	Maturity Bond			Split Maturity		
Surplus		10.327			10.918	
	YC1	YC2	YC3	YC1	YC2	YC3
$N_i'D_T$	−1.532	−2.328	−4.955	−1.451	−2.478	−5.289
$N_i'C_T\ N_i$	−1.976	15.667	25.489	−2.039	16.281	25.51
$2N_0'\Theta_i$	.07274	.06864	.07398	.07108	.06674	.07182
Cauchy Bound ($\lVert D_T\rVert$)		±26.06%			±39.90%	
$N^{*\prime}\ C_T\ N^*$		−85.39			−142.06	
$N_0'\ C_T\ N_0$		42.37			44.21	
Time Value = $2N_0'\Theta$		.0664			.0648	

TABLE 3

	Maturity Bond			Low Surplus		
Surplus		10.598			4.68	
	YC1	YC2	YC3	YC1	YC2	YC3
$N_i'D_T$	−0.874	−0.638	−1.360	−5.086	−5.287	−11.298
$N_i'C_T\ N_i$	−0.847	5.902	12.01	−5.039	36.520	62.55
$2N_0'\Theta_i$	.07736	.07408	.07592	.0616	.0600	.07156
Cauchy Bound ($\lVert D_T\rVert$)		±12.29%			±32.86%	
$N^{*\prime}\ C_T\ N^*$		21.61			267.49	
$N_0'\ C_T\ N_0$		11.96			109.34	
Time Value = $2N_0'\Theta$		.0724			.0510	

* See Notes to Tables 1-3. YC1 has the term structure flattening up, with the T=10 rate constant; YC2 has the term structure flattening down with the T=.5 (6 month) rate constant; and, YC3 has a flattening pivot with the T=5 year rate constant. The annualized term $2N_0'\Theta_i$ evaluates (3) using the N_i shifted spot rates.

NOTES

1. Various sources fail to distinguish 'yield curve' shift from 'term structure' shift. More precisely, the 'term structure' refers to the relationship between spot interest rates (implied zero coupon interest rates) and term to maturity while the 'yield curve' refers to the relationship between observed yields to maturity and term to maturity. Because spot interest rates are bootstrapped from the observed yield curve for riskless government fixed income securities, the notions are closely related.

2. D'Antonio and Cook (2004) provide a useful overview of various approaches to the classical convexity formulas.

3. This ***par bond*** curve has semi-annual yields from 6 months to 10 years, i.e., y = (.08, .083, .089, .092, .094, .097, .10, .104, .106, .108, .109, .112, .114, .116, .118, .119, .12, .122, 124, .125)'. The 6 month and 1 year yields are for zero coupon securities, with the remaining yields applying to par coupon bonds. This yield curve produces the associated spot rate curve, z = (.08, .083, .0893, .0925, .0946, .0979, .1013, .106, .1083, .1107, .1118, .1159, ,1186, .1214, .1243, .1256, .1271, .1305, .1341, .1358)'.

4. The relevant calculation for the change in surplus value with a shift magnitude of $\Delta i = .01$ using the extreme Cauchy bound for the low surplus portfolio would be (.01)(97.131)(4.66735) = 4.5334.

5. To see this, recall that the length of the shift vector is one. As a consequence, the sum of the squares will equal one and the square of each n^* is the percentage contribution of that particular rate to the extreme directional shift vector.

6. This follows from the previous discussion which identified the off-diagonal elements as being equal to zero.

7. In these one factor models, the spot interest rates for various maturities are constructed by taking the product of the generated one period interest rates along each specific path, treating each future short term interest rate as a forward rate. This will produce the predicted spot rate structure for each path, e.g., Fabozzi (1993, Chp.13). While useful for generating analytical results, it is well known that such processes cannot capture the full range of potential term structure behaviour. Without restrictions on the paths, some paths may wander to zero and the average over all the paths may not equal the actual observed spot interest rate curve.