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Semigroup Properties of Arbitrage Free Pricing Operators

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ABSTRACT

This paper provides a substantive connection between the infinitesimal generator of the semigroup of pricing operators and conditions required for absence of arbitrage in security prices. The practical importance of recognizing the semigroup properties is illustrated by considering the implications for empirical tests of discounted dividend models of stock prices. The convention in these empirical studies is to invoke some ad hoc method to detrend price-dividend data in order to achieve the statistical property of covariance stationarity. However, considering properties of the semigroup of arbitrage free pricing operators reveals that covariance stationarity alone is an insufficient statistical requirement. Under appropriate assumptions, further conditions on the underlying probability distributions must be satisfied to ensure absence of arbitrage requirements.

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In financial economics, the use of diffusion processes in solving continuous time valuation problems provides a potential connection to operator semigroup concepts. More precisely, application of the Hille-Yosida theory of operator semigroups permits a useful representation of the Kolmogorov backward and forward equations associated with Markov processes, e.g., Taira (1988). Developing this approach, Garman (1985), Duffie and Garman (1986), and Heaney and Poitras (1991, 1993) explore connections between the widespread use of diffusion processes in financial valuation problems and semigroup properties of bounded linear pricing operators {V(t)} which obey the semigroup condition: $V(t+s) = V(t)\,V(s)$.[1] This paper provides a substantive connection between specification of the infinitesimal generator of the semigroup and conditions required for arbitrage free prices (Harrison and Kreps 1979, Back and Pliska 1991). This connection is developed using the Radon-Nikodym (R-N) derivative associated with the equivalent martingale measure to impose absence of arbitrage restrictions on the semigroup of pricing operators.

The practical importance of recognizing the semigroup properties of arbitrage free pricing operators is considerable. This paper focuses on the implications for empirical tests of Gordon discounted dividend models, e.g., Shiller (1981), Campbell and Shiller (1987), Campbell and Kyle (1993), and tests of related null hypotheses, such as implied variance bounds tests.[2] The convention in these empirical studies is to invoke some ad hoc method to detrend price-dividend data in order to achieve the statistical property of covariance stationarity. However, considering properties of the semigroup of pricing operators reveals that covariance stationarity alone is an insufficient statistical requirement. Further conditions on the underlying probability distributions are required in order to satisfy absence of arbitrage requirements. In practice, this means that the detrending procedure cannot be selected arbitrarily but must reflect the equilibrium conditions associated with absence of arbitrage in security prices. This paper demonstrates how, under the appropriate conditions, it is possible to derive a closed form for the "optimal" detrender.

I. Introduction to Arbitrage Free Semigroups

Operator semigroups arise in continuous time valuation problems of the general form:

$$p(0) = V(T)\; p(T) + \int_0^T V(u)\; c(u)\; du \qquad (1)$$

where p(t) is the price at time t, c(t) is the instantaneous dividend or coupon paid at t, and T is the terminal or maturity date for the valuation problem. The semigroup structure is associated with the "valuation operators" {V(t)}. (1) applies to numerous valuation problems. For example, (1) represents a deterministic, continuous time bond pricing problem, where T is the maturity date, p(T) is the principal, c(t) is the coupon paid at t and {V(t)} is the associated discounting operator with the requisite property $(1+r)^{-(t+s)} = (1+r)^{-t}(1+r)^{-s}$.[3] For present purposes, a more relevant interpretation of (1) is as a continuous time version of the Gordon price-dividend (stock) pricing model, where p(T) is the anticipated (selling) price at time T and c(t) is the continuous dividend. Progressive substitution for p(T) provides the infinite horizon, discounted-dividend model.

In valuation problems where p(T) and c(t), $t \in [0,T]$, are uncertain at the t= 0 decision date, the {V(t)} take the form of expectations operators, which also possess a discounting feature. In terms of the price-dividend model, (1) can now be expressed:

$$p(0) = E\left[\frac{p(T)}{(1 + k_T)^T} \mid X(0)\right] + E\left[\int_0^T \frac{c(s)}{(1 + k_s)^s}\, ds \mid X(0)\right]$$

where X(t) is the conditioning information available at time t and k_i is the appropriate stochastic discounting factor for period i. The progressive substitution for E[p(T) | X(0)] in (1) required for the discounted dividend model involves applying the semigroup property to expected future prices:

$$E\left[\frac{p(T)}{(1+k_T)^T} \mid X(0)\right] = E\left[\frac{1}{(1+k_T)^T} \left(E\left[\frac{p(T+k)}{(1+k_{T+k})^k} + \int_T^{T+k} \frac{c(T+s)}{(1+k_{T+s})^s}\, ds \mid X(T)\right]\right) \mid X(0)\right]$$

$$= E\left[\frac{p(T+k)}{(1 + k_{T+k})^{T+k}} \mid X(0)\right] + E\left[\int_T^{T+k} \frac{c(T+s)}{(1 + k_{T+s})^{T+s}}\, ds \mid X(0)\right] \qquad (2)$$

To be practical, this formulation requires stochastic properties of the state variables to be identified in order to precisely specify {V(t)}. Selection of a Markovian, diffusion stochastic structure permits direct application of operator semigroup concepts, e.g., Karlin and Taylor (1981), to the analysis associated with differentiating (1).[4]

Development of the derivative of (1) as $t \to 0$ provides the basis for defining the **infinitesimal generator** of the semigroup as the linear operator A:

$$Ap = \lim_{t \to 0} \frac{1}{t}\{V(t)p - p\}$$

This connection between A and {V(t)} is fundamental. In effect, the operator semigroup {V(t)} which provides the {p(t)} solutions to (1) has, under applicable assumptions, an associated differential operator A. This infinitesimal generator is central to asset valuation equations of the form: $\partial p/\partial t = -[Ap + c]$. Such differential operators occur frequently in arbitrage free valuation problems based on diffusion state variables. A classical example of this formulation is the Black-Scholes (1973) PDE where p is the call option price, c = 0, and X is the underlying state variable, the nondividend paying stock price, which follows a geometric diffusion $dX = \mu X\, dt + \sigma X\, dB$. In this case, A takes the specific form of Φ:

$$\mathbf{\Phi} = r - rX\frac{\partial}{\partial X} - \frac{1}{2}\sigma^2 X^2 \frac{\partial^2}{\partial X^2}$$

where r is the continuously compounded rate of interest, which is assumed to be constant. Numerous other examples of A can be found in Ingersoll (1987) and Hull (1989). In the case of diffusion state processes, A takes the form of a partial differential equation operator which is second order in state.

For the price-dividend interpretation of (1), the infinitesimal generator of the semigroup can provide precise information about how the determination of p depends explicitly on time and the dividend. Considerable analytical effort has been devoted to deriving specific functional forms for A which are consistent with absence of arbitrage, e.g., Cox, Ingersoll and Ross (1985). The first recognition of a connection between absence of arbitrage in security prices and the infinitesimal generator of the semigroup of pricing operators was presented by Garman (1985):[5]

Proposition (Garman): **Semigroup Properties of Pricing Operators**

If A is the infinitesimal generator of some semigroup of arbitrage free pricing operators $\{V(t),\ t \geq 0\}$, then the dividend stream c(t) generates the price p(t) via the equation:

$$\frac{\partial p}{\partial t} = -[\ Ap(t) + c(t)\ ]$$

While useful, Garman's Proposition does not provide a direct connection between a specific A and the associated semigroup of operators {V(t)} which solve (1). As observed by Huang (1985), a substantive connection between the existence of the infinitesimal generator and conditions required for arbitrage free prices is not provided in the Proposition.

Embedded in the specification of an infinitesimal generator for a given valuation problem are a collection of assumptions and null hypotheses, including assumptions made about the stochastic behaviour of the state variables. From the initial structure, specific A are constructed to insure absence of arbitrage opportunities in {p(t)}. Under reasonably general conditions, the Hille-Yosida theorem states that the presence of an infinitesimal generator ensures the existence of an operator semigroup. The presence of arbitrage free prices permits introduction of the equivalent martingale measure associated with absence of arbitrage opportunities in security prices, e.g., Harrison and Kreps (1979), Back and

Pliska (1991). This martingale measure is related to the assumed empirical measure assigned to the state variables through the R-N derivative which transforms the assumed empirical measure into the equivalent martingale measure. Under appropriate conditions, the R-N derivative can be expressed in closed form. Heuristically, this is significant because the product of the R-N derivative and the observed price is the arbitrage free shadow price which, by construction, follows a martingale.

For empirical testing purposes, the martingale property is attractive because, with limited additional statistical structure, a martingale satisfies the essential requirement of covariance stationarity.[6] While there are many potential methods of detrending prices and dividends to achieve the covariance stationarity, the presence of an equivalent martingale measure dictates that there is only one method of detrending which is consistent with absence of arbitrage opportunities in security prices. It follows that the ad hoc methods conventionally used to detrend prices and dividends for empirical testing purposes may be inadequate. In semigroup terms: even though the semigroup structure of {V(t)} can be exploited to specify the arbitrage free detrending procedure, semigroup structure of the valuation operators alone does not guarantee the relevant prices are arbitrage free. In practical terms: even though there are numerous methods to detrend price and dividend data to achieve a martingale process, there is only one detrending process which ensures absence of arbitrage in the resulting covariance stationary process.

II. Arbitrage Free Semigroups with State Variables as Prices

Identifying the relevant state variables is a significant practical complication which arises in the development of equivalent martingale results. Following Harrison and Kreps (1979), it is conventional to assume that the requisite state variables are prices on non-dividend paying securities. In practical applications this structure is, typically, unnatural. For example, many securities pay "dividends" or coupons. For purposes of relaxing the no-dividends condition, it is convenient to work with the cumulative dividend adjusted price process. This requires specification of the stochastic price processes and a method for handling dividends. Because arbitrage free pricing requires each state variable to be associated with a risk premium, significant analytical simplification can be achieved by specifying dividends to be a constant proportion of stock prices, e.g., Merton (1973). Introducing distinct dividend processes produces complications similar to including other non-price state variables such as interest rates and volatility. This case is examined, by example, in Section III.

In addition to making security prices the only state variables, another substantive analytical simplification is achieved by working with processes which have been detrended by the riskless interest rate. This simplification is often achieved by assuming that interest rates are zero, directly suppressing consideration of issues associated with the numeraire. For present purposes, this permits the R-N derivative to be used, without further transformation, to derive arbitrage free shadow prices. For empirical applications, the primary implication of using this approach is that observed price data are not of direct concern. Rather, security prices and dividends detrended by the interest rate are the relevant state variables.[7] Given this, consider a security market with K interest rate detrended prices, S, which follow the Kx1 vector diffusion price process:

$$dS = \alpha(S,t)\ dt + \sigma(S,t)\ dB$$

Arbitrage free pricing requires the specification of K risk premia (λ) of the form: $\lambda = \alpha(S,t) + D$ where D is the interest rate detrended dividend and $\alpha(\cdot)$ is the drift coefficient for the diffusion process.

More formally, the basic analytical structure requires a probability space $(\Omega, \{F_t\}, P)$ where $0 \leq t < T$, and the filtration $\{F_t\}$ satisfies the usual conditions: F_0 contains all the null sets of P; and, $\{F_t\}$ is right continuous, meaning that $F_t = \cap_{s>t} F_s$. S(t) denotes a K-dimensional vector of $\{F_t\}$ measurable state variables defined on $[0,\infty) \times \Omega$. $\alpha(S,t) \in R^K$ and

$\sigma(S,t) \in R^K \otimes R^K$ both are F_t measurable and defined on $[0,T] \otimes R^K$. S(t) is assumed to be a continuous real valued diffusion process that takes on values on R^K. Given that B(t) is a K-dimensional vector of Brownian motions defined on $(\Omega, \{F_t\}, P)$, let Q be a martingale measure on Ω equivalent to P. The method of constructing Q from P depends fundamentally on the R-N derivative, Z. More precisely, for all $F_i \in \{F_t\}$:

$$\int_{F_i} dP = \int_{F_i} \frac{dP}{dQ}\, dQ = \int_{F_i} Z\, dQ$$

Necessary and sufficient conditions for the general existence of a measurable Z transformation is provided by the Radon-Nikodym Theorem. Further properties of the change of measure to security pricing situations can be derived from Girsanov' s Theorem, e.g., Duffie (1988), Cheng (1991).

The method of transforming observed prices and dividends into processes consistent with covariance stationarity is directly related to the transformation from the observed, P measure to the equivalent martingale, Q measure. Arbitrage free pricing requires that S(t) and D(t), the price and dividend processes, be such that the cumulative price-dividend process:

$$S(t) + \int_0^t D(u)\, du$$

is a martingale under Q.[8] The precise role played by Z in achieving this result can be formalized in:

<u>Proposition</u> 1: **Arbitrage Free Shadow Pricing**[9]

There exists a non-negative martingale Z(t) on $(\Omega, \{F_t\}, P)$, such that for the cumulative price dividend process, the transformed process:

$$Z(t)S(t) + \int_0^t Z(u)\, D(u)\, du$$

is a martingale on $(\Omega, \{F_t\}, P)$, i.e., for $t > 0$:

$$S(0) = E^{P}\,[\int_{0}^{t} Z(u)D(u)\;du \;+\; Z(t)S(t)\;|\;F_0] \qquad (3)$$

In terms of the semigroup {V(t)} defined by (2), Proposition 1 reveals that {Z(t)} corresponds to the stochastic discounting function, $\{(1 + k_t)^t\}$ where k has been adjusted to account for the interest rate detrending of S and D. The expectation in (3) is appropriate for empirical testing purposes because it is taken with respect to the empirical P measure. Assumptions made about the specific functional form of the state variable processes represent a nested null hypothesis under which a specific Z is the appropriate, arbitrage free detrender for prices and dividends.

For purposes of empirically testing price dividend models, Proposition 1 suggests a general outline for an arbitrage free detrending procedure. Observed prices and dividends, p and c, are initially detrended by the interest rate. The resulting S and D series are then multiplied by the Z applicable to the valuation problem at hand. The specific Z used will depend on both the parameters of the underlying state variable processes and the associated state variable risk premia. To be practical, implementation of this detrending procedure depends crucially on having a manageable closed form solution for Z. The significance of taking the S(t) to be diffusions is that operator semigroup notions can be employed to derive a specific functional form for Z. The diffusion structure of the underlying valuation problem permits Z(t) to be twice differentiable in the state variables, and once differentiable in time. From Proposition 1, since Z(t) obeys the requisite differentiability requirements and is a martingale, it follows by Ito's lemma that it satisfies a set of partial differential equations which can then be solved to get a specific Z.

Under the same assumptions as in Proposition 1, it follows:

Proposition 2: **Differential Properties of Z**

Z(t) obeys the K+ 1 first order partial differential equations:

$$\frac{\partial}{\partial s_i} Z(S,t) = b_i(t)\ Z(S,t) \qquad (4a)$$

$$\frac{\partial}{\partial t} Z(S,t) = f(t)\ Z(S,t) \qquad (4b)$$

where:

$$b = -\Sigma^{-1}\lambda$$

$$\Sigma = \sigma\sigma'$$

$$f = -(\alpha' b + \frac{1}{2} b'\Sigma b + \frac{1}{2} Tr\ (\Sigma \frac{\partial b}{\partial s}))$$

and Σ is a full rank K× K variance-covariance matrix associated with the Brownian motions of the security price diffusions. Tr denotes the trace (i.e., the sum of the diagonal elements) of the matrix in brackets and ∂b/∂s is a KxK matrix with components $\partial b_j/\partial s_i$.

Given the information in Σ and λ, it is possible to integrate (4) to obtain a specific solution for Z. This implies that a specific Z will depend on maintained null hypotheses about the parameters and functional form of the underlying price processes.

It is significant that the differential equations for Z given in Proposition 2 provide restrictions on both the risk premia and the parameters of the underlying state variable process. More precisely, Proposition 2 can be used to identify absence of arbitrage restrictions on the parameters of specified diffusion processes (Heaney and Poitras 1994).[10] These restrictions provide necessary and sufficient conditions for Z to satisfy (4). In particular:

Corollary 2.1: **Necessary and Sufficient Conditions for Z**

The following integrability conditions:

$$\frac{\partial f}{\partial s_i} = \frac{\partial b_i}{\partial t} \qquad and \qquad \frac{\partial b_i}{\partial s_j} = \frac{\partial b_j}{\partial s_i} \qquad (5)$$

are necessary and sufficient for Z to satisfy (4).

In addition to being required for consistency of the cross derivatives of (4a) and (4b), the restrictions provided by (5) also ensure it is possible to integrate (4) to solve for Z. More formally, the integrability conditions imposed on Z are required for the existence of an arbitrage free semigroup {V(t)}. This connection is directly facilitated by the use of diffusions to motivate the derivative of {p(t)} in (1).

In developing Propositions 1 and 2, the semigroup properties of arbitrage free pricing operators played an important, if somewhat abstract, role in connecting the {V(t)} solutions to (1) with the equivalent martingale measure. Absence of arbitrage in the {V(t)} is ensured by working with infinitesimal generators of the semigroup constructed to ensure absence of arbitrage. This connection between {V(t)} and A depends fundamentally on exploiting the derivative properties of operator semigroups. Specific generators are based on probabilistic assumptions about the stochastic behaviour of the price process. Associated with the assumed empirical measure is an equivalent martingale measure which is also consistent with absence of arbitrage. Under appropriate assumptions, the R-N derivative of the empirical and equivalent martingale measures plays an essential role in producing covariance stationarity in the observed cumulative price dividend process. Assuming that the empirical processes are diffusions permits derivation of closed form solutions for Z as well as the derivation of restrictions on the parameters of the underlying price processes.

Using the results of Proposition 2, it is possible to develop further properties of the semigroup {V(t)}. In particular, it is possible to derive a form for the infinitesimal generator of the arbitrage free pricing operator semigroup:

Proposition 3: **Infinitesimal Generator of the Arbitrage Free Semigroup**

If S(t) is Markov under P and conditions (5) are satisfied, then S(t) is Markov under the equivalent martingale measure Q and the infinitesimal generator A* of the semigroup of arbitrage free pricing operators is:

$$A* = \frac{1}{2} \sum_{i=1}^{K} \sum_{j=1}^{K} \sigma_{ij} \frac{\partial^2}{\partial s_i \partial s_j} + \sum_{i=1}^{K} (\alpha_i - \lambda_i) \frac{\partial}{\partial s_i} \qquad (7)$$

A* applies to arbitrage free pricing operators because (5) ensures that the λ are consistent with absence of arbitrage. In deriving (7), the diffusion property of path independence is essential to establishing that while $Z(y,t)$ and $Z(x,t+s)$ may depend on the initial conditions their ratio does not. This permits Q to be Markov when P is Markov.

The importance of correct specification of the risk premia in arbitrage free security valuation is captured in the role played by λ in (7). Consider the case where A* is simplified by choosing λ such that the second term on the rhs of (7) is zero. Because the risk premia have a general form of $\lambda = \alpha(S,t) + D$, a constant risk premia $\lambda = \alpha$ is not typically acceptable. However, the result can be achieved by using the integrability conditions (5) to provide further restrictions on λ. For example, in certain cases (5) will require the further condition that $\alpha = 0$. This implies that $\lambda = 0$ or, in other words, that arbitrage free pricing requires risk neutrality. Recalling that the relevant state variables have been interest rate detrended, this implies that for $c = 0$ the observed p series will have drift equal to the interest rate. In some situations, it may not be possible to arrive at an arbitrage free solution with $\lambda > 0$. Admitting dividend paying securities does not change this result appreciably, even though introducing dividends can provide for a larger number of possible solutions, e.g., Heaney and Poitras (1994).

III. Applications

Initially, it is useful to derive Z for a **non-dividend paying** asset price process. This case is not directly relevant to modelling the precise relationship between prices and dividends because, in addition to the absence of a dividend process, the use of only one state variable process has some conceptual shortcomings. However, this approach does have desirable analytical features which are relevant to understanding Z for cumulative price dividend processes. As discussed in Section II, derivation of a specific closed form for Z requires precise specification of the price process, which then becomes a nested null hypothesis under which the Z is appropriate. To this end, let the non-dividend paying price process, Y, follow the lognormal (Black-Scholes) process:

$$dY = \mu Y\, dt + \sigma Y\, dB \qquad (9)$$

The derivation of Z from the conditions associated with (4) require:

$$b = -\frac{\mu}{\sigma^2\, Y} \qquad and \qquad f = \frac{1}{2}[(\frac{\mu}{\sigma})^2 - \mu]$$

Verifying that (5) is satisfied, the R-N derivative can now be derived as:

$$Z(Y,t) = e^{\frac{1}{2}((\frac{\mu}{\sigma})^2 - \mu)(t-t_0)}\,(\frac{Y}{Y_0})^{-\frac{\mu}{\sigma^2}} \qquad (10)$$

Empirically, detrending the observed non-dividend paying asset price, firstly, by the interest rate and, secondly, by Z will produce a martingale process, under the null hypothesis (9).

An intuitive motivation for the use of Z is provided by contrasting (10) with results derived from representative investor models. In this context, Bick (1990) and He and Leland (1993) and others construct equilibrium which can support a given stochastic process. For example, it is demonstrated that a representative investor with constant proportional risk aversion is required to support a geometric Brownian motion in wealth. In other words, taking Y to be an aggregate wealth process it is possible to interpret Z as the indirect utility function of the representative investor (Heaney and Poitras 1994). Because the underlying price process for (10) is assumed to be geometric Brownian motion, this Z has a power utility representation, consistent with constant proportional risk aversion. This requires the investor to maintain the fraction of total wealth invested in the risky asset. Under appropriate conditions, it is possible to connect a given method of detrending prices and dividends embodied in Z with assumptions concerning the utility of the representative investor.

To illustrate the connection between investor utility and Z consider a more general form of (9), the constant elasticity of variance (CEV) process. Taking Y to be either the aggregate wealth or cumulative price dividend process:

$$dY = \mu Y\, dt + \sigma Y^{\frac{\beta}{2}}\, dB$$

For this process:

$$b = -\frac{\mu}{\sigma^2} Y^{1-\beta} \qquad and \qquad f = \frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2 Y^{2-\beta} + \frac{\mu}{2}(1-\beta)$$

To satisfy the integrability conditions (5) now requires that either $\beta = 2$ or $\mu = 0$. Of the class of processes covered by the CEV, only the limiting lognormal, $\beta = 2$ case is compatible with a non-zero drift, and $\lambda \neq 0$. For $\mu = 0$, because the asset involved does not pay dividends, this condition reduces to risk neutrality, $\lambda = 0$. The associated Z(t) = 1, a constant, indicating that detrending by the interest rate is all that is required to achieve covariance stationarity. In the absence of a dividend process, the $\beta \in [0,2)$ CEV processes does not appear compatible with a risk averse representative investor.

In order to provide a detrender for dividend paying securities, it is possible to extend the analysis to the Merton (1973) case where dividends are a constant fraction (δ) of the stock price: D = δ S. The asset price, S, is for simplicity assumed to follow a lognormal process: dS = θ S dt + σ S dB. Because there is still only one state variable, there is also only one $\lambda = \theta S + \delta S$. While this approach to incorporating dividends is not fully consistent with observed dividend behaviour, it does provide the significant analytical simplification of retaining only prices as state variables. Based on Proposition 1, absence of arbitrage for a **dividend paying asset** requires the cumulative price dividend process to be a martingale under Q. In this case:

$$b = -\frac{\theta + \delta}{\sigma^2 S} \equiv -\frac{\gamma}{S}$$

$$f = \gamma\theta - \frac{1}{2}\sigma^2 \gamma (1 + \gamma) = \frac{1}{2}\left(\frac{\theta^2 - \delta^2}{\sigma^2} - (\theta + \delta)\right)$$

Verifying that (5) are satisfied, it is possible to derive Z as:

$$Z(S,t) = e^{f\,(t-t_0)} \left(\frac{S}{S_0}\right)^{-\gamma} \qquad (11)$$

By construction, this Z is based on the null hypothesis of lognormal, interest rate detrended security prices and constant proportional dividends.

Incorporating a separate diffusion process for dividends, as in Campbell and Kyle (1993), substantively complicates the derivation of the arbitrage free detrender. Two dependent state variable processes must combine to produce a cumulative price dividend process which is a martingale under Q. In effect, the asset price (capital gains) and dividend processes are functionally related. In order to satisfy Proposition 1, specification of an asset price process imposes implicit restrictions on the dividend process. Ignoring the technical complications associated with introducing a non-price state variable, derivation of Z(t) depends on the specific diffusion processes selected to model actual empirical processes. The presence of two state variables means that b in Proposition 2 now has two elements. As a consequence, there are now three integrability conditions to satisfy, instead of one. However, there is still only one Z(t) applicable to all the state variables. The upshot is that the combined integrability conditions can usually be satisfied only in the $\lambda = 0$ or risk neutral case where the arbitrage free detrender Z(t) = 1.

V. Conclusions

The significance of semigroup concepts to the main results of this paper is decidedly abstract. Much of the paper is concerned with developing properties of the R-N derivative associated with the equivalent martingale measure. These properties are shown to have a practical application to the problem of deriving the appropriate detrending method for achieving covariance stationarity required in empirical tests of asset pricing models. Semigroup properties arise because the price processes are assumed to be Markov, particularly diffusions. Under the null hypothesis that the assumed price process is the actual empirical process, absence of arbitrage restrictions are embedded in the R-N derivative connecting the assumed empirical measure and the associated equivalent martingale measure. Differential properties of the R-N derivative provide restrictions on the coefficients of the price processes (Heaney and Poitras 1994). Because the first differences of a martingale process are orthogonal, it follows that the R-N derivative associated with the equivalent martingale measure is required to transform observed price processes to be covariance stationary.

Intuitively, the role of the R-N derivative can be motivated by comparing results from Bick (1990) and He and Leland (1993) which were generated using representative investor models. These studies derive equilibrium restrictions on investor preferences required to sustain a specific diffusion process. For example, geometric Brownian motion is supportable by a representative investor with constant proportional risk aversion (CPRA), typically power utility. Examining the closed form solutions for Z given in Section III, the connection with power utility functions is apparent. In effect, a maintained null hypothesis about the stochastic process for prices contains embedded assumptions about the equilibrium restrictions required to sustain this process. If the representative investor is CPRA then a constant fraction of total wealth will be held in the riskless asset. Given this, the geometric Brownian motion process will be generated by expected utility maximizing portfolio adjustments along a given price path. Within a representative investor framework, Z can be taken to be the investor's indirect utility function.

The claims made in this paper are substantial. For example, it is claimed that by using inappropriate detrending procedures significant errors may have been made in many of the empirical studies of price dividend models. Intuitively, this is due to ignoring the implications that absence of arbitrage has for the empirical behaviour of security prices. Implementation of the detrending procedure proposed here involves making specific stochastic process assumptions about prices in order to generate a Z. Failure of the detrending procedure to produce a martingale process is evidence against the assumed price process. In cases where both prices and dividends are assumed to follow separate but dependent stochastic processes, a risk neutral ($\lambda = 0$) solution is likely to emerge where $Z(t) = 1$. In this case, absence of arbitrage requires only detrending by the interest rate to achieve covariance stationarity. Even in the risk neutral case, it remains to be verified whether the Campbell and Kyle (1993) procedure of detrending by the inflation rate and the dividend growth rate produces a covariance stationary process which is not substantively different than the risk neutral process which is detrended only by the interest rate.

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APPENDIX

Proof of Proposition (Garman): Related proofs of this result and extensions can be found in Lamperti (1977), Pazy (1983) and Ethier and Kurtz (1986). The proof provided here will be avoid dealing with a number of technical issues. If T(t) is a C_0 semigroup there exist constants $\omega \geq 0$ and $M \geq 1$ so that:

$$\|T(t)\| \leq Me^{\omega t} \qquad (A.1)$$

It follows that T(t)f is continuous for all t since:

$$\begin{aligned} \|T(t+h)f - T(t)f\| &= \|T(t)T(h)f - T(t)f\| \\ &\leq \|T(t)\| \, \|T(h)f - f\| \\ &\leq M\, e^{\omega t} \|T(h)f - f\| \end{aligned}$$

tends to 0 as $h \to 0$. Now:

$$\begin{aligned} \frac{d}{dt}T(t)f &= \lim_{h\to 0}\frac{T(t+h)f - T(t)f}{h} \\ &= \lim_{h\to 0}\frac{T(t)T(h)f - T(t)f}{h} \\ &= T(t)\lim_{h\to 0}\frac{T(h)f - f}{h} \\ &= T(t)Af \qquad (A.2) \end{aligned}$$

Integrate (A.2) from 0 to t and make use of the continuity of T(t)f to obtain:

$$T(t)f - T(0)f = \int_0^t T(s)Af\, ds \qquad (A.3)$$

(1) in the text follows on rearranging (A.3) since T(0)= 1. To prove the Proposition now follows from $f \in L$ depending parametrically on time:

$$\frac{d}{dt}T(t)f(t) = \lim_{h\to 0}\frac{T(t+h)f(t+h)-T(t)f(t)}{h}$$

$$= \lim_{h\to 0}\frac{T(t+h)[f(t) + h\frac{\partial f}{\partial t}(t) + o(h)] - T(t)f(t)}{h}$$

$$= \lim_{h\to 0}\frac{T(t)[T(h)f(t) - f(t)] + T(t+h)[h\frac{\partial f}{\partial t}(t) + o(h)]}{h}$$

$$= T(t)Af(t) + T(t)\frac{\partial f}{\partial t}(t) \qquad (A.4)$$

Using the continuity of T(t)f, integration of (A.4) yields:

$$f(0) = T(t)f(t) + \int_0^t T(s)[-Af(s) - \frac{\partial f}{\partial t}(s)]\, ds \qquad (A.5)$$

Comparison of (A.5) with (1) reveals that when T(t) is a valuation semigroup, the dividend stream $-Af-\partial f/\partial t$ generates the price f.

<u>Proof of Proposition</u> 1:

Since the cumulative price dividend process is a martingale under Q:

$$S(t_0) = E^Q(S(t) + \int_{t_0}^t D(s)ds|F_{t_0}) \qquad (A.1)$$

Since S(t) is F_t measurable, then for $t > t_0$:

$$E^Q(S(t)|F_{t_0}) = \frac{E^P(\frac{dQ}{dP}S(t)|F_{t_0})}{E^P(\frac{dQ}{dP}|F_{t_0})} = \frac{E^P(E^P(\frac{dQ}{dP}S(t)|F_t)|F_{t_0})}{Z(t_0)}$$

$$= \frac{E^P(E^P(\frac{dQ}{dP}|F_t)S(t)|F_{t_0})}{Z(t_0)} = \frac{E^P(Z(t)S(t)|F_{t_0})}{Z(t_0)} \qquad (A.2)$$

where $Z(t) \equiv E^P[\, dQ/dP \mid F_t] = E^P[\, dQ/dP \mid S_t]$ due to the Markov property. From its definition Z(t) is a martingale under P. Also since D(s) is F_s measurable, then:

$$E^Q(\int_{t_0}^{t} D(s)ds|F_{t_0}) = \frac{E^P(\frac{dQ}{dP}\int_{t_0}^{t} D(s)ds|F_{t_0})}{E^P(\frac{dQ}{dP}|F_{t_0})} = \frac{E^P(E^P(\int_{t_0}^{t}\frac{dQ}{dP}D(s)ds|F_s)|F_{t_0})}{Z(t_0)}$$

$$= \frac{E^P(\int_{t_0}^{t} E^P(\frac{dQ}{dP}|F_s)D(s)ds|F_{t0})}{Z(t_0)} = \frac{E^P(\int_{t_0}^{t} Z(s)D(s)ds|F_{t_0})}{Z(t_0)} \qquad (A.3)$$

where in the third step above the order of integration over ω and t is interchanged, by Fubini' s Theorem. For the purpose of using Fubini' s theorem it is assumed that:

$$E^Q(\int_{t_0}^{t} |D(s)|ds) < \infty$$

Taking $Z(t_0) = 1$, substituting (A.2) and (A.3) into (A.1) and recalling the normalization assumption that $Z(t_0) = 1$ gives equation (3) of the text.

Proof of Proposition 2:

These equations follow from both Z(t) and transformed cumulative price dividend processes being martingales under P. From Proposition 1:

$$E(d\{\int_{0}^{t} Z(s)D(s)ds + Z(t)S(t)\}|F_t) = 0 \qquad (A4)$$

It follows on using Ito's lemma that Z(y,t) satisfies the following partial differential equation:

$$\frac{\partial}{\partial t}Z(y,t)S(y,t) + Z(y,t)D(y,t) + \sum_{i=1}^{K}\alpha_i(y,t)\frac{\partial}{\partial y_i}Z(y,t)S(y,t)$$
$$+ \frac{1}{2}\sum_{i=1}^{K}\sum_{j=1}^{K}A_{ij}\frac{\partial}{\partial y_i \partial y_j}Z(y,t)S(y,t) = 0 \qquad (A.5)$$

where $\sigma\sigma' = A$. Equation (A.5) holds for the cumulative price-dividend process. In particular for the k^{th} such asset:

$$y_k\frac{\partial Z}{\partial t} + ZC_k + y_k\sum_{i=1}^{K}\alpha_i\frac{\partial Z}{\partial y_i} + \alpha_k Z + \sum_{j=1}^{K}A_{kj}\frac{\partial Z}{\partial y_j}$$

$$+ y_k\frac{1}{2}\sum_{i=1}^{K}\sum_{j=1}^{K}A_{ij}\frac{\partial Z}{\partial y_i\partial y_j} = 0 \qquad (A6)$$

Since Z(t) is a martingale, $E(dZ(t)|F_t) = 0$ so that, using Ito's lemma once again:

$$\frac{\partial Z}{\partial t} + \sum_{i=1}^{K}\alpha\frac{\partial Z}{\partial y_i} + \frac{1}{2}\sum_{i=1}^{K}\sum_{j=1}^{k}A_{ij}\frac{\partial Z}{\partial y_i\partial y_j} = 0 \qquad (A7)$$

Equations (A6) and (A7) together imply that Z satisfies the K equations:

$$[\alpha_i(y,t)+C(y,t)]Z(y,t) = \sum_{i=1}^{k}A_{ij}(y,t)\frac{\partial}{\partial y_i}Z(y,t) \qquad (A8)$$

Inverting these yields equations (4a) of the text. Substituting these back into (A8) gives (4b) of the text.

Proof of Proposition 3:

This proof is derived for the more general non-price state variable case. To reproduce the price state variable case, substitute S(t) for Y(t) where applicable. Let I be the indicator function for Y(t+ s) starting from y at time t, i.e.:

$$I(Y_y(t+s) \in dx) = 1 \quad \textit{for } \{\omega\colon Y_y(t+s) \in R^K\}$$
$$= 0 \quad \textit{otherwise}$$

Then, keeping in mind that x is a fixed point, and that $Y_y(t+s)$ is F_{t+s} measurable:

$$E^Q(I(Y_y(t+s))\varepsilon dx|F_t) = \frac{E^P(ZI(Y_y(t+s)\varepsilon dx)|F_t)}{E^P(Z|F_t)}$$

$$= \frac{E^P(E^P(ZI(Y_y(t+s)\varepsilon dx|F_{t+s})|F_t)}{Z(y,t)} = \frac{E^P(E^P(Z|x,t+s)I(Y_y(t+s)\varepsilon dx|F_t)}{Z(y,t)}$$

$$= \frac{Z(x,t+s)E^P(I(Y_y(t+s)\varepsilon dx|F_t)}{Z(y,t)} = \frac{Z(x,t+s)P(y,t;dx,t+s)}{Z(y,t)}$$

Thus, since the transition function for the process starting from y at time t depends on y and t only and not on their past history, Y is Markov under Q with transition function given by Proposition 3. To derive the functional form for the infinitesimal generator, from the definition of A, (5) and Proposition 2:

$$Af(y) = \lim_{s\to 0^+} \frac{T(s)f(y) - f(y)}{s}$$

$$= \lim_{s\to 0^+} \frac{\int Q(y,t;dx,t+s)f(x) - f(y)}{s}$$

$$= \frac{1}{Z(y,t)} \lim_{s\to 0^+} \frac{\int P(y,t;dx,t+s)[Z(x,t+s)f(x) - Z(y,t)f(y)]}{s}$$

$$= \frac{1}{Z(y,t)} \lim_{s\to 0^+} \frac{E^P(d[Zf]|y,t)}{s}$$

However:

$$\lim_{s\to 0^+} E^P(d[Zf]) = Z\left[\sum_{i=1}^{K} \alpha_i \frac{\partial f}{\partial y_i} + \frac{1}{2}\sum_{i=1}^{K}\sum_{j=1}^{K} A_{ij} \frac{\partial f}{\partial y_i \partial y_j}\right]$$

$$+ f\left[\frac{\partial Z}{\partial t} + \sum_{i=1}^{K} \alpha_i \frac{\partial Z}{\partial y_i} + \frac{1}{2}\sum_{i=1}^{K}\sum_{j=1}^{K} Aij \frac{\partial Z}{\partial y_i \partial y_j}\right]$$

$$+ \sum_{i=1}^{K}\sum_{j=1}^{K} A_{ij}\left(\frac{\partial Z}{\partial y_i}\right)\left(\frac{\partial f}{\partial y_j}\right) = 0$$

The term proportional to f is zero by (A7). Substitute (A8) into the last term to obtain:

$$Af(y) = \frac{1}{Z(y,t)} \lim_{s\to 0^+} \frac{E^P(d[Zf]|y,t)}{s}$$

$$= \sum_{i=1}^{K} (\alpha_i(y,t) - \lambda_i(y,t)) \frac{\partial f}{\partial y_i} + \frac{1}{2}\sum_{i=1}^{K}\sum_{j=1}^{K} A_{ij}(y,t) \frac{\partial f}{\partial y_i \partial y_j}$$

This is the result stated in the Proposition.

NOTES

1. The specification of the semigroup often includes the condition: V(0) = I (where I is the identity operator). However, the presence of the unit element, which insures that S is not empty, is often used to distinguish a semigroup from a monoid. The presentation here will follow a looser convention by including the identity operator as part of the semigroup terminology.

2. A number of detrending procedures have been employed. A recent example is Campbell and Kyle (1993) which uses the Standard and Poors Composite and the associated dividend series both detrended, initially, by the producer price index, to get a "real stock price" and "real dividend". These series are then further detrended by "the mean dividend growth rate over the sample". The resulting series are, under the Campbell and Kyle method, required to be I(1) processes. With a number of minor improvements, this is the detrending procedure followed in Campbell and Shiller (1987). The failures of the less sophisticated detrending procedures used in early studies, e.g., Shiller (1981), contributed significantly to the "econometric difficulties" identified in Campbell and Kyle (1993).

3. It is also possible, perhaps preferable, to express the semigroup property of the discounting operators as exp{-r(t+ s)} = exp{-rt} exp{-rs}.

4. The development in Karlin and Taylor (1981, p.285-305) uses a regular, time homogeneous diffusion process. In terms of expectations operators, if (U(t+ s)p)(x) = E[p(T) | X(0)= x], then the semigroup property requires U(t+ s) = U(t) U(s) = U(s) U(t). It follows, E[p(t+ s) | X(0)= x] = E[E[p(t+ s) | X(t)] | X(0)= x]. This result is directly related to the Chapman-Kolmogorov equation.

5. The result requires that the semigroup be C_0. Garman's result is a straight forward application of the semigroup approach to solving boundary value problems for second order PDE's, e.g., Goldstein (1985). The C_0 semigroup condition refers to the formal definition of a semigroup of operators where V(t), t ≥ 0, is a family of linear operators on a Banach space B with norm $\|\cdot\|$. V(t) is a one parameter semigroup provided that:

$$\|V(t)\ p\| \leq \|p\| \quad \textit{for all } p \in B$$

$$V(t+s) = V(t)V(s)$$

$$\lim_{t \to 0^+} V(t)p = p$$

The first requirement of the definition ensures that the semigroup is bounded, and the third requirement ensures that it is (strongly) continuous. The set of operators thus defined are referred to as a strongly continuous semigroup or C_0 semigroup. The operator V(t), operates on functions belonging to B mapping them into the same space B. In this paper the Banach space B is the set of bounded continuous functions from R^k to R^1 under the supremum norm. That is the domain of the operators V(t) is B and its range is in B.

6. This follows because of the result: if {X(t)} is a martingale process, the first differences {ΔX(t)} are orthogonal. Hence, restricting observed random variables to be L_2, the process composed of the first differences of a martingale process will be covariance stationary.

7. Conventionally, detrending by the interest rate is accomplished by dividing the security price by the stochastic price of the relevant discount bond which matures at T with price equal to one, using a consol rate for the infinite horizon case. An alternative approach is to use the

instantaneous coupon on a perpetual floating rate bond. The use of interest rate detrended series is for analytical simplicity. Working directly with observed prices changes the presentation in a number of ways, e.g., because of the presence of a discounting factor, the detrender Z will not be equal to the R-N derivative.

8. The cumulative price dividend process could also be defined as the total return process. The change in notation from p and c to S and D is intended to recognize the difference between these variables: S and D are interest rate detrended while p and c are observed prices and dividends.

9. This Proposition assumes that Z(0) = 1. Interpreting Z as a density, this implies that the current state is known with certainty, a delta function centered at the current state in probabilistic terms. Because this assumption is not required, it is possible to generalize (3) to describe the behaviour of Z(t) S(t).

10. While the necessity of (5) involves a straight forward exercise of differentiating (4), verifying the sufficiency of these conditions is considerably more difficult. The derivation of sufficiency follows Apostle (1964, Chp. 10) Theorem 48.