Spread Options, Exchange Options and Arithmetic Brownian Motion

Since the early contributions of Black and Scholes (1973) and Merton (1973), the study of option pricing has advanced considerably. Much of this progress has been achieved by retaining the assumption that the relevant state variable follows a geometric Brownian motion. Limitations inherent in using this assumption for many option pricing problems have led to theoretical extensions involving the introduction of an additional state variable process.[1] Except in special cases, the presence of this additional process requires a double integral to be evaluated in order to solve the expectation associated with the option valuation problem. This complicates the European option pricing problem to the point where a closed form is usually not available and numerical techniques are required to solve for the option price. Such complications arise in the pricing of spread options, e.g., Shimko (1994), Pearson (1995), options which have a payoff function depending on the difference between two prices and an exercise value. For lognormally distributed state variables, a closed form for the spread option price is only available for the special case of an exchange option or, more precisely, an option to exchange one asset for another (Margrabe 1978, Carr 1988, Fu 1996).[2]

The objective of this paper is to develop pricing formulae for European spread options under the assumption that the prices follow arithmetic Brownian motions.[3] Significantly, unlike the lognormal case, assuming arithmetic Brownian motion does permit the derivation of simple closed forms for spread option prices. The potential generality of assuming arithmetic Brownian motion for single state variable option pricing problems has been demonstrated by Goldenberg (1991) which provides various option pricing results derived using arithmetic Brownian motion with an absorbing barrier at zero. Due to the complexities of using absorbed Brownian motion for pricing spread options, this paper argues that assuming arithmetic Brownian motion without an absorbing barrier at zero is appropriate for developing spread option pricing results. The resulting option price is a special case of a **Bachelier option**, an option price derived under the assumption of unrestricted arithmetic Brownian motion.[4] In this vein, Bachelier exchange option prices can be contrasted with the Black-Scholes exchange option in order to benchmark relative pricing performance. Even though the homogeneity property used to simplify the lognormal case does not apply to the Bachelier exchange option, the linearity property of arithmetic Brownian motion provides for a similar simplification.

In the following, Section I reviews previous studies which have assumed arithmetic Brownian motion to derive an option pricing formula. Arguments related to using this assumption in pricing spread options are reviewed. Section II provides European spread option prices where the individual security prices are assumed to follow arithmetic Brownian motion. Bachelier spread option prices for assets with equal and unequal proportional dividends, as well as spread options on futures contracts, are derived. Implications associated with different types of spread option contract design are also discussed. Section III gives results for alternatives to the Bachelier option. It is demonstrated that the commonly used Wilcox spread option formula does not satisfy absence-of-arbitrage requirements. Recognizing that the exchange option is a special type of spread option, closed form solutions are provided for Black-Scholes exchange options using securities with dividends as well as for futures contracts. In Section IV some simulated pricing scenarios are used to identify relevant features of the Bachelier spread option and to contrast the properties of Bachelier exchange options with the Black-Scholes exchange options. In conclusion, Section V provides a summary of the main results in the paper.

Section I: Background Information and Literature Review

Despite having received only limited empirical support in numerous distributional studies of financial prices, the analytical advantages of assuming geometric Brownian motion have been substantial enough to favour retaining the assumption in theoretical work. While much the same theoretical advantages can be achieved with arithmetic Brownian motion, this assumption has been generally avoided. Reasons for selecting geometric over arithmetic Brownian motion were advanced at least as early as Samuelson (1965) and some of the studies in Cootner (1964). In reviewing previous objections, Goldenberg (1991) recognizes three of practical importance: (i) a normal process admits the possibility of negative values, a result which is seemingly inappropriate when a security price is the relevant state variable; (ii) for a sufficiently large time to expiration, the value of an option based on arithmetic Brownian motion exceeds the underlying security price; and, (iii) as a risk-neutral process, arithmetic Brownian motion without drift implies a zero interest rate. Taken together, these three objections are relevant only to an unrestricted "arithmetic Brownian motion" which is defined to have a zero drift. As such, some objections to arithmetic Brownian motion are **semantic**, avoidable if the process is appropriately specified.

Smith (1976) defines arithmetic Brownian motion to be driftless and provides an option pricing formula which is attributed to Bachelier (1900) and is subject to all of the three objections. Smith (p.48) argues that objection (ii) for the formula is due to the possibility of negative sample paths, though this objection can be avoided by imposing an appropriate drift. Goldenberg (1991) reproduces the Smith-Bachelier formula and proceeds to alter the pricing problem by replacing the unrestricted driftless process with an arithmetic Brownian motion which is absorbed at zero. The resulting option pricing formula avoids the first two objections. The third objection is addressed by setting the drift of the arithmetic Brownian process equal to the riskless interest rate times the security price, consistent with an Ornstein-Uhlenbeck (OU) process, e.g., Cox and Miller (1965, p.225-8). Using this framework, Goldenberg generalizes an option pricing result in Cox and Ross (1976) to allow for changing variances and interest rates. With the use of appropriate transformations for time and scale, Goldenberg argues that a wide range of European option pricing problems involving diffusion price processes can be handled using the absorbed-at-zero arithmetic Brownian motion approach.

An alternative to using absorbed Brownian motion, adaptable to the study of spread options, is to derive the option price formula using unrestricted arithmetic Brownian motion with drift. While this does not address objection (i), it can handle the other two. An option pricing model for individual securities which uses this approach does appear in Brennan (1979), a study of the utility-theoretic properties of contingent claims prices in discrete time.[5] However, the intuitive limitations of arithmetic Brownian motion associated with objection (i) combined with the availability of simple closed forms solutions for absorbed Brownian motion have created a situation where results on the Bachelier option for **individual** securities are generally unavailable. The situation for spread options is somewhat different. Significantly, for the pricing of spread options, assuming that prices follow unrestricted arithmetic Brownian motions permits the derivation of substantively **simpler closed forms** than assuming that the prices forming the spread follow absorbed Brownian motions. Even though absorbed arithmetic Brownian motion has greater intuitive appeal due to the avoidance of negative sample paths for the individual prices, this advantage will not typically be of much practical relevance for pricing spread options.

Determining a spread option formula when the prices are assumed to be absorbed Brownian motions is complicated and **the resulting theoretical prices will only differ from the unrestricted case if there is significant probability of the price processes reaching zero** (Heaney and Poitras 1997). Hence, for pricing traded spread options, much of the concern about price processes being absorbed at zero is moot, because the probability of either process being absorbed is almost zero. For example, consider the following candidate variables for spread options: the difference in the price of heating oil and gasoline; the difference in the Nikkei and the Dow Jones stock price indices; and, the difference in the price of gold or copper futures contracts for different delivery dates. The practical likelihood of any of these price processes going to zero is negligible. While, in general, the validity of the unrestricted Brownian solution will depend on the type of spread being evaluated, cases for which it is not a plausible candidate process are difficult to identify in practice. In addition, direct evaluation of the spread option assuming lognormally distributed prices requires a complicated double-integration over the joint density of S_2 and S_1, which has to be evaluated numerically. Again, treating the spread as an unrestricted arithmetic Brownian motion has substantive analytical advantages.

Without precise empirical information on specific spread distributions, spread options are a security which could arguably provide a useful application of the Bachelier option. This insight was first exploited in a trade publication, Wilcox (1990), which employs arithmetic Brownian motion to derive a closed form spread option pricing formula. However, as demonstrated in Section III, the Wilcox formula is not consistent with absence-of-arbitrage and, as a result, is not a valid option pricing formula. Despite its theoretical limitations, the Wilcox model has been used as a benchmark pricing result in a number of studies. In particular, Shimko (1994) and Pearson (1995) both compare the Wilcox model with option prices derived from a double-integration approach involving lognormally distributed prices. Pearson (1995) contrasts the performance of the Wilcox spread option with a double-integration approach which is analytically simplified by providing a closed form solution to the first integration. A numerical algorithm is used to solve the second integral and arrive at exact prices. Evaluating option prices and delta for a number of specific examples, Pearson claims that the double-integral log-normal solution provides substantially more accurate pricing than the Wilcox approach, particularly for long maturity options.

Shimko (1994) applies the Jarrow and Rudd (1982) approximation technique to the Wilcox (1990) option price in order to approximate the "true" lognormal solution. In effect, the Wilcox formula is augmented with the addition of higher order moment terms which approximate the difference between the normal and lognormal cases. Indirect information on the relative performance of the Wilcox option is provided in a specific illustration of the "accuracy of analytical approximation" (p.211-212) which contrasts the prices from the approximation and an exact double-integral lognormal solution which encompasses stochastic convenience yield. However, because Shimko relies on the Wilcox spread option formula, the comparison between the double lognormal integration approach and the arithmetic Brownian motion spread option model is not fully developed. In addition, it is not clear to what extent the limitations of the Wilcox model have been incorporated in the Jarrow and Rudd approximation solution. Finally, Shimko makes an important, if debateable, statement about spread options: "...the behaviour of the spread option is affected by the behaviour of two traded contracts; a spread cannot be modelled as if it is a single asset" (p.184). This is precisely what assuming arithmetic Brownian motion permits.

Shimko (1994) recognizes that a fundamental difficulty in evaluating spread option pricing models is the limited number of traded securities. As a consequence, there are only a limited number of empirical studies on spread options. Grabbe (1995) provides some empirical information on copper spread options traded on the LME while Wilcox (1990) examines traded oil spread options and Falloon (1992) provides some practical examples. Despite the presence of these few studies, the data on spread options are, at this point, insufficient to support conclusions about the superiority of one pricing method over another. Related empirical evidence on the distribution of spreads is also limited. Poitras (1990) provides a detailed study of the distribution of gold futures spreads, together with a methodology for deconvolving the distribution into two component distributions. However, because gold tends to be at or near full-carry, the distributional information is of limited value for inferring the distribution of other types of spreads. Gibson and Schwartz (1990) use a time series approach to evaluate the behaviour of convenience yield for crude oil, providing useful information about the spread distribution for that commodity. Some limited empirical information is also available in other sources, e.g., Rechner and Poitras (1993) on the soy crush spread.

Section II: Bachelier Spread Option Pricing[6]

On the expiration date, the payout on a spread option has the form:[7]

$$C_T = \max[S_{2T} - S_{1T} - X, 0]$$

where T is the expiration date of the option, C_T is the call option price at time T, X is the exercise value (which can be either positive or negative) and $S_{2T} - S_{1T}$ is the difference between two prices S_2 and S_1 at time T. The complexity of the spread option pricing problem is reflected in studies which have taken the direct approach to valuation, Shimko (1994), Pearson (1995), Ravindran (1993), Bjerksund and Stensland (1994), and Grabbe (1995). The direct approach involves solving the risk neutral valuation problem for the European spread option price:

$$C_t = e^{-rt^*} E[\ \max[S_{2T} - S_{1T} - X,\ 0]\ ]$$

$$= e^{-rt^*} \{ \int \int \max[S_{2T} - S_{1T} - X,\ 0]\ g[S_{2,T} \mid S_{1,T}]\ f[S_{1,T}]\ \ dS_2\ dS_1 \}$$

where the risk-neutral expectation is taken with respect to a lognormal conditional density, g[·], and marginal density, f[·]. From this point, a number of solution techniques are available. However, with lognormally distributed price processes, it is only possible to achieve a closed form solution in the special case of an exchange option, where one of the assets can be used as a numeraire. Otherwise, some numerical technique must be implemented to evaluate the double integral. In this process, while it is possible to derive a closed form solution to the first integration where the expectation is taken with respect the conditional density, e.g., Pearson (1995), the second integral must be evaluated numerically.[8]

One advantage of having a closed form solution is the avoidance of having to numerically evaluate a double integral to determine option prices. In the absence of traded securities, it is difficult to assess relative pricing performance and, by implication, the validity of a given modelling approach. Moreover, it is reasonable to assume that the distributional assumption selected would depend on the specific type of spread being modelled. Unlike individual security prices, the spread distribution depends on the difference of two, possibly disparate, distributions. In general, evaluation of the resulting convolution is difficult, though it is possible to conclude that a wide range of distributions can result.[9] Given this, arithmetic Brownian motion is one potentially viable candidate process. Recognizing that the spread option pricing problem will have different solutions, depending on the empirical distribution of the spread being modelled, is in sharp contrast to the current modelling convention of assuming lognormally distributed prices and treating the spread option in a general fashion, making limited reference to either potential variations in the design of the spread option or to empirical characteristics of the underlying spread. This ignores the possibility that the solution to the spread option pricing problem can differ, depending on the types of spreads being considered. For example, S_2 and S_1 could be the prices of gold contracts for different delivery dates, an intracommodity futures spread option or the Nikkei and S&P stock indices or the prices of crude oil and gasoline.

One possible generic type of price spread occurs where the securities both pay the same proportional dividend ($\delta S\ dt$). If it is assumed that the individual price processes both follow arithmetic Brownian motion, then the price spread will follow the diffusion:

$$d(S_2 - S_1) = (r - \delta)(S_2 - S_1)\ dt + \sigma_S\ dW_s \qquad (1)$$

where the drift and volatility parameters are specified to be consistent with absence-of-arbitrage. This diffusion is constructed by taking S_2 and S_1 to both follow **unrestricted** arithmetic Brownian motions of the form:

$$dS_2 = (r - \delta)\ S_2\ dt + \sigma_2\ dW_2 \qquad dS_1 = (r - \delta)\ S_1\ dt + \sigma_1\ dW_1$$

where the variable of the joint process is specified as:

$$\sigma_S^{\,2} = \sigma_1^{\,2} - 2\sigma_{12} + \sigma_2^{\,2}$$

Hence, as a consequence of assuming the individual price process follow unrestricted arithmetic Brownian motions with appropriately specified coefficients, it is possible to construct a SDE for the spread option as (1), where the spread can be treated as a single random variable. Because the difference of lognormal variables is not lognormal, a similar simplification is not available if the price processes are assumed to be lognormal.

In what follows, derivation of the closed form solutions for the spread option prices proceeds by stating the PDE for the dynamic hedging problem and verifying that the stated solution satisfies the PDE. The procedure for deriving the PDE is not stated explicitly but does follow the standard procedure of identifying the relevant riskless hedge portfolio, which is composed of a long and a short position in the commodities or securities determining the spread. This cash position is dynamically hedged by writing an appropriate number of spread call options. This riskless hedge portfolio provides two conditions, one associated with applying Ito's lemma and another condition associated with the restriction that the net investment in the hedge portfolio must earn the riskless rate of interest. Equating these two conditions and manipulating provides the PDE associated with the dynamic hedging problem. The validity of the solutions given in the various Propositions is proved by evaluating the relevant partial derivatives of the stated option formula and verifying that the closed form satisfies the PDE. By construction, if the PDE is satisfied the result is consistent with absence-of-arbitrage.

In the special case where both S_2 and S_1 are assets which pay the same constant dividend (δ), the PDE associated with riskless hedge portfolio problem for the spread option can be motivated by treating the spread as a single random variable and using the well-known PDE for the single variable case which gives:

$$\frac{\partial C}{\partial t} = rC - (r - \delta)\frac{\partial C}{\partial (S_2 - S_1)}(S_2 - S_1) - \frac{1}{2}\{\frac{\partial^2 C}{\partial (S_2 - S_1)^2}\} \sigma_S^{\,2} \tag{2}$$

By treating the spread as a single random variable, this PDE involves only one delta hedge ratio and one gamma. In general, the riskless hedge portfolio for a spread option will involve two delta hedge ratios, one for each of the two spot (or futures) positions. Evaluating the riskless hedge portfolio for this dynamic hedging problem produces the PDE:

$$\frac{\partial C}{\partial t} = rC - (r - \delta)\{\frac{\partial C}{\partial S_1}S_1 + \frac{\partial C}{\partial S_2}S_2\} - \frac{1}{2}\{\frac{\partial^2 C}{\partial S_1^{\,2}}\sigma_1^{\,2} + 2\frac{\partial^2 C}{\partial S_1 \partial S_2}\sigma_{12} + \frac{\partial^2 C}{\partial S_2^{\,2}}\sigma_2^{\,2}\} \tag{3}$$

For the arithmetic Brownian diffusion process, the solutions to the PDE's (2) and (3) are equivalent, a result which can be verified by taking the relevant derivatives of the formula given in Proposition I.

Given this background, it is now possible to provide the following result:

Proposition I: The Bachelier Spread Option for Equal Dividend Paying Securities

Assuming perfect markets and continuous trading, for a price spread involving two prices making equal dividend payments and both obeying arithmetic Brownian motion, the absence-of-arbitrage solution that satisfies the PDE's (2) and (3) is the Bachelier spread option pricing formula:

$$C_{St}[S_2 - S_1,\ t^*;\ r,\ \delta,\ \sigma,\ X] = ((S_{2,t} - S_{1,t})e^{-\delta t^*} - Xe^{-rt*}\ )\ N[y] + V\ n[y] \tag{4}$$

where:

$$y = \frac{(S_{2,t} - S_{1,t})e^{-\delta t^*} - Xe^{-rt^*}}{V} \qquad V = \sigma_S \sqrt{\{\frac{e^{-2\delta t^*} - e^{-2rt^*}}{2(r - \delta)}\}}$$

C_{St} is the price of the Bachelier spread call option for equal dividend paying securities, and N[y] and n[y] represent the cumulative normal density and normal probability function, respectively, evaluated at y.

The proof of Proposition I given in the Appendix verifies by direct differentiation that this solution satisfies the PDE for the dynamic hedging problem. In Section III, it will be verified that (1), the SDE for the spread process associated with Proposition I, imposes the appropriate absence-of-arbitrage restriction on the drift coefficient. The special case of securities which pay no dividends is determined by setting $\delta = 0$ in (4).

The generalization of Proposition I to include securities making unequal dividend payments has considerable practical importance, e.g., for pricing cross-currency swaptions. The presence of unequal dividend payments involves a restatement of both the diffusion process and the PDE for the riskless hedge portfolio. Recognizing the absence-of-arbitrage restrictions on the drift, for the constant proportional dividends case the absence-of-arbitrage diffusions are:

$$dS_2 = (r - \delta_2)S_2 \, dt + \sigma_2 \, dW_2 \qquad dS_1 = (r - \delta_1)S_1 \, dt + \sigma_1 \, dW_1$$

And the PDE for the riskless hedge portfolio is:

$$\frac{\partial C}{\partial t} = rC - \frac{\partial C}{\partial S_1}S_1(r - \delta_1) - \frac{\partial C}{\partial S_2}S_2(r - \delta_2) - \frac{1}{2}\{\frac{\partial^2 C}{\partial S_1^{\,2}}\sigma_1^{\,2} + 2\frac{\partial^2 C}{\partial S_1 \partial S_2}\sigma_{12} + \frac{\partial^2 C}{\partial S_2^{\,2}}\sigma_2^{\,2}\}$$

It follows that:

Proposition II: The Bachelier Spread Option for Unequal Dividend Paying Securities

Assuming perfect markets and continuous trading, if the spread difference (S_2 - S_1) involves securities which pay constant proportional dividends δ_2 and δ_1, respectively, and the spread difference follows an appropriately defined arithmetic Brownian motion, then the absence-of-arbitrage solution to the spread call option valuation problem is given by:

$$C_{Dt}[S_2, S_1, t^*; r, \delta_1, \delta_2, \sigma, X] = (S_{2t} \, e^{-\delta_2 t^*} - S_{1t} \, e^{-\delta_1 t^*} - Xe^{-rt*}) \, N[z] + \Lambda \, n[z] \qquad (5)$$

where:

$$z = \frac{S_{2,t}\, e^{-\delta_2 t^*} - S_{1,t}\, e^{-\delta_1 t^*} - Xe^{-rt^*}}{\Lambda}$$

$$where: \qquad \Lambda = \sqrt{v_{11} + v_{22} - 2v_{12}}$$

$$v_{11} = \sigma_1^2 \{ \frac{e^{-2\delta_1 t^*} - e^{-2rt^*}}{2(r - \delta_1)} \} \qquad v_{22} = \sigma_2^2 \{ \frac{e^{-2\delta_2 t^*} - e^{-2rt^*}}{2(r - \delta_2)} \}$$

$$v_{12} = \sigma_{12} \{ \frac{e^{-(\delta_1 + \delta_2) t^*} - e^{-2rt^*}}{2(r - \delta_1 - \delta_2)} \}$$

C_{Dt} is the price of the Bachelier spread call option for securities with unequal dividend payments and N[z] and n[z] represent the cumulative normal density and normal probability function, respectively, evaluated at z.

As with previous results, the proof of Proposition II involves direct differentiation to verify that this solution does satisfy the PDE for the relevant riskless hedge portfolio problem.

The dynamic hedging problem for spread options on futures contracts results in a PDE where the restriction that r - δ = 0 in (2) and (3) is imposed due to the ability to create a futures position with no net investment of funds. For (2), it follows that:

$$\frac{\partial C}{\partial t} = rC - \frac{1}{2}\, \sigma_F^2\, \frac{\partial^2 C}{\partial (F_1 - F_2)^2}$$

where F_2 and F_1 are the prices for the relevant futures contract. A similar PDE is related to (3). Given this, the appropriate solution is:

Proposition III: The Bachelier Futures Spread Option

Assuming perfect markets and continuous trading, for a futures price spread following arithmetic Brownian motion, the absence-of-arbitrage solution to the spread call option problem is the Bachelier spread option pricing result:

$$C_{Ft}[F_2 - F_1, t^*; r, \sigma, X] = e^{-rt^*} \{ (F_{2t} - F_{1t} - X)\, N[u] + \sigma_F \sqrt{t^*}\, n[u] \} \qquad (6)$$

where:

$$u = \frac{F_{2t} - F_{1t} - X}{\sigma_F \sqrt{t^*}} \qquad \sigma_F = \sigma_{F_1}^{\ 2} - 2\ \sigma_{F_1, F_2} + \sigma_{F_2}^{\ 2}$$

C_{Ft} is the price of the Bachelier futures spread call option, and N[u] and n[u] represent the cumulative normal density and normal probability function, respectively, evaluated at u.

At present, futures spread options are specified using available contract units to determine the value of the commodities being exchanged. For example, one of the NYMEX crack spread contracts offers an option to exchange futures contracts for crude oil and heating oil. The LME copper calendar spread option has a similar configuration. However, judicious choice of the 'prices' used in the spread permits the payoff function to be more appropriately structured to facilitate speculative trading.

An important illustration of the benefits of associated with appropriate selection of units occurs when F_2 and F_1 refer to contracts for the same commodity but for different delivery dates, a calendar spread option. For this example, take F_2 and F_1 to be the total value of gold represented by, say, the June '99 and June '98 100 oz. gold contracts, respectively. Even though the quantity of gold for each contract is the same, because of the gold futures price contango, F_2 and F_1 will be different dollar values. For spreads using equal quantities, the change in the spread over time will be a function of the change in the net implied carry and the change in futures price levels, e.g., Poitras (1990).[10] However, when the spread option is initially written, F_2 and F_1 could be equated by tailing the spread; for the gold spread example, this involves taking (F_{2t}/F_{1t})*100 oz. of June '98 gold for each 100 oz. of June '99 gold. This will equate the dollar value of the two legs of the spread. This has at least two important implications. Firstly, it simplifies the payoff on the spread option by making changes in the spread dependent solely on changes in the net implied carry. Because payoffs depending on changes in price levels are available with other options, this would facilitate the market completion properties of the spread option, supporting demand for the contract. Secondly, it means an at-the-money option with an exercise value equal to zero would have a simple pricing solution, again supporting trade in the option.

Finally, consistent with an observation made in Section I, it is possible to redefine the prices used to specify the spread and model the problem as an arithmetic Brownian motion on one state variable. This is the fundamental theoretical advantage that assuming the price processes follow arithmetic Brownian motions has for solving the spread option pricing problem. Observing that the sum or difference of normally distributed variables is also normal permits the spread term in the risk neutral valuation problem to be redefined as a single random variable $y = S_{2T} - S_{1T}$. The resulting changes this redefinition would produce in Propositions I and III are apparent. For these two Propositions, modelling the spread using two distinct price processes serves primarily to clarify the precise form of the volatility process. However, the redefinition required to modify Proposition II is much less obvious. While it is still possible to make a redefinition of the price processes for Proposition II that is consistent with modelling the spread as a single random variable, the resulting pricing formula substantively obscures the form of the volatility process. On balance, for practical and pedagogic reasons, Propositions I-III are stated with using distinct price processes.

Section III: Other Types of Spread and Exchange Options

A. Black-Scholes Exchange Options

In order to compare the properties of the Bachelier spread options to the lognormal case, the spread options are converted to exchange options. An important advantage of spread option solutions based on arithmetic Brownian motion is that converting to an exchange option only involves setting X=0 in (4), (5) and (6). The advantage of examining exchange options for geometric Brownian motion is that, while $X \neq 0$ requires a numerical solution to a double integral when S_2 and S_1 or F_2 and F_1 are jointly lognormal, the $X = 0$ lognormal case has a closed form solution. The Black-Scholes futures exchange option price differs somewhat from the Margrabe (1978) result, due to the inability to generate cash flows from the futures contracts when constructing the riskless hedge portfolio. As a consequence, the Black-Scholes futures exchange option still retains the property of linear homogeneity, but a net investment of funds is required to establish the hedge portfolio leading to the PDE:[11]

$$\frac{\partial C}{\partial t} = rC - \frac{1}{2}\{\frac{\partial^2 C}{\partial F_1^{\,2}}\ \sigma_1^{\,2}\ F_1^{\,2} + \frac{\partial^2 C}{\partial F_2^{\,2}}\ \sigma_2^{\,2}\ F_2^{\,2} + 2\frac{\partial^2 C}{\partial F_1 \partial F_2}\ \sigma_{12}\ F_1\ F_2\}$$

This leads to:

Proposition IV: Black-Scholes Futures Exchange Option

Assuming perfect markets and continuous trading, if the two prices in the spread difference (F_2 - F_1) involve futures prices which follow constant parameter geometric Brownian motions, then the solution to the futures exchange option valuation problem is given by:

$$C_{Bt} = e^{-rt*}\{F_2\ N[f_1] - F_1\ N[f_2]\} \qquad (7)$$

$$f_1 = \frac{\ln[F_2/F_1] + (\sigma_f^2/2)\ t^*}{\sigma_f\ \sqrt{t^*}} \qquad f_2 = f_1 - \sigma_f\sqrt{t^*}$$

$$\sigma_f^2 = \sigma_1^{\,2} - 2\rho_{12}\sigma_1\sigma_2 + \sigma_2^{\,2}$$

C_{Bt} is the price of the Black-Scholes futures exchange option and $N[f_i]$ represent the cumulative normal density evaluated at f_i.

This solution can be proved by direct differentiation of the Black-Scholes futures exchange option formula and verifying that the PDE is satisfied.

Where the securities pay unequal proportional dividends and follow separate geometric Brownian motions, a generalization of Margrabe (1978) provides the perfect markets, continuous trading result for an European exchange option:

Proposition V: Lognormal Exchange Option with Unequal Dividend-Paying Assets

Assuming perfect markets and continuous trading, if the two prices in the spread difference (S_2 - S_1) involve securities which pay constant proportional dividends δ_2 and δ_1, respectively, and the prices follow constant parameter geometric Brownian motions, then the solution to the exchange option valuation problem is given by:

$$C_{Ut} = S_2\, e^{-\delta_2 t^*}\, N[d_1] - S_1\, e^{-\delta_1 t^*}\, N[d_2] \qquad (8)$$

$$d_1 = \frac{\ln[S_2/S_1] + (\delta_1 - \delta_2 + \sigma_u^2/2)\, t^*}{\sigma_u \sqrt{t^*}} \qquad d_2 = d_1 - \sigma_u \sqrt{t^*}$$

$$\sigma_u^2 = \sigma_1^2 - 2\rho_{12}\sigma_1\sigma_2 + \sigma_2^2$$

C_{Ut} is the price of the Black-Scholes exchange option for securities with unequal dividend payments and $N[d_i]$ represent the cumulative normal density evaluated at d_i.

As with the Black-Scholes futures exchange option, Proposition V is proved by direct differentiation of (8) and verifying the PDE is satisfied. As with Proposition II, Proposition V has considerable practical value for pricing cross-currency exchange swaptions and can be adapted to pricing cross-currency warrants, e.g., Dravid et al. (1994).

Deriving the PDE for the riskless hedge portfolio relevant to securities with unequal dividend payments depends on the linear homogeneity of the Black-Scholes exchange option. This permits the riskless hedge portfolio to be constructed with no net investment of funds. Recognizing that the two securities will pay unequal proportional dividends over time leads to the PDE:

$$\frac{\partial C}{\partial t} = -\frac{1}{2}\{\frac{\partial^2 C}{\partial S_1^2}\, \sigma_1^2 S_1^2 + \frac{\partial^2 C}{\partial S_2^2}\, \sigma_2^2 S_2^2 + 2\frac{\partial^2 C}{\partial S_1 \partial S_2}\, \sigma_{12} S_1 S_2\} - \delta_1\, \frac{\partial C}{\partial S_1} S_1 - \delta_2\, \frac{\partial C}{\partial S_2} S_2$$

The hedge ratios being ($\exp\{-\delta_2\}$ $N[d_1]$) for S_2 and -($\exp\{-\delta_1\}$ $N[d_2]$) for S_1, to form the hedge portfolio ($S_2 \exp\{-\delta_2\}$ $N[d_1]$) will be sold short and ($N[d_1]\exp\{-\delta_1\}$ S_1) purchased with the balance being just sufficient to purchase the spread call option of Proposition V. The self-financing property does not apply to the Bachelier exchange option for securities with constant but unequal proportional dividends. The solution provided by Proposition II reveals hedge ratios of ($\exp\{-\delta_2\}$ $N[z]$) and -($\exp\{\delta_1\}$ $N[z]$) for S_2 and S_1, respectively. The resulting hedge portfolio for X=0 cannot be constructed without a net investment. Hence, it is not possible for r = 0 requiring rC to appear in the relevant PDE for the riskless hedge portfolio.

B. The Wilcox Spread Option

The Wilcox spread option is important because it has been acknowledged, e.g., Pearson (1995), Shimko (1994), as the spread option pricing formula for the case where prices follow arithmetic Brownian motion. However, while the formula does have a theoretical foundation in that the solution can be motivated by using the risk neutral valuation problem for a European call option on a non-dividend paying stock, it is possible to demonstrate that, as conventionally stated, the Wilcox formula is not consistent with absence-of-arbitrage. More precisely, for the single state variable case:[12]

$$C_t = e^{-rt^*} E\{\max\ [0,\ S_T - X]\}$$

$$= e^{-rt^*}\{\ E[S_T - X \mid S_T \geq X] + E[0 \mid S_T < X]\}$$

$$= e^{-rt^*}\{E[S_T - X \mid S_T \geq X]\}$$

$$= e^{-rt^*}\ \{E[S_T \mid S_T \geq X] - X\ Prob[S_T \geq X]\} \qquad (10)$$

where t* = T - t, E[·] is the time t expectation taken with respect to the risk neutral density Prob[·] and r is the riskless interest rate. For arithmetic Brownian motion, extending this result to spread options involves substitution of $(S_2 - S_1)_t$ for S_t in (10) to get:

$$C_t = e^{-rt^*}\ \{E[(S_{2T} - S_{1T}) \mid (S_{2T} - S_{1T}) \geq X] - X\ Prob[(S_{2T} - S_{1T}) \geq X]\} \qquad (11)$$

where the E[·] and Prob[·] are for the arithmetic Brownian motion spread process:

$$d(S_2 - S_1) = \alpha_s\ dt + \sigma_s\ dW \qquad (12)$$

where the drift is specified only as some arbitrary constant α_S, which may or may not be some function of S_2, S_1 and t.

Evaluation of the expectation in (11) **using the probability density associated with (12)** leads to Wilcox spread option formula:

Proposition VI: The Wilcox Spread Option Formula

Assuming perfect markets and continuous trading, for a price spread following (12), the solution to the valuation problem (11) provides the Wilcox spread option pricing result:

$$C_W(S_2 - S_1,\ t^*;\ r,\ \sigma,\ X) = e^{-rt^*}\{((S_{2t} - S_{1t}) + \alpha_S t^* - X)\ N[w] + \sigma_S \sqrt{t^*}\ n[w]\} \qquad (14)$$

where:

$$w = \frac{(S_{2t} - S_{1t}) + \alpha_s t^* - X}{\sigma_s \sqrt{t^*}}$$

and N[w] and n[w] represent the cumulative normal density and normal probability function, respectively, evaluated at w.

Evaluating the appropriate derivatives of this solution and comparing with the PDE (3) where $\delta = 0$ reveals that the Wilcox formula does not conform to absence-of-arbitrage due to the presence of the **arbitrary** parameter alpha. This inconsistency raised by the presence of α_s in Proposition VI is intuitive because α_s does not appear in either the PDE (3) or the associated boundary condition, indicating that the absence-of-arbitrage solution will be independent of α_s.

It is possible to develop the Wilcox solution further to identify the restrictions which are imposed on (12) for absence-of-arbitrage. To be consistent with absence-of-arbitrage, it is necessary for (14) to satisfy (2). This leads to:

Corollary VI.1: Absence-of-Arbitrage Coefficient Restrictions on the Bachelier Option

In order for the solution (14) to satisfy the PDE (2), the drift coefficient in (12) must satisfy the restriction $\alpha = r(S_2 - S_1)$.

More precisely, Corollary VI.1 indicates that (12) has to be in the form of an Ornstein-Uhlenbeck (OU) process to satisfy absence-of-arbitrage. The requirement of a non-constant drift also impacts the volatility used in Proposition VI, which must be rescaled and will be non-constant. The absence-of-arbitrage restriction is reflected in Propositions I-III. In other words, using the risk neutral valuation approach associated with (11) would have provided the correct solution if the probability space had been correctly defined.

C. Spread Option Prices for Absorbed Brownian Motion

If unrestricted arithmetic Brownian motion has the appealing feature of providing simpler and more readily implementable closed form solutions for option prices, how do these closed forms compare to solutions obtained using absorbed Brownian motion? Addressing this question requires some consideration of the transition probability density for the spread option, under both absorbed and unrestricted Brownian motion. The importance of the transition density can be identified by considering the risk neutral valuation procedure. On the expiration date, the payout on a spread option has the form: $C_T = \max[S_{2T} - S_{1T} - X, 0]$. Risk neutral valuation determines the spread option pricing formula by evaluating the discounted expected value of the expiration date payout, where the expected value calculation is taken over the assumed probability space for the S_2 and S_1 state variables, a bivariate normal transition probability density. Evaluating this expected value is decidedly less complicated for unrestricted than for absorbed Brownian motion (Heaney and Poitras 1997).

The transition probability density for a spread option calculated using two price processes which both follow absorbed Brownian motion can be motivated by generalizing the single variable case given in Cox and Ross (1976, p.162), Goldenberg (1991, p.7) and, more rigorously, in Karlin and Taylor (1975, p.354-5). In this case, the transition probability density is specified as the difference between the unrestricted density and the density associated with the paths where the state variable goes negative. By ignoring the economic restriction that state variable prices must be non-negative, call option valuation assuming unrestricted Brownian motion provides a higher theoretical price than for absorbed Brownian motion. The unrestricted solution attaches value to those state variable paths which reach zero and continue on to exceed the exercise price at a later time. Under absorbed Brownian motion, these paths will be absorbed at zero and will be assigned a zero value. It follows that the difference between the absorbed and unrestricted solutions, if any, will depend on the density associated with these paths which, in turn, will depend on the exercise price level and the probability of absorbtion, a function of the maturity of the option, the initial price and the volatility of the state variable process.

For the spread option, taking the expectation using the joint density associated with the absorbed Brownian motions can be modelled using a four part density. As in the single variable case, one part applies to the unrestricted paths of S_{1t} and S_{2t}. Hence, as in the single variable case, the unrestricted call option price provides an upper bound on the call price determined using absorbed Brownian motion. The unrestricted density is adjusted by differencing out the second and third parts of the density which apply to paths where one of S_{1t} or S_{2t} is absorbed at zero and the other path is unrestricted. The fourth part of the density accounts for the bias in the second and third parts of the density introduced by ignoring cases where both S_{1t} and S_{2t} are absorbed. This density is conditional, depending on the probability that one price reaches zero given that the other variable also reaches zero. While useful for illustrating the relationship between the univariate and spread option solutions, this four part decomposition of the joint density required for risk neutral valuation of the spread option is not the only method for determining the price formula (Heaney and Poitras 1997).

In general, when the risk neutral expectation of the spread option is evaluated, each of the four parts of the decomposed joint density will produce corresponding terms in the closed form option price. Even in the case of independent random variables, specification of the relevant densities and evaluation of the risk neutral integrations is complex. Allowing the random variables to be dependent makes the problem even more difficult. The resulting closed form solution will be complicated, whatever the approach used to modelling the joint density. Taking the expectation for unrestricted Brownian motion does not require decomposition of the joint density, substantively reducing the number of terms in the closed form option price. This gain in analytical simplicity is desirable if the intuitive disadvantages of assuming unrestricted Brownian motion do not have a significant impact on the pricing of spread options. In particular, if it is not possible for the prices composing the spread (S_2 - S_1) to have negative values before the maturity date of the option the solutions for the unrestricted and absorbed cases will be identical for practical purposes. In terms of the four part decomposition of the joint density, those parts associated with one or both of the prices being absorbed will be zero and the call option solution will be reduced to the unrestricted case.

Section IV: Simulation of Spread and Exchange Option Prices

The closed form spread option prices provided in Section II have the desirable characteristic of simplicity. Is this simplicity achieved at the expense of pricing accuracy? Because a spread option is being priced, answering this question is more complicated than for options involving one security price. In particular, the correlation between the prices in the spread introduces an additional dimension to the comparisons. When evaluating option prices associated with Propositions I and II, another complication also arises in determining the volatility for the arithmetic process. In order to initially avoid this complication, Table 1 provides a summary of simulated futures spread call prices for Proposition III with selected values of F_1, t* and ρ_{12}, the correlation between F_1 and F_2. Considerable variation in pricing is observed as the absolute value of ρ increases. One practical implication of this result is that calendar spread options will typically have ρ_{12} in the .9 region and many intercommodity spread options will often have ρ_{12} in the 0-.5 region. Hence, even though the same dollar values may be traded, the prices for different types of spread options will have considerable variation, depending on the correlation in the state variable prices.

The parameters in Table 1 have been selected to provide rough comparability with Pearson (1995, Exhibit 5): F_2 is fixed at 100 with X=4 and r=.1. Unfortunately, direct comparison is not possible because Pearson evaluates a Wilcox option solution similar to (14) and, in the process, uses an inappropriate method of determining the volatility. More precisely, in order to simulate the Wilcox prices, Pearson estimates α_S empirically and, in order to determine σ_S, Pearson assumes both S_2 and S_1 are lognormal. More precisely, Pearson assumes:

$$\sigma_S^2 = \sigma_1^2 S_1^2 - 2\sigma_{12} S_1 S_2 + \sigma_2^2 S_2^2 \qquad \alpha_S = (r - \delta_2) S_2 - (r - \delta_1) S_1$$

However, this method of specifying the volatility is inconsistent with the assumption of constant parameters used to specify (12) and, as a result, the simulated Wilcox prices will not satisfy the relevant PDE for the dynamic hedging problem if these parameter values are used. Pearson compares the resulting Wilcox option price estimates with prices determined from a double-integral lognormal solution which is evaluated numerically. Pearson reports substantial deviations between the Wilcox option prices and the more complex double integral solution when t^* is large.

Using the same F_2, T and σ_{12} values as in Table 1, the Black-Scholes and Bachelier futures exchange option prices are calculated and reported in Table 2. By retaining the same parameters as for Table 1, the impact of changing from X=4 to X=0 can be determined by examining the Bachelier prices. As expected, the impact is largest for short maturity, in-the-money options, e.g., an increase of \$3.38 for $t^* = .08$, $S_1 = 95$ and $\rho_{12} = .9$. Much smaller differences are observed for the longest maturity options, e.g., an increase of \$1.50 for $t^* = 5$, $S_1 = 95$ and $\rho_{12} = .9$. At any given maturity, the difference increases with ρ_{12}. Examining the differences between the Black-Scholes and Bachelier exchange option prices reported in Table 2, the general similarity of the prices is striking. The primary source of difference is that Black-Scholes prices are less affected as ρ_{12} increases. The absolute size of the difference between the Black-Scholes and Bachelier prices increases with maturity. At any given t^* and ρ_{12}, price differences are larger for out-of-the-money than in-the-money spread options.

The results in Table 2 are significantly different than those provided in Pearson (1995, Exhibit 2 and 5), where much wider differences between Black-Scholes and Bachelier spread option prices were observed. There are a number of possible reasons for the discrepancy. Volatility misspecification is one potential reason for this discrepancy, though results in Table 1 indicate that this is not likely to be a substantive source of pricing error. The use of exchange options instead of spread options is also not likely to be a source of **differences** in the Bachelier and Black-Scholes prices, e.g., due to the linear impact of X on the Bachelier prices. Another possible difference is that Pearson uses the Wilcox solution which differs from (6) in a number ways, such as the need to provide an estimate for the drift of the spread process. Finally, a fundamental parametric difference is that Table 2 is calculated for futures prices, while Pearson evaluates a physical security which incorporates dividends. Comparison of (6) with (4) and (5) reveals a number of important differences. Significantly, (4) and (5) involve volatility estimates which depends on t*, r and δ in a complicated fashion.

Table 3 reveals that the impact of introducing dividend payments on the difference between Bachelier and Black-Scholes exchange option prices is substantial. Determining volatility (Λ) was a major difficulty in calculating the Bachelier exchange options in Table 3. One source of difficulty is the singularity point in v_{12} of Proposition II which prevented the use of the same δ_1 and δ_2 as in Pearson. There are similar singularity points in v_{11} and v_{22}. No attempt was made to determine a specific volatility to calibrate the exchange option prices, though there appears to be calibration at $\rho_{12} = .9$, $t^* = .08$ and $S_1 = 95$. The v_{11} and v_{22} values used were determined by solving the implied V for an individual security using the same call price as for a constant proportional dividend Black-Scholes call calculated using the parameter values associated with Table 3, e.g., $\sigma = .2$. To determine the appropriate Λ for $\rho_{12} = .5$ and .9, the relevant value for v_{12} was directly calculated. Bachelier prices are relatively unchanged from Table 2 while Black-Scholes prices exhibit much larger changes, particularly with small ρ_{12} and large t*.

The advantage of selecting the specific r, δ and σ values used in Table 3 is that a rough comparison with Pearson is permitted. In assessing the differences in the Black-Scholes and Bachelier prices, implications of the method used to determine Bachelier volatility (Λ) and the implied calibration of prices have to be recognized. Calibrating prices for a specific volatility would involve selecting some initial parameter values and determining the volatilities required for the Black-Scholes and Bachelier prices to be equal.[13] Given this, Table 3 generally confirms the differences reported by Pearson. Substantial pricing differences are observed, particularly for the long maturity, low ρ_{12} cases. This difference would be even larger if negative ρ_{12} results were reported. Comparison of the Black-Scholes results of Table 3 with Pearson (Exhibit 2) can also provide an assessment for the impact of exercise price changes on the Black-Scholes spread option. Results similar to those from the comparison of Bachelier prices in Tables 1 and 2 are observed. This indicates a possibility for developing easily calculated heuristic prices for Black-Scholes spread options by using the Black-Scholes exchange option solution, with an adjustment for the exercise price effect.

V. Conclusion and Summary

Spread trading is an important source of liquidity in both cash and futures markets. Wide variation in the types of spreads being traded can be identified. Just in the futures market there are: credit spreads, such as the TED spread between Eurodollar and Treasury bill rates; production spreads, such as the soybean crush spread between prices for soybeans, soybean meal and soybean oil or the crack spread between prices for crude oil, heating oil and gasoline; tailed or untailed calendar spreads between the prices for the same commodity on different delivery dates, e.g., Yano 1989; and, maturity spreads, such as the NOB spread between prices for Treasury notes and Treasury bonds. In addition to these trades, the cash market also features spreads for other commodities as well as spreads based on variations in grade and location. Despite the considerable potential for spread option trading to support cash and futures market activity, there is only relatively restricted trading in options on price differences. There are only a few exchange traded contracts and activity in the OTC market is limited.

Spread option pricing is complicated by the presence of two random variables. Because the spread distribution will be the convolution of two distributions, closed form solutions for spread options are difficult to determine in general. For the conventional options pricing assumption of lognormally distributed random variables and nonzero exercise price, closed form solutions are not available and numerical methods are required to determine option prices. This paper exploits the linearity properties of arithmetic Brownian motion to specify closed forms for three types of spread option prices: securities paying equal dividends; securities paying unequal dividends; and, futures prices. The solutions provided are referred to as Bachelier spread options. In order to provide a pricing comparison with solutions assuming lognormally distributed prices, relevant exchange option prices for the lognormal case are derived. Bachelier and Black-Scholes exchange option prices are then compared and differences identified. For spreads involving the difference of two futures prices, the Black-Scholes and Bachelier spread option prices are similar. However, sizable pricing differences are observed for spread options involving securities making dividend payments. These differences are partly due to difficulties in determining volatility for the arithmetic processes and, in turn, calibrating that volatility to the lognormal case.

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Appendix

Proof of Proposition I:

The derivations in the following Propositions require the result that:

$$\frac{\partial n[g]}{\partial x} = \frac{\partial}{\partial x} \frac{1}{\sqrt{2\pi}} e^{-\frac{g^2}{2}} = n[g] \frac{\partial\{-g^2/2\}}{\partial g} \frac{\partial g}{\partial x} = n[g] \ (-g) \ \frac{\partial g}{\partial x}$$

Given this, the proof proceeds by treating the spread as a single random and evaluating the PDE (2). For ease of notation, let $(S_2 - S_1) = S$ and observe that the relevant derivatives of (4) are now:

$$\frac{\partial C}{\partial S} = e^{-\delta t^*} N[h] + (Se^{-\delta t^*} - Xe^{-rt^*}) \frac{\partial N}{\partial S} + V \frac{\partial n}{\partial S} = e^{-\delta t^*} N[h]$$

$$\frac{\partial^2 C}{\partial S^2} = e^{-\delta t^*} \frac{\partial N}{\partial S} = e^{-\delta t^*} n[h] \frac{\partial h}{\partial S} = \frac{e^{-2\delta t^*}}{V} n[h]$$

$$\frac{\partial C}{\partial t^*} = (-\delta Se^{-\delta t^*} + rXe^{-rt^*}) N[h] + (Se^{-\delta t^*} - Xe^{-rt^*}) \frac{\partial N}{\partial t^*} + \frac{\partial V}{\partial t^*} n[h] + V \frac{\partial n}{\partial t^*}$$

$$= -\frac{\partial C}{\partial t} = (-\delta Se^{-\delta t^*} + rXe^{-rt^*}) N[h] + \frac{\partial V}{\partial t^*} n[h]$$

Substitution back into the PDE and cancelling leads to:

$$-\frac{\partial V}{\partial t^*} n[h] = r \ V \ n[h] - \frac{1}{2}\sigma^2 \frac{e^{-\delta t^*}}{V} n[h]$$

$$-\frac{\sigma^2}{2V} \left\{\frac{-\delta \ e^{-2\delta t^*} - r \ e^{-2rt^*}}{r - \delta}\right\} = r\left[\frac{\sigma^2 \ (e^{-2\delta t^*} - e^{-2rt^*})}{2(r - \delta)}\right]^{1/2} - \frac{1}{2} \sigma^2 \frac{e^{-\delta t^*}}{V}$$

Multiply through by V and then by $[2(r - \delta)]/\sigma_2$ and cancelling proves the Proposition. Derivatives for verifying that (4) also obeys (3) can be obtained by examining and, where appropriate, simplifying the solutions to Proposition II.

Proof of Proposition II:

For ease of notation, let C_i denote partial differentiation with respect to S_i. Second derivatives are similarly defined. The relevant derivatives of (5) are:

$$C_1 = e^{-\delta_1 t^*} N[z] \qquad C_2 = -e^{-\delta_2 t^*} N[z]$$

$$C_{11} = \frac{e^{-2\delta_1 t^*}}{V} n[z] \qquad C_{22} = \frac{e^{-2\delta_2 t^*}}{V} n[z] \qquad C_{12} = -\frac{e^{-(\delta_1 + \delta_2)t^*}}{V} n[z]$$

$$\frac{\partial C}{\partial t^*} = -\frac{\partial C}{\partial t} = (-\delta_2 S_2 e^{-\delta_2 t^*} + \delta_1 S_1 e^{-\delta_1 t^*} + rXe^{-rt^*})\ N[z]$$

$$+ (S_2 e^{-\delta_2 t^*} - S_1 e^{-\delta_1 t^*} - Xe^{-rt^*}) \frac{\partial N}{\partial t^*} + n[z] \frac{\partial V}{\partial t^*} + V \frac{\partial n}{\partial t^*}$$

$$= (-\delta_2 S_2 e^{-\delta_2 t^*} + \delta_1 S_1 e^{-\delta_1 t^*} + rXe^{-rt^*})\ N[z] + n[z] \frac{\partial V}{\partial t^*}$$

Substituting these results back into the PDE and cancelling terms leaves:

$$n[z] \frac{\partial V}{\partial t^*} = rVn[z] - \frac{1}{2V}(\sigma_1^2\ e^{-2\delta_1 t^*} + \sigma_2^2\ e^{-\delta_2 t^*} - 2\sigma_{12}\ e^{-(\delta_1 + \delta_2)t^*})\ n[z]$$

Evaluating $\{\partial V/\partial t^*\}$, cross-multiply by -2V and cancel n[z] which is common to all terms. All terms involving r e^{-rt^*} now cancel. Collecting the remaining terms from rVn[z] and $\{\partial V/\partial t^*\}$ and cancelling the denominators where appropriate, the remaining terms all cancel which proves the Proposition.

Proof of Proposition III:

The relevant derivatives of (6) are:

$$\frac{\partial C}{\partial F} = e^{-rt^*} N[k] \qquad \frac{\partial^2 C}{\partial F^2} = \frac{e^{-rt^*}}{\sigma \sqrt{t^*}} n[k]$$

$$\frac{\partial C}{\partial t^*} = -r\ e^{-rt^*}\ \{(F-X)\ N[k] + \sigma \sqrt{t^*}\ n[k]\} + e^{-rt^*}(F-X) \frac{\partial N}{\partial t^*}$$

$$+ e^{-rt^*} n[k] \frac{\sigma}{2\sqrt{t^*}} + e^{-rt^*} \sigma \sqrt{t^*} \frac{\partial n}{\partial t^*}$$

$$= -\frac{\partial C}{\partial t} = -r\ e^{-rt^*} \{(F-X)\ N[k] + \sigma \sqrt{t^*}\ n[k]\} + e^{-rt^*}\ n[k] \frac{\sigma}{2\sqrt{t^*}}$$

Substitution back into the PDE and cancelling proves the Proposition.

Proof of Proposition VI:

From (12), over the time interval starting at t and ending at T, with t* = T - t:

$$S_T = S_t + \alpha t^* + \sigma \sqrt{t^*}\, Z \qquad or \qquad Z = \frac{S_T - S_t - \alpha t^*}{\sigma \sqrt{t^*}}$$

where Z is N(0,1). Evaluating (10) gives:

$$X\ Prob[S_T \geq X] = X\ N[\frac{S_t + \alpha t^* - X}{\sigma\sqrt{t^*}}]$$

$$E[S_T\ |S_T \geq X] = \int_{(X - S_t - \alpha t^*)/\sigma\sqrt{t^*}}^{+\infty} (S_t + \alpha t^* + \sigma\sqrt{t^*}\ Z)\ n[Z]\ dZ$$

$$= (S_t + \alpha t^*) \int_{-\infty}^{(S_t + \alpha t^* - X)/\sigma\sqrt{t^*}} n[Z]\ dZ + \sigma\sqrt{t^*} \int_{(X - S_t - \alpha t^*)/\sigma\sqrt{t^*}}^{+\infty} Z\ n[Z]\ dZ$$

$$= (S_t + \alpha t^*)\ N[\frac{S_t + \alpha t^* - X}{\sigma\sqrt{t^*}}] + \sigma\sqrt{t^*}\ n[\frac{S_t + \alpha t^* - X}{\sigma\sqrt{t^*}}]$$

Substituting these results into (10) and observing the definition of g gives Proposition VI.

Proof of Corollary VI.1:

Given the rule for differentiating n[·], the relevant derivatives of (12) are:

$$\frac{\partial C}{\partial S} = e^{-rt^*}\{N[g] + (S + \alpha t^* - X)\frac{\partial N}{\partial S} + \sigma\sqrt{t^*}\ \frac{\partial n}{\partial S}\} = e^{-rt^*}\ N[g]$$

$$\frac{\partial^2 C}{\partial S^2} = e^{-rt^*}\frac{\partial N}{\partial S} = e^{-rt^*}\ n[g]\ \frac{\partial g}{\partial S} = e^{-rt^*}\ \frac{n[g]}{\sigma\sqrt{t^*}}$$

$$\frac{\partial C}{\partial t^*} = -\frac{\partial C}{\partial t} = -rC + e^{-rt^*}\{\alpha N + (S+\alpha t^* - X)\frac{\partial N}{\partial t^*} + \frac{1}{2}\frac{\sigma}{\sqrt{t^*}}n[g] + \sigma\sqrt{t^*}\frac{\partial n}{\partial t^*}\}$$

$$= -rC + e^{-rt^*}\{\alpha N + \frac{1}{2}\frac{\sigma}{\sqrt{t^*}}n[g]\}$$

Substituting these results into (2) and cancelling proves the Corollary.

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Spread Options, Exchange Options and

Arithmetic Brownian Motion*

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ABSTRACT

This paper is concerned with exploiting the properties of arithmetic Brownian motion to specify closed form prices for spread options. Previous results derived from assuming arithmetic Brownian motion are reviewed and spread option pricing problems for some relevant securities are solved using this assumption. Implications of significant variations in futures spread option contract design are also considered. Recognizing the simplifications provided by exchange options for the lognormal case, a number of futures exchange option pricing solutions are simulated for the normal and lognormal cases and compared across a range of pricing scenarios. Solutions are also provided for a variety of other spread option pricing situations.

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NOTES

1. Examples include many of the exotic options as well as the stochastic convenience yield and stochastic volatility models, e.g., Rubinstein (1991a,b), Gibson and Schwartz (1990), Ball and Roma (1994).

2. It is also possible to use lognormality to solve the spread option for the redundant case where the spread is treated as a single random variable. While this case is potentially applicable to a range of spread options, e.g., credit spreads such as the TED spread, this approach is inconsistent with the assumption that the individual prices are lognormally distributed. This follows because the difference of lognormal variables will not be lognormal.

3. This class of processes includes all untransformed prices with diffusions having stationary distributions which are normal. Other terminology such as **absolute** Brownian motion and Gaussian process is also used.

4. This terminology follows Smith (1976) and Goldenberg (1991). Austin (1990) associates a Bachelier option with absorbed Brownian motion.

5. As in Goldenberg (1991) and Smith (1976), Brennan (1979) neglects to make an obvious simplification in the formula involving the argument entering the cdf and pdf. As indicated in Cox and Ross (1976), the N and n function arguments are the same.

6. Relevant distribution-free properties of spread options including put-call parity conditions are examined in Shimko (1994) and Grabbe (1995). One fundamental result provided by Shimko (p.191) is that the value of a spread option with exercise price X will be less than or equal to any combination of a call on S_2 with exercise price X_2 and a put on S_1 with exercise price X_1, given $X_2 - X_1 = X$. In effect, a spread option will be a less expensive than trading puts and calls on the underlying commodities in the spread.

7. This form of the spread option suppresses consideration of the method of specifying units of the securities or commodities being exchanged. In many cases, the number of units being exchanged will be equal and the option price can be considered as a per unit price. In other cases, the units being exchanged will differ and the prices will represent the value of the items being exchanged.

8. Hybrid approaches are also possible, as in Grabbe (1995) or Shimko (1994). An alternative pricing methodology is provided by Brooks (1995) which uses a lattice approach to valuing spread options.

9. For example, the difference or sum of two lognormally distributed distributions will not usually be lognormal, though the product will be. Similarly, while the sum of two exponentially distributed variables will be gamma, the same is not true about the difference. A key advantage of using normal random variables to model spreads is that the convolution of the difference of two normal distributions is also normal.

10. Net implied carry is defined as interest and other carry charges net of pecuniary carry returns and convenience yield.

11. In specifying the Black-Scholes exchange options the volatility parameters, σ^2_i, are for lognormal diffusions and are not the same as those used in Section II, which apply to arithmetic Brownian motion. While the same notation is being used for different parameters, the difference will be apparent from the context.

12. To be implemented, risk neutral valuation requires a transition probability density to be specified for evaluating the expectation. Assuming risk neutrality imposes certain conditions on the admissible form of the transition probability density, typically the restrictions required to apply Girsanov's theorem. Risk neutrality also permits discounting by the riskless rate, e.g., Cox

and Ross (1976, p.153). Stoll and Whalley (1993, Chap. 11) provides a useful introduction to risk neutral valuation with (10) being given on p.214.

13. Some method of calibration is required because Black-Scholes volatility is for returns while Bachelier volatility is for prices.