# 6. The Valuation of Life Annuities

## Origins of Life Annuities

The origins of life annuities can be traced to ancient times. Socially determined rules of inheritance usually meant a sizable portion of the family estate would be left to a predetermined individual, often the first born son. Bequests such as usufructs, maintenances and life incomes were common methods of providing security to family members and others not directly entitled to inheritances.[1] One element of the Falcidian law of ancient Rome, effective from 40 BC, was that the rightful heir(s) to an estate was entitled to not less than one quarter of the property left by a testator, the so-called "Falcidian fourth' (Bernoulli 1709, ch. 5). This created a judicial quandary requiring any other legacies to be valued and, if the total legacy value exceeded three quarters of the value of the total estate, these bequests had to be reduced proportionately.

The Falcidian fourth created a legitimate valuation problem for jurists because many types of bequests did not have observable market values. Because there was not a developed market for life annuities, this was the case for bequests of life incomes. Some method was required to convert bequests of life incomes to a form that could be valued. In Roman law, this problem was solved by the jurist Ulpian (Domitianus Ulpianus, ?-228) who devised a table for the conversion of life annuities to annuities certain, a security for which there was a known method of valuation. Ulpian's Conversion Table is given by Greenwood (1940) and Hald (1990, p.117):

*Age of annuitant in years*

| 0-19 | 20-24 | 25-29 | 30-34 | 35-39 | 40…49 | 50-54 | 55-59 | 60- |
|---|---|---|---|---|---|---|---|---|

*Comparable Term to maturity of an annuity certain in years*

| 30 | 28 | 25 | 22 | 20 | 19…10 | 9 | 7 | 5 |
|---|---|---|---|---|---|---|---|---|

The connection between age and the pricing of life annuities is a fundamental insight of Ulpian's table.

Using the duration of the annuity certain together with the value of the annuity payment and an assumed interest rate, the value of life annuities and other types of usufructs can be determined. However, it is not apparent how calculations were actually made using Ulpian's table. Nicholas Bernoulli in the *De Usu Artis Conjectandi in Jure* (1709) indicates that values were often determined by taking the annual value of the legacy, and multiplying this value by the term to maturity of the annuity certain to get the associated legacy value (Hald 1990, p.117). For example, if the individual was 37 years old and was receiving a life income of £100 per year, then the legacy value according to Ulpian's Table would be £2000. This naive valuation method would be consistent with the market practice of quoting life annuity prices using 'years' purchase'. Bernoulli correctly identifies the method of multiplying the table value by the size of the payment as faulty due to the omission of the value of interest. Bernoulli observes that at, say, 5% interest the value of the legacy would be only £1246.22.

Ulpian's table has attracted considerable attention over the years. The conversion method underlying the table has not been discovered, so only speculative conclusions are available. Nicholas Bernoulli (1709) interpreted the table to represent expectations of life, but this point has been disputed by Greenwood (1940) who observes that the main concern of Ulpian was to ensure that the heir received at least the Falcidian fourth as set out in Roman law. As a result, the table values are likely to be conservative, determining only the legal maximum value. In any event, the valuation method of determining the price of a life annuity using an annuity certain with the expectation of life as the term to maturity is known as Ulpian's approximation. As will be shown, this valuation method does not produce an actuarially correct price for a life annuity.

The acceptance of life annuities in Roman law was carried forward into the *census* contracts of medieval Europe. However, despite evidence that Ulpian's table had not been forgotten, prices of life annuities were typically quoted without reference to age (Daston 1988, p.121). By the early 15th century, life annuities had become an established method of municipal finance in Italy, Germany and the Low Countries. For example, from 1402 Amsterdam offered life annuities, priced without reference to age. Prices charged varied considerably across time and from location to location. Despite being unaffected by age, prices were influenced by the prevailing rate of interest. Because the price was not affected by age, it was common practice for purchasers to chose healthy children as nominees.

### The Genesis of Modern Contingent Claims Pricing

One of Simon Stevin's lesser known contributions involved drawing up the statutes and curriculum for a new mathematical school for engineers that was created at Leiden in 1600 (van Berkel 1988). The focus of the mathematical instruction was decidedly more practical than conventional university instruction and it was in Leiden that Frans van Schooten (1615-1660) instructed a number of important students including Christian Huygens (1629-1695), Jan Hudde (1628-1704) and Jan de Witt (1625-1672).[2] Of these three, Huygens is well-known, justifiably recognized for writing the 'first published work on probability' (Hald 1990, p.68) that is credited with providing the first precise presentation of mathematical expectation. While Huygens dedicated his life to academic pursuits, both de Witt and Hudde became involved in Dutch political life.

Jan de Witt was not a professional mathematician. He was born into a burgher-regent family and at Leiden was a student of jurisprudence.

While at Leiden, de Witt lived in the house of van Schooten who, while a professor of jurisprudence, was also deeply involved in mathematical studies. Van Schooten encouraged Huygens, Hudde and de Witt in their mathematical studies and published their efforts as appendices to two of his mathematical books. De Witt's contribution on the dynamics of conic sections was written around 1650 and published as an appendix to van Schooten's 1659 exposition of Cartesian mathematics, *Geometria a Renato Des Cartes*. From the perspective of the history of mathematics, de Witt's contribution is an interesting and insightful exposition on the subject but 'marks no great advance' (Coolidge 1990, p.127).

Around 1650, de Witt began his career in Dutch politics as the pensionary of Dordrecht. In 1653, at the age of 28, de Witt became the grand pensionary or prime minister of Holland. During his term as grand pensionary, de Witt was confronted with the need to raise funds to support Dutch military activities, first in the Anglo-Dutch war of 1665-1667 and later in anticipation of an invasion by France that, ultimately, came in 1672. Life annuities had for many years been a common method of municipal and state finance in Holland and de Witt also proposed that life annuity financing be used to support the war effort. However, de Witt was not satisfied that the convention of selling of life annuities at a fixed price, without reference to the age of the annuitant, was a sound practice. Instead de Witt proposed a method of calculating the price of life annuities that would vary with age. This remarkable contribution can be considered the start of modern contingent claims analysis.

More precisely, aided by contributions from Huygens in probability and Hudde in mortality statistics, in *Value of Life Annuities in Proportion to Redeemable Annuities* (1671, in Dutch) de Witt provided the first substantive analytical solution to the difficult problem of valuing a life annuity.[3] Unlike the numerous variations of fixed term annuity problems that had been solved in various commercial arithmetics, the life annuity valuation required the weighting of the relevant future cash flows by the probability of survival for the designated nominee. De Witt's approach, which is somewhat computationally cumbersome but analytically insightful, was to compute the value of a life annuity by applying the concept of mathematical expectation advanced by Huygens in 1657.[4]

De Witt's approach involved making theoretical assumptions about the distribution of the number of deaths. To provide empirical support for his calculations, he gave supplementary empirical evidence derived from the register at The Hague for life annuitants of Holland and West Friesland for which he calculated the average present values of life annuities for different age classes, based on the actual payments made on the annuities. This crude empirical analysis was buttressed by the considerably more detailed empirical work of Hudde on the mortality statistics of life annuitants from the Amsterdam register for 1586-1590. For the next century, the development of pricing formula for life annuities is intimately related to progress in the study of life contingency tables, a subject that is central to the development of modern statistical theory and actuarial science.

Not long after submitting his *Value of Life Annuities* to the States General, de Witt's life came to a tragic end. The invasion of the Dutch Republic by France in 1672 led to a public panic that precipitated de Witt's forced resignation and his replacement by the Stadholder William III. However, the demand for public retribution for the Grand Pensionary's perceived failings did not end with his resignation. Later in 1672, de Witt was set upon by a mob and shot, publicly hanged and his body then violated. However, despite the tragic demise of de Witt, Jan (Johan) Hudde had been consulted by him on various aspects of the results contained in the *Value of Life Annuities*, particularly the validity of the calculations, the empirical evidence on mortality of annuitants and the theoretical procedures required to calculate annuities on two or more lives. Hudde continued and expanded de Witt's work on life annuity valuation.[5]

The similarities between the lives of Hudde and de Witt are striking. Both were students of jurisprudence under van Schooten at Leiden from whom both acquired interest and ability in mathematics. Both also went on to undertake important political positions. Like de Witt, Hudde also published mathematical papers, in Hudde's case as an appendix to van Schooten's *Exercitationum Mathematicarum* (1657). Hudde's mathematical work on the solution to algebraic equations has received more recognition than de Witt's work on conic sections. In particular, Hudde derived necessary and sufficient conditions for an 'ingenious method' (Coolidge 1990, p.135) of finding out when an algebraic equation has two equal roots. Hudde is also credited with an important notational contribution: providing the first instance where a single letter is used to denote a variable having both positive and negative values (D. Smith 1958, v.2, p.466).

Hudde was not a professional mathematician, spending the bulk of his life as a civil servant and politician. The highlight of Hudde's political career is being chosen in 1672 by William III to be a burgomaster of Amsterdam, a position Hudde was to hold intermittently for 21 years, switching at times to serve as chancellor of the admiralty. In 1672 Hudde also was selected to direct the destruction of dykes in order to creating flooding to slow the advance of the French army. Hudde's contributions to financial economics originate from the active correspondence he had with de Witt on the problems associated with valuing life annuities. Even though the formulation and solution of the problem can be attributed to de Witt, Hudde clarified numerous empirical points and, almost certainly, assisted de Witt in working through various aspects of the problem.

Prior to engaging with de Witt on the specific problem of pricing life annuities, Hudde had been pursuing his own related work on mortality statistics. Hudde used a data set containing 1,495 Amsterdam annuitants associated with purchases from the period 1586-1590. Using this data, Hudde calculated average present values for all annuitants and for the 1 to 10 year age class, coming up with values similar to de Witt. Though Hudde's work on the empirical distribution of deaths has been recognized by statisticians as an important early contribution to the calculation of life tables, the value of the statistical contribution is limited because the data he presented were based only on the lives of life annuity nominees and, as a result, are not representative of the whole population. However, as contributors to the history of financial economics, the theoretical work of de Witt and the empirical work of Hudde must be considered *seminal*. Important elements of modern financial economics, such as contingent claim pricing and risk neutral valuation, are reflected in this early work.

**Graunt and Halley**

It is difficult to assess the impact of de Witt's contribution to the practice of pricing life annuities. Based on his recommendation, in 1672 the city of Amsterdam began offering life annuities with prices dependent on the age of the nominee. However, this practice did not become widespread and by 1694, when Edmond Halley (1656-1742) published his influential paper 'An Estimate of the Degrees of Mortality of Mankind, drawn from the curious Tables of the Births and Funerals at the City of Breslaw; with an Attempt to ascertain the Price of Annuities upon Lives', the English government was still selling life annuities at seven years' purchase, independent of age.[6] Halley's paper is remarkable in providing substantive contributions to both demography and financial economics. The importance of this paper reinforces the intellectual stature of an individual who is recognized in modern times primarily for his contributions to astronomy.

Halley's 'Estimate…' is mainly concerned with presenting a life table calculated from the detailed birth and death registers of Breslau in Silesia. At the time, the most important source of statistical demography was John Graunt's *Natural and Political Observations Made Upon the Bills of Mortality* (1662) which was limited by the incomplete records that the bills of mortality for London provided, for example, Pearson (1978, ch. II).[7] Graunt's *Natural and Political Observations* is justly recognized as being a seminal contribution to the history of statistics and demography. Insofar as life tables are a necessary, but not sufficient, element in the calculation of the price of life annuities, Graunt also is so some importance in the history of financial economics. The *Observations* is much more than an honest statistical compilation of an interesting demographic data set. The arguments and reasoning used in the book are remarkable and Hald's (1990, p.87) recommendation is warranted: 'Graunt's fascinating book should be read by anyone interested in the history of statistics.'

**Selection from *Natural and Political Observations*, Graunt (1662)**

Of the Number of Inhabitants

I have been several times in company with men of great experience in this City, and have heard them talk seldom under millions of people to be in London, all which I was apt enough to believe until, on a certain day, one of eminent reputation was upon occasion asserting that there was in the year 1661 two millions of people more than in the year 1625, before the great Plague; I must confess that, until this provocation, I had been frighted with that misunderstood example of David, from attempting any computation of the people of this populous place; but hereupon I both examined the lawfulness of making such enquiries and, being satisfied thereof, went about the work itself in this matter: viz.

2. First, I imagined that, if the conjecture of the worthy person aforementioned had any truth in it, there must needs be about six or seven millions of people in London now; but repairing to my Bills I found that not above 15,000 per annum were buried, and consequently, that not above one in four hundred must die per annum if the total were but six millions.

3. Next considering, that it is esteemed an even lay whether any man lives ten years longer, I supposed it was the same, that one of any 10 might die within one year. But when I considered, that of the 15,000 aforementioned about 5,000 were Abortive, and Stillborn, or died of Teeth, Convulsion, Rickets, or as Infants, and Chrysoms, and Aged. I concluded that of men and women, between ten and sixty, there scarce died 10,000 per annum in London, which number being multiplied by 10, there must be 100,000 in all, that is not the one-sixtieth part of what the Alderman imagined. These were but sudden thoughts on both sides, and both far from truth, I thereupon endeavoured to get a little nearer, thus: viz.

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**Graunt (1662) ... (cont'd)**

4. I considered, that the number of child-bearing women might be double to the births: forasmuch as such women, one with another, have scarce more than one child in two years. The number of births I found, by those years wherein the registries were well kept, to have been somewhat less than the burials. The burials in these late years at a medium are about 13,000 and consequently the christenings not above 12,000. I therefore esteemed the number of teeming women to be 24,000: then I imagined, that there might be twice as many women aged between 16 and 76m as between 16 and 40, or between 20 and 44; and that there were about eight persons in a family, one with another, viz. the man and his wife, three children and three servants, or lodgers: now 8 times 48,000 makes 384,000.

5. Secondly, I find by telling the number of families in some parishes within the Walls, that 3 out of 11 families per annum have died: wherefore, 13,000 having died in the whole, it should follow there were 48,000 families according to the last mentioned account.

6. Thirdly, the account which I made of the trained bands and auxiliary soldiers, doth enough justify this account.

7. And lastly I took the map of London set out in the year 1658 by Richard Newcourt, drawn by a scare of yards. Now I guessed that in 100 yards square there might be about 54 families, supposing every house to be 20 foot in the front: for on two sides of the said square there will be 100 yards of housing in each, and in the two other sides 80 each; in all 360 yards: that is 54 families in each square, of which there are 220 within the Walls, making in all 11,880 families within the Walls. But forasmuch as there die within the walls about 3,200 per annum, and in the whole about 13,000; it follows that the housing within the Walls is 1/4 part of the whole, and consequently, that there are 47,520 families in and about London, which agrees well enough with all my former computations: the worst whereof doth sufficiently demonstrate that there are no millions of people on London, which nevertheless most men do believe, as they do, that there be three women for one man, whereas there are fourteen men for thirteen women, as elsewhere hath been said.

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The bills of mortality, as the name suggests, were concerned with causes of death. Starting in 1538, the Church of England had instituted parish registers throughout England in order to record all christening, weddings and burials. While the registers do not record events for those of other faiths, the numbers of these individuals probably did not exceed 15% of the population that was recorded (Hald 1990, p.83). The bills of mortality started in 1604 when the Company of Parish Clerks began to publish weekly bills of mortality for the parishes of London. The bills contained information on the causes of death, that were determined by two 'Searchers, who are ancient Matrons, sworn to their Office'. These matrons went to the place of death, made inquiries and inspections and reported back to the parish clerk. In Graunt's time, the bills did not provide information on the age of the deceased and, given the somewhat limited medical knowledge of the times, about 80 different causes of death were recorded.

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**Graunt (1662) … (cont'd)**

8. We have (though perhaps too much at random) determined the number of the inhabitants of London to be about 384,000: the which being granted, we assert that 199,112 are males and 184,866 females.

9. Whereas we have found that of 100 quick conceptions about 36 of them die before they be six years old, and that perhaps but one surviveth 76, we, having seven decades between six and 76, we sought six mean proportional numbers between 64, the remainder living at six years, and the one which survives 76, and find that the numbers following are practically near enough to the truth; for men do not die in exact proportions, nor in fractions: from whence arises this Table following:

| | | | |
|---|---|---|---|
| Viz. of 100 there dies | | The fourth | |
| 6 | | | |
| Within the first six years | 36 | The next | |
| 4 | | | |
| The next ten years, or | | The next | |
| 3 | | | |
| Decade 24 | The next | | 2 |
| The second decade 15 | The next | | 1 |
| The third decade | 9 | | |

10. From whence it follows, that of the said 100 conceived there remains alive at six years end 64.

| | | | |
|---|---|---|---|
| At sixteen years end | 40 | At fifty-six | |
| 6 | | | |
| At twenty-six | 25 | At sixty-six | |
| 3 | | | |
| At thirty-six | 16 | At seventy-six | 1 |
| At forty-six | 10 | At eighty | |
| 0 | | | |

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**Graunt (1662) ... (cont'd)**

11. It follows also, that of all which have been conceived, there are now alive 40 per cent above sixteen years old, 25 above twenty-six years old, & sic deniceps, as in the above Table: there are therefore of aged between 16 and 56, the number of 40, less by six, viz. 34; of between 26 and 66, the number of 25 less by three, viz. 22: & sic deniceps.

Wherefore, supposing there be 199,112 males, and the number between 16 and 56 being 34. It follows, there are 34 per cent of all those males fighting men in London, that is 67,694, viz. 13,539, is to be added for Westminster, Stepney, Lambeth, and the other distant parishes, making in all 81,233 fighting men.

12. The next enquiry shall be, in how long time the City of London shall, by the ordinary proportion of breeding and dying, double its breeding people. I answer in about seven years, and (Plagues considered) eight. Wherefore since there be 24,000 pair of breeders, that is one-eighth of the whole, it follows that in eight times eight years the whole people of the City shall double without the access of foreigners: the which contradicts not our account of its growing from two to five in 56 years with such accesses.

13. According to this proportion, one couple viz. Adam and Eve, doubling themselves every 64 years of the 5,610 years, which is the age of the world according to the Scriptures, shall produce far more people than are now in it. Wherefore the world is not above 100 thousand years old as some vainly imagine, nor above what the Scripture makes it.

Graunt's work was much more than a statistical compilation of the data contained in the parish registers and the associated bills of mortality. Perhaps the most remarkable example of this arises with the construction of Graunt's Life Table. The stated motivation for the preparation of this table was to answer the immediate question of estimating the number of 'fighting men' in London. This task requires the construction of an age distribution for the surviving population based on the data at hand, primarily the number of deaths classified by causes. No direct information is available on the age of death, the size of the underlying population or the rate at which population is changing due to migration. The task of preparing a life table seems insurmountable.

Graunt was able to accomplish the remarkable task of constructing a life table by making some plausible assumptions about the types of diseases that occur during childhood and during old age. For example, causes such as 'infants', 'abortives', and 'thrush' can be associated with childhood. From this, Graunt estimates about 36% mortality in the 0-6 age group. Deaths classified as 'aged', about 7%, are handled by making assumptions about the age required to die from old age. The final result is Graunt's life table, expressed showing the progress for a cohort of 100 newborns:

From whence it follows, that of the said 100 conceived there remains alive at the end of six years 64.

| | | | | | |
|---|---|---|---|---|---|
| At Sixteen years end | 40 | At Twenty six | 25 | At Thirty six | 16 |
| At Forty six | 10 | At Fifty six | 6 | At Sixty six | 3 |
| At Seventy six | 1 | At Eighty | 0 | | |

Contemporaries recognized the significance of this remarkable table and were stimulated to verify its validity using more precise data. The modern reader will likely be more fascinated by the detailed listing of causes of death that Graunt provides. This list, which is compiled from the bills of mortality, include such unusual categories as 'Fainted in a Bath', 'Frighted', 'Found Dead in the Streets', 'Grief', and 'Itch'.

**Halley (1693, pp.602-3) on the Valuation of Life Annuities**

**Use V.** On this depend the Valuation of Annuities upon Lives; for it is plane that the Purchaser ought to pay for only such a part of the value of the Annuity, as he has Chances that he is living; and this ought to be computed yearly, and the Sum of those yearly Values being added together, will amount to the value of the Annuity for the Life of the Person proposed. Now the present value of Money payable after a term of years, at any given rate of Interest, either may be had from Tables already computed; or almost as compendiously, by the Table of Logarithms: for the Arithmetical Complement of the Logarithm of Unity and its yearly Interest (that is, of 1,06 for Six per Cent. being 9,974694.) being multiplied by the number of years proposed, gives the present value of One Pound payable after the end of so many years. Then, by the foregoing Proposition, it will be as the number of Persons living after that term of years, to the number dead; so are the Odds that any one Person is Alive or Dead. And by consequence, as the Sum of both or the number of Persons living of the Age first proposed, to the number remaining after so many years, (both given by the Table) so the present value of the yearly Sum payable after the term proposed, to the Sum which ought to be paid for the Chance the person has to enjoy such an Annuity after so many Years. And this being repeated for every year of the persons Life, the Sum of all the present Values of those Chances is the true Value of the Annuity. This will without doubt appear to be a soft laborious Calculation, but it being one of the principal Uses of this Speculation, and having found some Compendia for the Work, I took the pains to compute the following Table, being the short Result of a not ordinary number of Arithmetical Operations; It shews the Value of Annuities for every Fifth Year of Age, to the Seventieth, as follows:

| Age | Years' Purchase | Age | Yr's P. | Age | Yrs' P. |
|---|---|---|---|---|---|
| 1 | 10 | 25 | 12,27 | 50 | 9,21 |
| 5 | 13,40 | 30 | 11,72 | 55 | 8,51 |
| 10 | 13,44 | 35 | 11,12 | 60 | 7,60 |
| 15 | 13,33 | 40 | 10,57 | 65 | 6,54 |
| 20 | 12,78 | 45 | 9,91 | 70 | 5,32 |

**The Life of Edmond Halley**

The Breslau data used in the preparation of Halley's 'Estimate…' was much better suited to construction of a life table than the bills of mortality. Thanks to Leibnitz, the data set came to attention of the Royal Society and Halley, the editor of the Society's journal, was selected to analyze the data. From the end of the 16th century, Breslau, a city in Silesia, had maintained a register of births and deaths, classified according to sex and age. For the purposes of constructing a precise life table, only the population size is missing. The paper is primarily concerned with constructing Halley's life table and touches on the valuation of life annuities only as an illustration of applying the information in the life table. In the process, Halley presents a somewhat different approach than de Witt to the valuation of a life annuity. This paper was Halley's primary effort in both demographics and financial economics.

It is a great credit to financial economics that an intellectual giant such as Halley dedicated analytical efforts to problems of interest to the subject. Halley's interests in financial economics were not limited to the life annuity valuation problem of the 'Estimate…'. Another example is Halley's work on the use of logarithms to solve for yield to maturity in present value problems. Yet another example is provided by Pearson (1978, p.89):

> One ingenious idea of Halley's may be mentioned from these years. Mr. John Houghton desired to know the total acreage of England and its counties and asked Halley's advice. Halley got the best map of England, cut off all the sea and struck the largest circle he could in the remainder. He then weighed the circle and the remainder on delicate scales and thus obtained the ratio of the whole of England to the circle, which had a known area of paper, and its area could be at once obtained by the known scale of the map. By the same method Halley computed the area of all the separate counties. Rough, but adequate for the purpose in days when planimeters were unknown.

While seemingly of incidental interest to financial matters, this example is of interest due to the connection to John Houghton, FRS, producer of the *A Collection for the Improvement of Husbandry and Trade*.

It is tempting to cover Halley's life in the conventional fashion, for example, Ronan (1972): Halley was born in Haggerston, England on 29 October, 1656; the eldest son of a well-to-do landowner, soapmaker and salter from the City of London, also known as Edmond Halley. The father had sufficient means to ensure an impressive education on his son, who showed an interest in astronomy from an early age. Together with an impressive and valuable collection of astronomical instruments that had been purchased by his father and in part made for himself, the younger Edmond Halley set of to study at Oxford at the age of 17. After three productive years of study, which included three papers published in the *Philosophical Transactions* of the Royal Society, at the age of 20 Halley moved from the overachieving to the remarkable (Pearson 1978, p.82):

> at the age of 20 an idea occurred to this young undergraduate. Why should he not go to the Southern Hemisphere and catalogue the stars which never rose above the horizon of either Dantzig or Greenwich? No sooner thought of than carried out. Halley packed up his telescope, left Oxford without a degree … and sailed under the auspices of the East India Company to St. Helena, where he arrived after three months' (!) voyage and set up his telescope, sticking to the work for eighteen months, until he had completed his star catalogue, reaching England again, exactly two years after he had left it, to be hailed as the Tycho Brahe of the Southern Hemisphere.

The star map by itself was considered sufficient for the King, Charles II, to issue a *mandamus* to Oxford for granting Halley a Master of Arts. In 1678, at the age of 22, Halley was made a Fellow of the Royal Society.

Unfortunately, there is so much in the life of Edmond Halley that a conventional historiography quickly becomes many pages, the writer becomes overwhelmed and the process of sifting out important details becomes unmanageable. For example, Halley had an important relationship with Sir Isaac Newton. Some of the connections between Halley and Newton were immediate, such as Halley being instrumental in getting the *Principia* published: 'There is little doubt that we owe its publication to the good offices of Halley' (Pearson 1978, p.86). This aid came both in financial support for publication from both the Royal Society and Halley, as well as 'important editorial aid' (Ronan 1972, p.68) in preparing the manuscript. Newton was a reluctant author, if only because he was not fully satisfied with the results that were being published.

The connections between Halley and Newton were not all so apparent. For example, in addition to the star map, Halley also returned with some puzzling observations about the behaviour of an English clock pendulum in St Helena. 'Halley found that his clock pendulum, which kept good time in England, had to be shortened to do so in St Helena.' When this information was passed on to Newton, he was able to interpret Halley's observations as being due to gravity. From this Newton drew the conclusion that the earth was not a sphere, but rather is an oblate spheroid. In another instance, Halley designed a diving bell and a diver's helmet. In experiments on this equipment, Halley reports 'on the colour of sunlight that he observed at various depths were sent to Newton, who incorporated them in his *Opticks*'.[8]

Halley is best known for his work on the periodicity of comet orbits. The naming of Halley's comet was a posthumous recognition for his theoretical and empirical work on a particular bright comet that exhibited a periodicity of 75 years. Though Halley's observations were well known to astronomers, 'it was not until the 1682 comet reappeared as predicted in 1758 that the whole intellectual world of western Europe took notice. By then Halley had been dead fifteen years; but his hope that posterity would acknowledge that this return "was first predicted by an Englishman" was not misplaced, and the object was named "Halley's comet"' (Ronan 1978, p.69). This recognition was a fitting tribute for someone who had contributed to so many fields, from astronomy and mathematics to history and philology.

## The Contributions of Abraham de Moivre[9]

In assessing Halley's contribution to the history of financial economics, it is natural to immediately mention Abraham de Moivre (1667-1754), an expatriate Frenchman transplanted to London following the Repeal of the Edict of Nantes. Halley and de Moivre were first acquainted in 1692 and in 1695 de Moivre's first paper contributed to the Royal Society was presented by Halley. Unlike Halley who touched only briefly on the pricing of securities, de Moivre spent much of his productive life studying the practical problem of pricing life annuities. By the time de Moivre undertook his work on life annuities, the basic groundwork had been laid. However, Halley and others recognized that the brute force approach to calculating tables for valuing life annuities would require 'a not ordinary number of Arithmetical operations'. Halley attempted to develop simplifying mathematical procedures, 'to find a Theorem that might be more concise than the Rules there laid down, but in vain'.

Selection from *Treatise of Annuities on Lives*,
de Moivre (1756b)

The *Treatise* is composed of two Parts, with Part I being composed of problems which de Moivre states and then solves. Part II is concerned with 'the Demonstrations of some of the principal Propositions in the foregoing Treatise'. The problems of Part I follow a natural progression in terms of complexity and application. Following is the first of the thirty three problems:

**Problem I**

*Supposing the Probabilities of Life to decrease in Arithmetic Progression, to find the Value of an Annuity upon a Life of an Age given.*

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de Moivre **(1756b) ... (cont'd)**

The solution to Problem II provided by de Moivre was:

**Solution**

Let the Rent or Annuity be supposed = 1, the Rate of Interest = $r$, the Complement of Life = $n$, the Value of an Annuity certain to continue during $n$ Years = $P$, then will the Value of the Life be $\{[1 - (r/n) P]/(r - 1)\}$, which is thus expressed in Words at length;

*Take the Value of an Annuity certain for so many Years, as are denoted by the Complement of Life; multiply this Value by the Rate of Interest, and divide the Product by the Complement of Life, then let the Quotient be subtracted from Unity, and let the Remainder be divided by the Interest of 1 l. then this last Quotient will express the Value of an Annuity for the Age given.*

Thus suppose it were required to find the present Value of an Annuity of 1 l. For an Age of 50, Interest being at 5 per Cent.

The Complement of Life being 36, let the Value of an Annuity certain, according to the given Rate of Interest be taken out of the Tables annex to this Book, this Value will be found to be 16.5468.

Let this Value be multiplied by the Rate of Interest 1.05, the Product will be 17.3741.

Let this Product be divided by the Complement of Life, viz. by 36, the Quotient will be 0.4826.

Subtract this Quotient from Unity, the Remainder will be 0.5174.

Lastly, divide this Quotient by the Interest of 1 l. viz. By 0.05, and the new Quotient will be 10.35; which will express the Value of an Annuity of 1 l. Or how may Years Purchase the said Life of 50 is worth.

And in the same manner, if Interest of Money was at 6 per Cent. an Annuity upon an Age of 50 would be found worth 9.49 Years Purchase.

But as I have annexed to this Treatise the Values of Annuities for an Interest of 3, 3½, 4, 5, 6 per Cent. it will not be necessary to calculate those Cases, but such only as require a Rate of Interest higher or lower, or intermediate; which will seldom happen but in case it does, the Rule may easily be applied.

In the history of financial economics de Moivre can be recognized for fundamental contributions involving the application of applied probability theory to the valuation of life annuities. This work laid the theoretical foundation for Richard Price, James Dodson and others to develop the actuarially sound principles required to implement modern life insurance. The immediate incentive for de Moivre was to value the various aleatory contracts that became increasingly popular as the 18th century progressed. Being (together with Laplace) one of two giants of probability theory in the 18th century (Pearson 1978, p.146), de Moivre was singularly well suited to the task of developing the foundations of insurance mathematics. It is one of the quirks of intellectual history that de Moivre's most significant contributions, which lay primarily in the area of probability theory and applied mathematics, contributed little to his personal comfort while his contributions to financial economics managed to help de Moivre maintain body and soul.

To the modern reader, it is strange that a person of de Moivre's stature had to endure most of his life in 'the hardest poverty'. Never able to secure an academic position, de Moivre earned a living as an 18th century reckoning master and algorist, tutoring mathematics, calculating odds for gamblers and reckoning values for underwriters and annuity brokers. Pearson (1978, p. 143) observes that:

> this seamy side of life had a golden lining. Every evening (Sir Isaac) Newton would come and fetch de Moivre from (Slaughter's) Coffee House, and take him for philosophical discussion to his own house in Golden Square. I picture De Moivre working at a dirty table in the coffee house with a broken-down gambler beside him and Isaac Newton walking through the crowd to his corner to fetch out his friend. It would make a great picture for an inspired artist.

De Moivre began his close friendships with Newton and Halley around the same time in the early 1690s. The timing of the 1694 publication of Halley's 'Estimate…' and Halley's subsequent presentation of de Moivre's first paper to the Royal Society in 1695 make it possible that de Moivre played some role in the inclusion of the life annuity valuation problem in the 'Estimate…'. It is not difficult to conceive enlightened interaction between the two on the subject of applying Halley's life table and de Moivre suggesting and explaining the important problem of life annuities. However, de Moivre's primary contribution to pricing life annuities did not appear until much later in the *Annuities Upon Lives* (1725) with a second edition (1743). Also important is the 1756 edition of his *The Doctrine of Chances* that contains a section titled 'A Treatise of Annuities on Lives' together with discussion of the life tables of Halley, Kersseboom, Simpson and Deparcieux.

Selection from *Treatise of Annuities on Lives*,
de Moivre **(1756b)**

As noted, the *Treatise* is composed of two Parts, with Part I being composed of problems which de Moivre states and then solves. Following is the second of the thirty three problems:

**Problem II**

*The Values of two single Lives being given, to find the Value of an Annuity granted for the Time of their joint continuance.*

(cont'd)

de Moivre **(1756b) ... (cont'd)**

**Solution (to Problem II)**

Let $M$ be the Value of one Life, $P$ the Value of the other $r$ the Rate of Interest; then the Value of an Annuity upon the two joint Lives will be, in Words thus;

*Multiply together the Values of the two Lives, and reserve the Product.*

*Let that Product be again multiplied by the Interest of Il. and let that new Product be subtracted from the Sum of the Values of Lives, and reserve the Remainder. Divided the first Quantity reserved by the Second, and the Quotient will express the Value of the two joint Lives.*

Thus, supposing one Life of 40 Years of Age, the other of 50, and Interest at 5 per Cent. The Value of the first Life will be found in the Tables to be 11.83, the Value of the second 10.35, the Product will be 122.4405, which Product must be reserved.

Multiply this again by the Interest of 1 l. viz. by 0.05, and this new Product will be 6.122025.

This new Product being subtracted from the Sum of the Lives which is 22.18, the Remainder will be 16.057975, and this is the second Quantity reserved.

Now dividing the first Quantity reserved by the second, the Quotient will be 7.62 nearly; and this expresses the Vales of the two joint Lives.

If the Lives are equal, the Canon for the Value of the joint Lives will be shortened and be reduced to $\{M/[2 - (r - 1)M]\}$, which in words may be thus expressed;

*Take the Value of one life, and reserve that Value.*

*Multiply this Value by the Interest of 1l. and then subtract the Product from the Number 2, and reserve the Remainder.*

*Divide the first Quantity reserved by the second, and the Quotient will express the Value of the two equal joint Lives.*

In *Annuities*, de Moivre examined a wide variety of the life annuities available in the early 18th century: single life annuities, joint annuities (annuities written on several lives), reversionary annuities, and annuities on successive lives. His general approach to these valuation problems involves two steps: first, to develop a general valuation formula for each type of annuity based on Halley's approach; and secondly, to produce an approximation to the general formula suitable for calculating prices without the considerable efforts involved in evaluating the more exact formula. In order to implement some of the approximations, de Moivre developed a mathematical formulation, a piecewise linear approximation, of the information contained in the life table.

The computational advantages of de Moivre's approximations were considerable and the methods became widely used in day-to-day commercial practice. The ensuing development of actuarial science and insurance mathematics progressed by working with the more tedious exact formulae, estimating more accurate life tables and calculating tables with exact prices for different situations and levels of interest rates. The next important person in the intellectual linkage developing life insurance mathematics was James Dodson, a pupil and friend of de Moivre. While admitting that Dodson's interest in life contingencies almost surely originated with de Moivre, Ogborn (1962, p.23) speculates that it 'is an interesting question whether [Dodson and de Moivre] ever discussed the mathematics of life assurance but there is no published evidence that they did so and it seems that the work is wholly Dodson's'.

Because most of the substantive problems in the exact theory of life annuities had been solved by de Moivre, subsequent initial contributions to the valuation of life annuities were primarily empirical and computational. Thomas Simpson (1710-1761) produced *The Doctrine of Annuities and Reversions* (1742) and *The Valuation of Annuities for Single and Joint Lives* (1752) which calculate a number of useful valuation tables for both single and joint lives using different rates of interest. Simpson is perhaps better known for being accused, in numerous sources (for example, Hald 1990; Pearson 1978), of shamelessly plagiarizing the contributions of de Moivre, both on life annuities and in probability theory. Simpson also took liberties with the contributions of other writers such as John Smart.

### Contributions after de Moivre

Other substantive contributions were made by the Dutchmen Nicholas Struyck (1687-1769) and Willem Kersseboom (1691-1771), the former a mathematician and the latter a statistician. In 1738, Kersseboom published an article in the spirit of Hudde's work on life tables for annuitants and provides a valuation table for single life annuities. Struyck also examines the valuation of life annuities in a memoir that is part of *Introduction to General Geography, besides certain astronomical and other memoirs* (1740, in Dutch). Similar to Hudde and Kersseboom, Struyck recognizes the importance of basing valuation tables on life tables for annuitants and not on life tables for the general population as Halley did. Using the registers for the Amsterdam life annuitants for 1672-1674 and 1686-1689, Struyck is the first to construct separate valuation tables for males and females.

In his *Essay on the Probabilities of the Duration of Human Life* (1746, in French) Antoine Déparcieux (1705-1768) acknowledges and extends the work of Kersseboom to lists of annuitants for the French tontines of 1689 and 1696. He conclusively demonstrates the atypical mortality of the French rentier, compared to both general populations as in Halley's life table and to Kersseboom's Dutch annuitants (Pearson 1978, p.200). Based on his life table, he provides numerous tables calculating the present values of various types of life annuities: single life annuities, joint annuities, tontines, compound tontines and so on.

Deparcieux was careful to provide accurate explanations of his life tables and present value calculations, a feature that distinguishes his work from similar efforts around the same time. Deparcieux continued the work of the *Essay* with *Addition à l'Essai sur les probabilitiés de la* *durée de la vie humaine* (1760) which provides the first available tables for the value of postponed life annuities, fundamental to evaluating pension fund cash flows. From this point it can fairly be argued that contributions to the study of life annuity valuation were concerned with application, extension, improvement and clarification, rather than in producing initial theoretical pricing results.

While it is appealing to conclude that the substantial theoretical development in the pricing of aleatory contracts was accompanied by a similar improvement in commercial practices, this was not the case. For example, in the area of life insurance, England, perhaps the most progressive nation in incorporating theoretical advances into commercial practice, did not establish the legal preconditions for a life insurance industry until the Gambling Act of 1774. Up to this point, many insurance schemes were targeted more at gambling outcomes than risk reduction. The small number of reputable insurance companies did not use actuarial principles in determining either premiums or payouts. Premiums were usually charged at a flat rate per period and payouts determined by dividing the available premium pool between eligible claimants for that period. The first insurance company to apply actuarial techniques was the Society for the Equitable Assurance on Lives and Survivorships (estab. 1762). Much as in modern insurance, the Equitable established premiums based on age, created a fund with which to make future claims and provided for a guaranteed fixed payout in the event of claim.

The resistance of market practice to adopting theoretical pricing results is also reflected in the pricing of life contingent claims. In pricing tontines and life annuities prior to the French revolution, the French government demonstrated only limited ability to set actuarially accurate prices, though Weir (1989, pp.118-19) attributes this to factors other than ignorance, such as the desire to disguise the true cost of the debt. Another rationale for underpricing life annuities was political: to provide a retirement subsidy for the increasingly powerful urban bourgeoisie, the primary purchasers of the government's life contingent debt. These motivations may also partly explain the French and English government practice of not accurately accounting for the age of the nominee in pricing life annuities. For example, even though the English government demonstrated much better understanding of theoretical pricing for life annuities and tontines than the French government, when the English government ended the obviously inaccurate practice of issuing life annuities without regard to age in the latter part of the 18th century, mispricing associated with the practice of permitting selection of nominees continued until 1852.

## De Witt's Calculation of Life Annuities

Useful sources for this material are Alter and Riley (1986), Hald (1990) and Pearson (1978). The first to propose a substantive, actuarially sound solution to the problem of pricing life annuities was de Witt where an age interval between 3 and 80 is considered. Hence, de Witt is considering the value of a life annuity written on the life of a three year old nominee. As the practice up to his time was to sell life annuities at the same price, regardless of the age of the nominee, it was conventional to select younger nominees from healthy families. Based on Hudde' s data for 1586-1590 Amsterdam life annuity nominees, approximately 50% were under 10 years of age, and 80% under 20 (Alter and Riley, p.33). *Throughout the following annuities will be assumed to make a payment of 1 unit of currency (florin, dollars, etc.) each period.*

Instead of assuming a uniform distribution where death at each age would be equally likely, De Witt divided the interval between 3 and 80 into four subperiods: (3,53), (53,63), (63,73) and (73,80). Within each subperiod, an equal chance of mortality is assigned. The number of chances assigned to each subgroup is 1, ⅔, ½, ⅓. It is of some interest that these assigned values do not correspond to de Witt's assumptions about the chance of dying in a given year in the second subgroup as 1½ that of the first group, 2 times the first for the third and 3 times the first for the fourth, being instead the reciprocal of these values. The chance of living beyond 80 is assumed to be zero. While de Witt corresponded with Hudde about mortality data he was collecting and tabulating for the 1586-1589 Amsterdam annuitants, these probabilities were assumed and not directly derived from a life table.

From these assumptions, de Witt constructs a distribution for the number of deaths and calculates the life annuity price as the expectation of the relevant annuity present values. In doing this, de Witt explicitly recognizes that life annuities were paid in semi-annual instalments, requiring time to be measured in half years and for survivors to be living at the end of the half year in order to receive the payment. The 77 year period translates into 154 half years. Using a discount rate of 4% per annum, De Witt uses his assumed chances of mortality in any half year to calculate a weighted average of the present values for the certain annuities associated with each half year. The resulting value is the expected present value of the life annuity that is the recommended price at which the annuity should be sold.

Algebraically, de Witt' s technique can be illustrated by defining $A_n$ to be the present value of an annuity with a 4% annual rate to be paid at the end of the half year *n* (Hald, pp.124-5):

$$A_n = \sum_{t=1}^{n} \frac{1}{(1 + r)^t} = \frac{1}{r} - \frac{1}{r(1 + r)^n} \qquad \textit{where} \quad (1 + r) = \sqrt{1.04}$$

To evaluate the expected present value of the life annuity, de Witt performs the calculation:

$$E[A_n] = \frac{\sum_{n=1}^{99} A_n + \frac{2}{3}\sum_{n=100}^{119} A_n + \frac{1}{2}\sum_{n=120}^{139} A_n + \frac{1}{3}\sum_{n=140}^{153} A_n}{128}$$

Interpretation of the sums is aided by observing that individuals must be alive at the end of the half year to qualify for annuity payments. For example, dying in the first half year means that no payments will be received. The divisor of 128 is calculated by determining the total number of chances as:

$$(100)1 \ + \ (20)\frac{2}{3} \ + \ (20)\frac{1}{2} \ + \ (14)\frac{1}{3} \ = \ 128$$

where the number in brackets is the number of half years in each subgroup. De Witt's solution can be compared to the less realistic case where the distribution of deaths is assumed to be uniform:

$$E[A_n] \ = \ \frac{\sum_{n=1}^{153} A_n}{154} \ = \ \frac{1}{154}(0) \ + \ \frac{1}{154}\,\frac{1}{1 \ + \ r} \ + \ \frac{1}{154}\sum_{t=1}^{2}\frac{1}{(1 \ + \ r)^t}$$

$$+ \ \ldots \ + \ \frac{1}{154}\sum_{t=1}^{153}\frac{1}{(1 \ + \ r)^n} \ + \ 0$$

By assigning less weight to the largest cash flows, de Witt's calculated expected value of 16.0016 florins for annual payments of 1 florin differs from the expected value of 17.22 florins calculated using a uniform distribution.

## Calculations of Halley and de Moivre

It is possible to reexpress de Witt's formula in more general terms by observing that the total number of annuities sold on a life starting at year $x$, $\ell_x$, equals the sum of $d_x + d_{x+1} + \ldots + d_{w-x}$ where $d_i$ is the number of annuities that terminate in period $i$ due to the death of annuitant nominees in that half year and that $d_i = 0$ for $x \geq w$. Taking $\ell_{x+t}$ to be the number of nominees, starting in year $x$ surviving in period $x+t$, it follows that: $d_{x+t} = \ell_{x+t} - \ell_{x+t+1}$ and that the probability of death in any given half year $j$ is $d_{x+j}/\ell_x$. The general pricing formula for a life annuity follows:

$$E[A_n] = \frac{1}{\ell_x}\sum_{n=1}^{w-x-1} A_n\, d_{x+n} = \frac{1}{\ell_x}\sum_{n=1}^{w-x-1} d_{x+n} \sum_{t=1}^{n} \frac{1}{(1+r)^t}$$

$$= \frac{1}{\ell_x}\sum_{t=1}^{n}\sum_{n=1}^{w-x-1} d_{x+n}\frac{1}{(1+r)^t} = \frac{1}{\ell_x}\sum_{n=1}^{w-x-1} \ell_{x+n}\frac{1}{(1+r)^n}$$

The last step in the derivation comes from progressively collecting terms associated with $(1+r)^{-t}$. For example, the $(1+r)^{-1}$ term will appear in each annuity and will, as a result, have coefficients that are the sum of $d_{x+1}, d_{x+2}, \ldots.\ d_{w-x}$. Recalling the definition of $d$ in terms of $\ell$, this sum returns $\ell_{x+1}$. This is the life annuity pricing formula presented by Halley (1693).

De Moivre provided an important simplification for the value of a single life annuity under the assumption that the 'Probabilities of Life ... decrease in Arithmetic Progression' or, in other words, are uniformly distributed starting at year $x$ up to some terminal year $w$, n = $w - x$. Generalizing the uniformly distributed formula given previously, de Moivre's result is derived by observing that for the uniform case:

$$E[A_n] = \frac{n-1}{n}\frac{1}{1+r} + \frac{n-2}{n}\frac{1}{(1+r)^2}$$

$$+ \ldots. + \frac{n-(n-1)}{n}\frac{1}{(1+r)^{n-1}} + \frac{n-n}{n}\frac{1}{(1+r)^n}$$

$$= \sum_{t=1}^{n}\frac{1}{(1+r)^t}\left(1 - \frac{t}{n}\right) = \sum_{t=1}^{n}\frac{1}{(1+r)^t} - \sum_{t=1}^{n}\frac{t}{n\,(1+r)^t}$$

From this point, de Moivre provides an obscure derivation in the first edition of *Annuities upon Lives* and a more tedious demonstration in later editions. A more modern derivation is provided in Pearson (1978, pp. 147-8) where it is observed that:

$$\sum_{t=1}^{n}\frac{t}{n\,(1+r)^t} = \frac{1+r}{n}\sum_{t=1}^{n}\frac{t}{(1+r)^{t+1}} = -\frac{1+r}{n}\frac{dA_n}{d(1+r)}$$

It follows that:

$$E[A_n] = \left[A_n + \frac{1+r}{n}\frac{dA_n}{d(1+r)}\right] = \frac{A_n}{n}\left[n + \frac{1+r}{A_n}\frac{dA_n}{d(1+r)}\right]$$

The last term, not provided by Pearson, contains the familiar Macaulay duration for the annuity applicable to the longest life.

Substituting the relevant expressions back into $E[A_n]$ and evaluating the derivative gives:

$$E[A_n] = \left[A_n + \frac{1+r}{n}\,\frac{dA_n}{d(1+r)}\right]$$

$$= \left[\frac{1}{r} - \frac{1}{r(1+r)^n}\right] + \frac{1+r}{n}\left[\frac{n}{r(1+r)^{n+1}} + \frac{1}{r^2(1+r)^n} - \frac{1}{r^2}\right]$$

$$= \frac{1}{r} - \frac{1+r}{n}\left\{\frac{1}{r}\left[\frac{1}{r} - \frac{1}{r(1+r)^n}\right]\right\} = \frac{1}{r}\left(1 - \frac{1+r}{n}A_n\right)$$

The final rhs expression is de Moivre's approximation to the value of a single life annuity. If only for the computational savings provided, this formula is a considerable advance. From the tedious calculation of a long weighted sum, with weights extracted from the not completely accurate life tables available at his time, de Moivre provides a calculation that could be done in a matter of seconds with or without the aid of an appropriate table for annuities certain. While the derivation provided is not precisely de Moivre's (see Hald 1990, pp. 521-2), the connection to the familiar notion of Macaulay duration is instructive. Similar to the improvement of the duration measure provided by the introduction of convexity, the accuracy of de Moivre's formula can be improved by considering higher order derivative terms (Pearson 1978, pp.150-52). In this case, the higher order terms improve the inaccuracy associated with the assumption of uniformly distributed death rates.

De Moivre provided numerous approximations relevant for other cases, such as joint life annuities, where two lives are nominated and the annuity payments continue until both are dead. Some of de Moivre's approximations were more successful than others and Simpson expended considerable effort showing that direct calculation making use of life tables was substantially better for pricing the joint life annuity (Hald 1990, p.532). De Witt also considered the problem of joint life annuities and, implementing an early version of Pascal's triangle, provided an insightful solution, considered to be his ‘most important contribution to mathematics’ (Coolidge 1990, p.131). De Moivre, Simpson and later writers used a more direct approach to the price of a joint life annuity,

$E[A_{mn}]$ for two joint lives, involving the price of the single life annuities $E[A_n]$ and $E[A_m]$ and an annuity for joint life continuance that makes payments only when both nominees are alive $E[{}_mA_n]$. Because the pricing problem for single life annuities was solved, the joint life annuity problem involved solving for $E[{}_mA_n]$.

The de Moivre approach to solving a joint life annuity written on two lives involved the relationship:

$$E[A_{mn}] = E[A_n] + E[A_m] - E[{}_mA_n]$$

This result follows from observing that the probability of having survival of at least one of the two lives at time $t$ is [1 - (1 - Prob[$x,t$]) (1- Prob[$y,t$])] = Prob[$x,t$] + Prob[$y,t$] - Prob[$x,t$]Prob[$y,t$] where Prob[$x,t$] is the probability of $x$ ($y$) surviving at time $t$ which can be related to $\ell_{x+t}/\ell_x$ in the $E[A_n]$ formula given previously. Multiplying by $(1 + r)^{-t}$ and summing gives the required result. From this point de Moivre used two approaches to solve for approximations to $E[{}_mA_n]$, one involved taking Prob[·] to be arithmetically declining and the other geometrically declining. While the former leads to a more exact result for $E[A_{mn}]$, the latter has a less complicated formula (Hald 1990, pp.528-30). An example of market prices for joint annuities are the 14%, 12% and 10% (7, 8.5 and 10 years' purchase) rates offered on annuities for one, two and three lives, irrespective of age, in a 1694 issue by the Government of England.

**Valuation of Tontines**

In the case of the tontine, approximations are not needed from the borrower's perspective because the payouts are based on the maximum mortality in a group. Assuming large numbers of nominees in each age group, the *tontine* has the desirable feature of being similar to a term annuity for the borrower, where w is set equal to the expected duration of the longest life and $x$ is determined by the average age associated with the specific class of borrowers. This creates a technical problem with determining the duration of the longest life, but the furthest cash flows are so deeply discounted there is little impact on pricing, except for the older age classes and relatively large differences in expected maximum life. For example, at 5% compounding the pricing impact of assuming 85 or 100 years as the maximum age, for the {5, 20, 40} year age groups is {19.60, 19.16, 17.77} versus {19.81, 19.60, 18.93} (Alter and Riley 1986, tables T1 and T4).

From the borrowers perspective, the tontine is similar to a term annuity. The situation is somewhat different for lenders. By offering increasing payments to surviving group members, the tontine introduces an element of lottery for the purchaser. Unlike life annuities, this lottery depends on the life expectancy of other nominees in the age class and on the percentage of total shares held in that group (Weir 1989). If all the shares are held by one individual written on one nominee, the return is identical to a life annuity. These features will impact the pricing of the tontine, even if the pricing method is accurate, assuming that investors do not pool risks. In the case of the French tontines, inaccurate underpricing by the government combined with the benefits of pooling risk led to the emergence of investment trusts that sold units backed by tontines (Alter and Riley, pp. 27-8). The risk pooling was done by purchasing tontines with different nominees. This process was further refined by selecting nominees for the tontines who were typically young, healthy and vital females, from families with a history of longevity.

While it is possible to generate differences under certain conditions, the value of a tontine for borrower and lender will be approximately equal under reasonable conditions. For example, take the case of uniformly distributed death rates and assume a starting date at birth with maximum possible life of n. If death occurs in the first period, no income will be received. If death occurs in the second period, income will be $[n/(n-1)](1+r)^{-1}$. If death occurs in the third period, income will be:

$$\left(1 + \frac{1}{n-1}\right)\frac{1}{1+r} + \left(1 + \frac{2}{n-2}\right)\frac{1}{(1+r)^2}$$

For death in the fourth period:

$$\left(1 + \frac{1}{n-1}\right)\frac{1}{1+r} + \left(1 + \frac{2}{n-2}\right)\frac{1}{(1+r)^2} + \left(1 + \frac{3}{n-3}\right)\frac{1}{(1+r)^3}$$

and so on. Observing that death occurring in each of the $n$ periods is equal to $1/n$ and summing gives:

$$\frac{n-1}{n}\left(1 + \frac{1}{n-1}\right)\frac{1}{1+r} + \frac{n-2}{n}\left(1 + \frac{2}{n-2}\right)\frac{1}{(1+r)^2} + \ldots = \sum_{t=1}^{n-1}\frac{1}{(1+r)^t}$$

Which is the required result.

**Bernoulli's Problem**

A final point to be considered is the relationship between $E[A_n]$ and the value of a term annuity for the expected duration of life from a given starting age $x$. The difference between these two valuations was recognized by de Witt but the point was still a revelation to Nicholas Bernoulli (1709) who stated: 'I notice that the value of (life annuity) incomes is not correctly calculated by supposing that the return will last as many years as someone is supposed probably to live.' To illustrate this problem, for simplicity assume all deaths occur at the beginning or end of period. This assumption permits the exclusion of the problem of evaluating where the average time of death will be in the year for persons that die after $s$ but before $s+1$, for example, averaging would give $s+\frac{1}{2}$ years. Given this, the expectation of life at birth, $D$, can be compared with $E[A_n]$ starting at birth and the associated value of the term annuity with length $D$:

$$D = \sum_{t=1}^{w-x-1} t \frac{d_{x+t}}{\ell_x} \qquad E[A_n] = \sum_{t=1}^{w-x-1} A_n \frac{d_{x+t}}{\ell_x}$$

$$A_d = \sum_{t=1}^{D} \frac{1}{(1+r)^t}$$

Comparing $D$ with $E[A_n]$ and $A_d$ it is apparent that $D > E[A_n]$ and $D > A_d$, due to the impact of discounting on the terms in $E[A_n]$ and $A_d$. Even if interest rates are zero and $D = A_d$, $E[A_n]$ and $D$ are still not equal due to the $E[A_n]$ only crediting the cash flow if the end of period is reached. (This is the point that was suppressed for simplicity.)

The difference between $D$ and $E[A_n]$ is well known, for example, Alter and Riley (1986, p.9), Hald (1990, p.128). However, the comparison between $A_d$, a certain annuity with term equal to expected life, and $E[A_n]$, the expected value of an annuity lasting for the duration of a life, is not as obvious. Under the simplifying assumption, these values will be equal if interest rates are zero. However, for $r > 0$, $A_d \geq E[A_n]$ with = only when all deaths occur at $n$. To see this, consider the uniformly distributed case where:

$$E[A_n] = \frac{1}{r}\left(1 - \frac{1+r}{n} A_n\right) \qquad A_d = \frac{1}{r} - \frac{1}{r\,(1+r)^D}$$

It follows:

$$A_d - E[A_n] = \frac{1}{r(1+r)^D} - \left(\frac{1+r}{n}.\frac{1}{r}\right)\{\frac{1}{r} - \frac{1}{r\,(1+r)^n}\}$$

$$= \frac{1}{r^2}\left\{\frac{r}{(1+r)^D} - \left[\frac{1+r}{n}\left(1 - \frac{1}{(1+r)^n}\right)\right]\right\}$$

$$= \frac{1}{r^2}\left[\frac{nr\,(1+r)^{n-D} + (1+r) - (1+r)^{n+1}}{n(1+r)^n}\right] > 0$$

In more general form, this was the relationship observed by Nicholas

Bernoulli.

**A Note on the Bernoullis**

The Bernoulli family of Basel, Switzerland includes many important politicians, merchants, jurists and mathematicians. At least four Bernoullis made important contributions to probability theory, James (1654-1705), John (1667-1748), Nicholas (1687-1759) and Daniel (1700-1782). Of these four, Nicholas and Daniel made substantive contributions to financial economics. Daniel has been recognized for introducing the notion of expected utility as method of solving what has come to be known, thanks to Laplace, as the St Petersburg Paradox (Jorlan 1987). The Bernoullis corresponded widely with other scholars involved in leading the probabilistic revolution. Nicholas Bernoulli's considerable correspondence with Pierre de Montmort (1678-1719) and de Moivre is particularly noteworthy. It was a 1713 letter between Nicholas and Montmort that contained the first statement of the Saint Petersburg problem.

The St Petersburg problem has many interesting facets. While Nicholas Bernoulli originally formulated the problem as a dice game, he described the game to the mathematician Gabriel Cramer (1704-1752) who defined the modern form of the problem in a 1728 letter to Nicholas. The game involves two players. Player A tosses a coin until a head appears at which time the game ends. Player B receives one coin if a head appears on the first toss, two coins if on the second toss, four coins if on the third toss, continuing until, in general, $2^{n-1}$ coins would be paid on the nth toss. The question is: how much would B be willing to wager to play this game? Using the conventional notion of mathematical expectation, the expected payoff would be infinite. The problem is often referred to as a paradox because common sense indicates that the value of the game to player B would not be infinite.

Yet another of the subtle quirks of intellectual history is that the 1728 letter of Cramer proposed a solution to the problem making explicit use of expected utility of wealth.[10] Daniel Bernoulli's result was produced shortly thereafter, apparently independently of Cramer. Daniel's result involved using the more appropriate log utility function instead of the square root utility used by Cramer. Daniel's work was communicated to the Imperial Academy of Sciences of St Petersburg and published in 1731, with Cramer's letter to Nicholas included as an Appendix, hence the reference to the St Petersburg problem. Further work was done on the solutions to this problem by both Euler and Laplace, among others. However, by the 19th century, the law of large numbers had become the motivation for mathematical expectation and the St Petersburg problem was of interest only 'to exemplify the idea that probability was valid only for repeatable events' (Jorlan 1987, p.171).

In comparison to the work of Daniel, the contributions of Nicholas Bernoulli were more substantial. Even more so than de Moivre, Nicholas was responsible for making the problem of pricing life annuities 'part of the probabilist's repertoire of applications by the first decade of the 18th century' (Daston 1987, p.242). Building on notions introduced in James Bernoulli's *Ars Conjectandi* (published posthumously in 1713), Nicholas produced the Latin treatise, *De Usu Artis Conjectandi in Jure* (1709). The contents of the book are suggested by the English translation of the title, 'On the Use of the Art of Conjecturing in Law'. Though the main thrust of the book is concerned with applying mathematical probability theory to legal problems, chapters 4-6 are directly concerned with problems of interest in financial economics.

The titles of the relevant chapters are: chapter 4, 'Of the purchase of an expectation, and in particular of the purchase of life incomes'; chapter 5, 'Concerning the means of deducting the Falcidian fourth from the bequests of maintenances, usufructs, life incomes, etc.'; and chapter 6, 'Concerning assurance and nautical interest' (Hald 1990, p.376). Though decidedly more developed than de Witt and Hudde, the general approach used in these chapters is along the same lines of applying mathematical expectation to either known probabilities, in the case of marine insurance, or from the probabilities estimated from Graunt's life table, in the case of life annuities. Nicholas makes no reference to the work of Halley or other contributions that had emerged in England by that time. Despite the improvement in form, Hald (1990, p.378) concludes that 'there is nearly nothing new' in this work.

## Appendix: Excerpts from *Valuation of Life Annuities...*, de Witt (1671).

Though a somewhat obscure historical document, *Valuation of Life Annuities in Proportion to Redeemable Annuities*, Jan de Witt (1671), is of sufficient importance that it was long ago translated from the Dutch and published in English. This secondary source, Hendricks (1852, 1853), provides a delightful and concise treatment of the 'history of insurance and the theory of life contingencies', providing the translation of de Witt (1671) as part of a much larger contribution. However, this source is itself somewhat obscure and it seems appropriate to provide an excerpt from the sections of de Witt (1671) most relevant to the life annuity valuation problem. This is only for purposes of brevity. The whole text of *Valuation* is itself remarkably coherent. After the fashion of a mathematician, de Witt starts the discussion with: 'First Presupposition ... Second Presupposition ... Third Presupposition ... First Proposition ... Demonstration ... Corollary ... Second Proposition ...' and so on to the Corollary associated with the Third Proposition where de Witt deals directly with the life annuity valuation problem:

### Corollary

It results from what precedes, and from the third presupposition, that as life annuities are paid in all the Offices of Holland and West Friesland by half-yearly instalments, or from six months to six months, that the annuitant loses all his capital, and receives no return whatever from it, if the life upon which the annuity is sunk happen to die in the first half-year after the purchase, or do not live six whole months. The annuity sunk is here supposed to be 1,000,000 of florins, or 20,000,000 stuyvers, per annum, in order that an exact calculation may be made without fractions: therefore, if the above-mentioned life survive a complete half-year, and do not die until in the course of the second half-year, the annuitant has then drawn 10,000,000 stuyvers, from which a deduction being made of 4 per cent. per annum for a half-year, it would have been worth to him in ready cash (that is to say, on the day of purchase of the said annuity) 9,805,807 stuyvers, which he would have had to pay, if taken at the true value. If the above life survive so long as two complete half-years, and die in the third half-year, the annuitant has then drawn 10,000,000 stuyvers after the expiration of the first half-year, and after that of the second half-year likewise 10,000,000 stuyvers; which sums, deduction being made at 4 per cent. per annum, one for a half-year or six months, and the other for a complete year, would have been worth to him in ready cash, or upon the day of purchase of the said annuity, 19,421,1992 stuyvers, and so on, according as the day of decease were to occur in the fourth, fifth, sixth, or further number of half-years, which would have been worth to him each time as many terms or half-yearly sums of 10,000,000 stuyvers as complete half-years had elapsed from the time of the purchase of the annuity, deduction being made as above of the respective discounts. The computed amounts are specially given in the following table:--

If the Nominee survive
the following Term of Life.

| Half-years. | Stuyvers. |
|---|---|
| 0 | 0 |
| 1 | 9,805,807 |
| 2 | 19,421,192 |
| 3 | 28,849,853 |
| 4 | 38,095,415 |
| 5 | 47,161,435 |
| 6 | 56,051,398 |
| | |
| 98 | 431,055,833 |
| 99 | 432,490,825 |
| 100 | 433,897,951 |
| 101 | 435,277,751 |
| | |
| 118 | 455,030,042 |
| 119 | 455,999,472 |
| 120 | 456,950,076 |
| 121 | 457,882,220 |
| | |
| 138 | 471,226,168 |
| 139 | 471,881,080 |
| 140 | 472,523,275 |
| 141 | 473,152,998 |
| | |
| 152 | 479,322,884 |
| 153 | 479,820,563 |
| | |
| 199 | 494,754,836 |
| 200 | 494,952,836 |

['The above table having been calculated very accurately by us the undersigned, Bookkeepers to My Lords the States-General, each separately, and having been collated by us, we find that a perfect agreement exists, without there being any errors in the figures.

(Signed) 'T. BELLECHIERE — JACOB LENSE.']

Thus, then, since an annuitant, having purchased and sunk a life annuity upon a young nominee, has in possession, or in his favour, as many different expectations or chances as there are half-years in which the death of the nominee may occur; — since the first 100 different expectations or chances (comprising the term of 50 years, reckoning from the day of the constitution or purchase of the annuity) may result with the same facility, and relatively to their probability are equal; —since during this term each half-year of the aforesaid nominee's life is equally destructive or mortal (which is demonstrated in the third proposition); —since the following 20 chances or expectation (comprising the first 10 years after the expiration of the 50 years above cited), considered one with the other, each in proposition to each of the first 100 chances, are not in a lower ration than 2 to 3 (according to the presupposition); — since the 20 expectations or chances of the 10 following years (comprising the second series of 10 years after the expiration of the first 50 years), also considered one with the other, each in proportion to each of the first 100 expectations or chances, are not in a lower ration than 1 to 2 (according to the third presupposition); — since the 14 following expectations or chances (comprising the 7 years after the expiration of the two preceding decennial terms, the epoch at which we here suppose the man to terminate his life), taken one with the other, each in proportion to each of the first 100 expectations or chances, are not in a lower ration than 1 to 3; — it follows that the aforesaid annuitant has in possession, or in his favour, more chances or expectations than there are the following table: —

| Chance | of Stuyvers | The Life to Survive Half-years |
|---|---|---|
| 1 | 0 | 0 |
| 1 | 9,805,807 | 1 |
| 1 | 19,421,192 | 2 |
| 1 | 28,849,853 | 3 |
| 1 | 38,085,415 | 4 |
| 1 | 47,161,435 | 5 |
| 1 | 56,051,398 | 6 |
| | | 7 to 97 given in original |
| 1 | 431,055,833 | 98 |
| 1 | 432,490,825 | 99 |
| | Sum 28,051,475,578 | Once = 28,051,475,578* |
| 2/3 | 433,897,951 | 100 |
| 2/3 | 435,277,751 | 101 |
| | | 102-117 in original |
| 2/3 | 455,030,042 | 118 |
| 2/3 | 455,099,472 | 119 |
| | Sum 8,911,946,713 | Two-thirds = 5,941,297,809 |
| 1/2 | 456,950,076 | 120 |
| 1/2 | 457,882,220 | 121 |
| | | 122-137 in original |
| 1/2 | 471,226,168 | 138 |
| 1/2 | 472,881,080 | 139 |
| | Sum 9,297,075,282 | One-half = 4,648,537,641 |

| | | | |
|---|---|---|---|
| 1/3 | 472,523,275 | | 140 |
| 1/3 | 473,152,998 | | 141 |
| | | | 142-151 in original |
| 1/3 | 479,322,884 | | 152 |
| 1/3 | 479,802,563 | | 153 |
| 128 | Sum 6,668,408,125 | One-Third = | 2,222,802,708 |
| | | Total | 40,864,113,736 |

*40,964,113,736, divided by 128, gives 320,032,130 8 9/16, which divided by 20 gives 16,001,606 18 9.[11]

Whence it follows that we can immediately determine, by a mathematical calculation, according to the principle of the second proposition above enunciated, the worth to the aforesaid annuitant of all the above-mentioned chances, taken together, always presupposing that such value is payable in ready money on the day of purchase of the annuity; and the method is as follows: —

Since the first 100 items, each taken once, or each multiplied by the number 1, form the sum of 28,151,475,578 stuyvers; — since the 20 following items, two-thirds of each being taken, or each multiplied by 2/3 (or, which is the same thing, two-thirds of the sum of the aforesaid 20 items), produce a sum of 5,941,297,809 stuyvers; —since then the half of the 20 following items gives a sum of 4,648,537,641 stuyvers, and the third of the 14 following and last items that of 2,222,802,708 stuyvers; —these sums, being combined, amount together to the sum of 40,964,113,736 stuyvers; which being divided by 128 (the number of chances added together), we find for a quotient (that is to say, the real and exact value of all the collective chances,) the sum of 320,032,130 9 stuyvers, or 16,001,607 florins: so that 1,000,000 per annum of life annuities, sunk or purchased on a young life, is worth in fact more in ready money, and should consequently be sold for more than 16,001,607 florins,[12] preserving the right proportion above mentioned; i.e., that proportionately each florin per annum of life annuity is worth more at 16 florins than the interest of a redeemable annuity a 4 per cent. per annum, —and consequently the person who for 16 florins has purchased 1 florin per annum on a young, vigorous, and healthy life, has made a remarkably advantageous contract; —I assert it to be remarkably advantageous for the following reasons: —

Because, in the first place, we have not been able to rate at a certain price, by perfect calculation or correct estimation, the power which the annuitant possesses (power which is of very great value to him) of choosing a life, or person in full health, and with a manifest likelihood of prolonged existence, upon whom to constitute or purchase his annuity; and there is much less risk or danger of a select, vigorous, and healthy life dying in the first half-year than in some of the following half-years at the beginning of which the aforesaid life might perhaps prove to be a weak state of health or even in a fatal illness; and such greater likelihood of prolongation of life in the purchase of an annuity upon a select, healthy, and robust life, may further extend itself to the second, third, and some other following terms or half-years.

In the second place, the advantage resulting from the aforesaid selection is so much the more considerable, as one half-year of life, at the commencement of and shortly after the purchase of the life annuity, is of greater value to the annuitant, with respect to the price of such purchase, than eighteen half-years during which the person upon whom the annuity is purchased might live after the said purchase, from the age, for example, of 70 to 79 years, — a circumstance which, although at first sight it appears strange and paradoxical, is nevertheless real and susceptible of demonstration.

In the third place, although each of the first 100 half-years expiring after the purchase be considered equally destructive or mortal, according to the principle of the before-established calculation, by reason of the scarcely appreciable difference existing between the first and second half of each year, it is however certain, when we examine the matter very scrupulously, that the likelihood of decease of the nominees upon who life annuities are usually purchased is less considerable, and smaller in the first years after the purchase than in the subsequent years, seeing that the said life annuities are oftenest purchased and sunk upon the lives of young and healthy children of 3, 4, 5, 6, 7, 8, 9, 10 years, or thereabout. During that time, and for some years ensuing, these young lives, having become more robust, are less subject to mortality than about 50 years afterwards, and than for some years anterior to these 50 years; and so much the more, as during the first aforesaid years that either are not, or are but little, exposed to external accidents and extraordinary causes of death, such as those from war, dangerous voyages, debauch, or excesses of drink, of the sex, and other dangers; — for females, there are also confinements and other like causes; — so that the first years after the purchase or foundation of the annuity are the least dangerous, which is a considerable advantage for the annuitant, particularly if we reflect, as I have above stated, that one of the said first years may, as regards the original price of purchase, balance a great number of the subsequent years.

Finally, and in the fourth place, it might also evidently occur, that the life upon which the annuity has been sunk were to live more than 77 years after the purchase, being the time supposed in the above calculation as the term of human life, although such considerations cannot be of much importance; for, notwithstanding that by presupposing the aforesaid nominee living still longer than the expiration of the said term, and preserving life up to the hundredth year inclusive, so that the annuitant or his heirs were to receive 46 more entire half-years of annuity, after the expiration of the term of the aforesaid 77 years, this could not, however, increased the price of the life annuity (calculated, as precedes, at above 16 years' purchase, i.e., at more than 16 florins of capital for 1 florin of annuity per annum) by more than 14 1/2 stuyvers of the same capital; and even if the annuitant could be assured that his heirs were, after the expiration of the above 100 years, to enjoy the life annuity from half-year to half-year, and that perpetually, the value of the capital at the time of first purchase would not thereby be increased by 10 stuyvers.

Whence likewise, although it may be considered that the latter years are not established as sufficiently destructive and mortal in the aforesaid presuppositions, and in the calculations upon which I have based them, when compared with the anterior years and the time of life's vigour, we easily conclude that it could not cause an appreciable rise in the price of the purchase found by the above calculation, which in fact is true, even on the presupposition of each half-year of the 10 years after the sixtieth year of the purchase being, instead of twice, three times more destructive and mortal than each half-year of the first 50 years, and of each half-year of the 7 subsequent years being, instead of three times, five times more destructive and mortal than each of the aforesaid first years; and even on the presupposition again, as above, that the said nominee would not survive beyond 77 years after the first purchase. All these presuppositions (which, however, manifestly represent the life as subject to too high mortality) could scarcely reduce by 6 stuyvers the aforesaid 16 florins or value of the before-described annuity. In consequence of all these reasons, we may assume it as established and demonstrated, that the value of a life annuity, in proportion to the redeemable annuity at 25 years' purchase, is really not below, but certainly above, 16 years' purchase; so that a person, wishing to purchase a life annuity in such proportion and according to its real value, ought to pay more than 16 florins for 1 florin of annuity per annum.

Besides the consideration that this calculation has been made on the principle of a deduction of 4 per cent. per annum at compound interest, and this with such benefit to the purchaser of the life annuity that he would realize not only the interest per annum, but also, without any intermission, interest upon interest at 4 per cent. per annum, as thought he could always thus advantageously make use of his money in purchase of annuity; it is constant that one could not always find such opportunity of investing it, and that one is sometimes obliged to let it lie fallow for some time, and often to lend it at a materially smaller interest, to provide against a greater loss.

Even besides this, as the capital of life annuities is not subject to taxation, nor to a reduction to a lower amount of annuity or interest, it follows, that if the blessing of the Almighty continue to be vouchsafed to this country, we may consider the life annuity as much more advantageous to the annuitant than the redeemable annuity, as may manifestly be judged by the example of foregoing times, —by reflecting, in fact, that My Lords the States of Holland and West Friesland have in the course of a few years not only increased the charge for life annuities from 11 years' purchase to 12 years' purchase, and from 12 years' purchase to 14 years' purchase, but that these annuities have been sold, even in the present century, first at first 6 years' purchase, then at 7 and at 8, and that the majority of all life annuities now current and at the country's expense were obtained at 9 years' purchase; which annuities, by reason of the successive reductions of the rate of interest from 6 1/4 to 5 per cent., and then from 5 per cent. to 4 per cent., produce to the annuitants an actual profit of nearly one-half of each half-year's payment, and of more than one-half in the case of those annuities which were obtained at 8 years' purchase or under.

'J. DE WIT.'

## Notes

1. An usufruct is the right of temporary possession, use or enjoyment of the advantages of property belonging to another, so far as may be had without causing damage or prejudice to the property.

2. As is common at this time, a number of spelling variants appear in place of Jan de Witt. The spelling Jan de Witt is found in Hald (1990), Coolidge (1990) and Pearson (1978). Hald also gives the variant Johan de Witt while Pearson reports John de Witt. Heywood (1985) uses Johannes de Wit while Hendricks (1852-3) uses John de Wit. In the *Valuation*, the author is listed as "J. de Wit".

3. Karl Pearson, who had strong views on a number of individuals involved in the history of statistics, depreciates de Witt's work by claiming: "…the data are uncertain and the method of computation is fallacious" (Pearson 1978, p.100). This is at variance with Hald (1990), Alter and Riley (1986) and others. Pearson (1978, p.702) also appears to have been unaware of Hudde's contribution, "I was unaware that (Hudde) had contributed to the theory of probability." Hecksher (v.1, p.214) also raises the possibility that de Witt might not have written all the works which are credited to him by referring to "the Dutchman, Pieter de la Court, whose main work often went under the name of the well-known statesman Jan de Witt." The practice of contracting-out of intellectual contributions was not uncommon around this time, e.g., Joshua Child and the work of Philopatris (Letwin 1963). However, it is highly unlikely that there were more than a handful of individuals both informed and capable enough to appreciate the relevance of Huygens's contribution on mathematical expectation to pricing life annuities. De Witt must be included in this handful of individuals.

4. There are various sources on the valuation of life contingencies, e.g., Alter and Riley (1986), Hald (1990) and Pearson (1978).

5. De Witt's submission to the State's General was "a prime minister's attempt to convince the State's General that the price of annuities should be raised from 14 to 16 years' purchase. Typical of other prime ministers in critical situations, de Witt was short of time, and he had presumably no hope of getting the price raised to more than 16 years' purchase. This may explain the inconsistencies in the paper" (Hald 1990, p.130). This situation speaks to the importance of Hudde's contribution in checking and expanding the original work of de Witt. In a modern setting, it is possible that de Witt and Hudde would have combined to produce a finished publication in which both were co-authors.

6. Halley's paper also did not have any impact on English government borrowing practices as life annuities continued to be sold at seven years' purchase without reference to the age of the annuitant (Hald 1990, p.139).

7. William Petty (1623-1687), one of the early founders of English political economy, was a contemporary and friend of Graunt, with Graunt being four years senior to Petty. Their acquaintance was a least partly the result of both being from Hampshire and both having migrated to London. It is known that in 1651 Graunt was able to obtain a professorship in music for Petty at Gresham College. Petty's *Political Arithmetick* which was written in 1671-2 but not published until 1690 was profoundly influenced by Graunt's *Observations*. There appears to be little practical evidence supporting the legend, e.g, Letwin (1963, p.142), that Petty made a substantive contribution to the writing of Graunt's *Observations* (Pearson 1978, p.12-9).

8. In an interesting development, Halley formed a public company for the purposes of developing commercial applications of the bell and helmet, in particular wreck salvaging. Shares in Halley's company were quoted from 1692-6.

9. There are a number of sources which contain substantial biographical information on de Moivre. Of these, an appendix to the 1967 reprint of the *Doctrine of Chances*, H. Walker (1967), is relatively thorough, if overly conservative. Pearson (1978), though not complete on all details, provides an essential biographical reference. Walker provides a list of different spellings of de Moivre, all of which were used by de Moivre himself. In particular, Moivre, Demoivre, De Moivre and de Moivre were used a various times.

10. Cramer is also involved on the other side of another quirk associated with the Cramer's rule for solving 2x2 determinants. Cramer's publication of the determinant result appeared in 1750, some two years after the posthumous publication of Colin Maclaurin's *Treatise on Algebra* (1748). Maclaurin may have worked out the actual result as early as 1729 (Boyer 1968, p.471). Cramer's main contribution to developing the result seems to have been the introduction of notation to more readily indicate the linear coefficients being manipulated.

11. There is some confusion here in the calculations. According to Hendricks (1853), the partial sums reflect a 'clerical error', which accounts for the discrepancy with the actual total stated. However, the value of 40,964,113,736, which is 'divided by 128 ...', is correctly stated, so the error does not carry forward. The values given for the division are as stated by de Witt. Presmably, 8 9/16 and 18 9 refer to fractions of a stuyver.

12. Hendricks (1853) provides a detailed footnote outlining a calculation method for arriving at this value.