15/6/06

Transition Density Decomposition and Generalized Pearson Distributions

Geoffrey Poitras and John Heaney*
Faculty of Business Administration
Simon Fraser University
Burnaby B.C.
CANADA V5A lS6

email: poitras@sfu.ca

ABSTRACT

This paper explores the implications of decomposing the transition probability density of a one-dimensional diffusion process subject to regular upper and lower reflecting barriers. The decomposition divides the density into a limiting stationary density which is time independent and a power series of time and boundary dependent transient terms. The classical Sturm-Liouville approach used to derive the decomposition results for diffusions provides a connection to the derivation of the Pearson system of distributions. Generalization of the Pearson system to include quartic exponential stationary densities is considered.

Keywords: Sturm-Liouville problem; diffusion; quartic distribution; transition probability density

* The authors are Professor and Associate Professor of Finance at Simon Fraser University. This paper was partially written while G. Poitras was a visiting Senior Fellow at the National University of Singapore.

Transition Density Decomposition and Generalized Pearson Distributions

The theoretical and intuitive distributional implications of boundary restrictions have long been recognized in economics (e.g., Cootner 1962) and have continued to attract attention (e.g., Poitras 1988; Krugman 1991; Ball and Roma 1998; Farnsworth and Bass 2003). Depending on the specific random variables being examined, there are a number of possible approaches that can be used to deal with the presence of boundaries on the sample space. For example, in the estimation of frontier production functions (e.g., Greene 1982) the one-sided effect of the production possibilities boundary is incorporated directly into the likelihood function by making a half-normal distributional assumption. In contrast, Svennson (1991) and de Yong (1994) model the distribution for an exchange rate contained within a target band using regulated Brownian motion arriving at results for the transition probability density that are derived using classical Sturm-Liouville methods. Following Hansen et al. (1998) and Nicolau (2002), diffusion processes can provide a potentially valuable theoretical framework within which the empirical implications of boundary restrictions can be explored. The use of Sturm-Liouville methods to solve for the transition density of a diffusion process subject to boundary restrictions has been used in a wide range of related fields, such as physics, engineering, biology and mathematical statistics, e.g., Ricciardi and Sacerdote (1987), Linetsky (2005).

Working within the classical Sturm-Liouville (S-L) framework, this paper presents a number of results on the distributional implications of imposing regular reflecting boundaries on diffusion processes. De Jong (1994), Lewis (1998), Veerstraeten (2004), Gorovoi and Linetsky (2004) and Linetsky (2005) identify numerous empirical applications for such a theoretical framework. Wong (1964) was seminal in using the S-L approach to solve the forward equation for the one-dimensional diffusion to determine specific closed form transition probability densities associated with the Pearson system of stationary distributions. This paper generalizes this approach to admit stationary densities that are bimodal. In the following, Section I specifies the S-L approach and reviews various studies that have used this approach to model boundary behavior. Section II examines how boundary restrictions, modeled as upper and lower reflecting barriers constraining a diffusion process, interact with the initial condition to produce a particular decomposition of the transition probability density into stationary and transient components. Section III explores results for cases where other types of boundary conditions, e.g., non-attracting, are used to determine workable solutions for the transition probability density. Section IV considers generalizing the Pearson system of distributions to include higher order exponential densities. Section V details properties of the quartic exponential stationary density and develops the implications for fitting the non-linear drift of a diffusion process. Finally, Section VI provides a summary of the main results.

I. The SL Approach

The distributional implications of boundary restrictions, derived by modeling the random variable as a diffusion process subject to reflecting barriers, have been studied for many years, e.g., Feller (1952,1954). The diffusion process framework is useful because it imposes a functional structure that is sufficient for known partial differential equation (PDE) solution procedures to be used to derive the relevant transition probability densities. Wong (1964) demonstrated that with appropriate specification of parameters in the PDE, the transition densities for popular stationary distributions such as the exponential, uniform, and normal distributions can be derived using SL methods. Following Karlin and Taylor (1981), the transition probability density function U is associated with the random (economic) variable x at time t ($U = U[x,\ t \mid x_0]$) that follows a regular, time homogeneous diffusion process with a state space that is either a possibly infinite open interval I_o = $(a,b\colon \infty \leq a < b \leq \infty)$, a finite closed interval I_c=[a,b: $-\infty < a < b < +\infty$], or the specific interval I_s= $[0 = a < b = \infty)$.[1] Assuming that U is twice continuously differentiable in x and once in t and vanishes outside the relevant interval, then U obeys the forward equation (e.g., Gihhman and Skorohod 1972, p.102-4):

$$\frac{\partial^2}{\partial x^2} \left\{ B[x] \;\; U \right\} - \frac{\partial}{\partial x} \{A[x] \;\; U\} = \frac{\partial U}{\partial t} \qquad (1)$$

where: $B[x]$ ($= ½\,\sigma^2[x] > 0$) is the one half the infinitesimal variance and $A[x]$ the infinitesimal drift of the process. $B[x]$ is assumed to be twice and $A[x]$ once continuously differentiable in x. Being time homogeneous, this formulation permits state, but not time, variation in the drift and variance parameters.

The specific problem of deriving the transition probability density for a diffusion process starting at an interior point $x_0 > 0$ with constant parameters $A[x] = \mu$ (≤ 0) and $B[x]$ = ½ σ^2 subject to a regular, fixed lower reflecting barrier at $x = 0$ is well known (e.g., Cox and Miller 1965). Because the process can reach but not pass below the barrier this imposes a restriction on the density to integrate to 1 over the specific interval I_s = $[0, \infty)$ or the open interval I_o = $(0 < a < b < \infty)$, depending on singularities at x=0 in $B[x]$ arising from, say, natural boundary restrictions. Differentiating with respect to time the condition that the density integrate to 1 over the state space then switching the order of integration and differentiation, permits the forward equation to be substituted for the time derivative. Letting $U = U[x, t \mid x_0] = U[x,t]$ for ease of notation, evaluating the remaining integral gives the reflecting boundary condition:

$$\frac{\partial}{\partial x} \left\{ B[x] \;\; U[x,t] \right\}\Big|_{x=0} - A[0] \;\; U[0,t] = 0 \qquad (2)$$

In effect, reflecting barriers can be represented as first derivative restrictions at the boundaries, in this case a lower boundary at $x = 0$. The drift term is required in the boundary condition to ensure conservation of probability. When the drift is zero, (2) reduces to the 'flux zero' condition.

More generally, if the diffusion process is subject to upper and lower reflecting boundaries that are regular and fixed ($-\infty < a < b < \infty$), the "Sturm-Liouville problem" involves solving (1) subject to the separated boundary conditions:[2]

$$\frac{\partial}{\partial x}\ \{\ B[x]\ \ U[x,t]\ \ \}\big|_{x=a} - A[a]\ \ U[a,t]\ =\ 0 \qquad (3)$$

$$\frac{\partial}{\partial x}\ \{\ B[x]\ \ U[x,t]\ \ \}\big|_{x=b} - A[b]\ \ U[b,t]\ =\ 0 \qquad (4)$$

And the initial condition:

$$U[x,0]\ =\ f[x_0] \qquad where\text{:} \quad \int_a^b f[x_0]\ =\ 1 \qquad (5)$$

and $f[x_0]$ is the continuous density function associated with x_0 where $a \le x_0 \le b$. When the initial starting value, x_0, is known with certainty, the initial condition becomes the Dirac delta function: $U[x,0] = \delta[\ x - x_0\ ]$ and the resulting solution for U is referred to as the 'principal solution'. Recognizing time homogeneity of the process eliminates the need to explicitly consider the location of t_0, for ease of notation it is assumed that $t_0 = 0$. In practice, solving (1) combined with (3)-(5) requires a and b to be specified. While a and b have ready interpretations in physical applications, e.g., the heat flow in an insulated bar, determining these values in economic applications can be more challenging. Some situations, such as the determination of the distribution of an exchange rate subject to control bands, are relatively straight forward. Other situations, such profit distributions with arbitrage boundaries or output distributions subject to production possibility frontiers, may require the basic SL framework to be adapted to the specifics of the modeling situation.

In general, solving the forward equation (1) for U subject to (3), (4) and some admissible form of (5) is difficult, e.g., Feller (1952), Risken (1989). In such circumstances, it is expedient to restrict the problem specification to permit closed form solutions for the transition density to be obtained. Wong (1964) provides an illustration of this approach. The PDE (1) is reduced to an ODE by only considering the non-trivial stationary distributions arising from the Pearson system. More precisely, the processes considered obey:

$$\lim_{t \to \infty} U[x,t \mid x_0] = \int_a^b f[x_0] \quad U[x,t \mid x_0] \; dx_0 = \Psi[x]$$

where only the principal solution ($f[x_0] = \delta[\, x - x_0 \,]$) is considered. Restrictions on the stationary distributions $\Psi[x]$ are constructed by imposing the fundamental ODE condition for the unimodal Pearson system of distributions:

$$\frac{d\Psi[x]}{dx} = \frac{e_1 x + e_0}{d_2 x^2 + d_1 x + d_0} \, \Psi[x]$$

The transition probability density U can then be reconstructed by working back from a specific closed form for the stationary distribution using known results for the solution of specific forms of the forward equation. In this procedure, the d_0, d_1, d_2, e_0 and e_1 in the Pearson ODE are used to specify the relevant parameters in (1). The U for important distributions that fall within the Pearson system, such as the normal, beta, central t, and exponential, can be derived by this method.

The solution procedure employed by Wong (1964) depends crucially on restricting the PDE problem sufficiently to apply classical S-L techniques. Using S-L methods, various studies have generalized the set of solutions for U to cases where the stationary distribution is not a member of the Pearson system or U is otherwise unknown, e.g., Linetsky (2005). While the conventional method is to employ an eigenfunction expansion solution, Veerstraeten (2004) demonstrates that a more revealing solution is provided for the special case where $B[x]$ and $A[x]$ are constants if the Green's function is used to solve the S-L problem[3]. In order to employ the separation of variables technique used in solving S-L problems,(1) has to be transformed into the canonical form of the forward equation. To do this, the following important function has to be introduced:[4]

$$r[x] = B[x] \exp\left[-\int_a^x \frac{A[s]}{B[s]} \, ds \right]$$

Using this function, the forward equation can be rewritten in the form (see Appendix):

$$\frac{1}{r[x]} \frac{\partial}{\partial x}\left\{ p[x] \frac{\partial U}{\partial x} \right\} + q[x] \; U = \frac{\partial U}{\partial t} \qquad (6)$$

$$\textit{where:} \qquad p[x] = B[x] \; r[x] \qquad q[x] = \frac{\partial^2 B}{\partial x^2} - \frac{\partial A}{\partial x}$$

Equation (6) is the canonical form of equation (1). The S-L problem now involves solving (6) subject to appropriate initial and boundary conditions.

Because the methods for solving the S-L problem are ODE-based, some method of eliminating the time derivative in (1) is required. The eigenfunction expansion approach applies separation of variables, permitting (6) to be specified as:

$$U[x,t] = e^{-\lambda t} \; \varphi[x] \qquad (7)$$

Where $\varphi[x]$ must satisfy the ODE:

$$\frac{1}{r[x]} \frac{d}{dx}\left[p[x] \frac{d\varphi}{dx} \right] + [q[x] + \lambda] \; \varphi[x] = 0 \qquad (1')$$

Transforming the boundary conditions involves substitution of (7) into (3) and (4) and solving to get:

$$\frac{d}{dx}\{B[x]\ \varphi[x]\ \}\Big|_{x=a} - A[a]\ \varphi[a] = 0 \qquad (3')$$

$$\frac{d}{dx}\{B[x]\ \varphi[x]\ \}\Big|_{x=b} - A[b]\ \varphi[b] = 0 \qquad (4')$$

Significant analytical advantages are obtained by making the S-L problem 'regular' which involves assuming: $[a,b]$ is a closed interval with $r[x]$, $p[x]$ and $q[x]$ being real valued and $p[x]$ having a continuous derivative on $[a,b]$; and, $r[x] > 0$, $p[x] > 0$ at every point in $[a,b]$. 'Singular' S-L problems arise where these conditions are violated due to, say, an infinite state space or a vanishing coefficient in the interval $[a,b]$. The separated boundary conditions (3) and (4) ensure the problem is self-adjoint (Berg and McGregor 1966, p.91).

The S-L problem of solving (6) subject to the initial and boundary conditions admits a solution only for certain critical values of λ, the eigenvalues. Further, since equation (1) is linear in U, the general solution for (7) is given by a linear combination of solutions in the form of eigenfunction expansions. Details of these results can be found in Hille (1969, ch. 8), Boyce & De Prima (1977) and Birkhoff and Rota (1989, ch. 10). When the S-L problem is self-adjoint and regular the solutions for the transition probability density can be summarized in the following:

Proposition I:

The regular, self-adjoint Sturm-Liouville problem has an infinite sequence of real eigenvalues, $0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots$ with:

$$\lim_{n \to \infty} \lambda_n = \infty$$

To each eigenvalue there corresponds a unique eigenfunction $\varphi_n \equiv \varphi_n[x]$. Normalization of the eigenfunctions produces:

$$\psi_n[x] = \left[\int_a^b r[x]\ \varphi_n^2\ dx \right]^{-1/2} \varphi_n$$

The $\psi_n[x]$ eigenfunctions form a complete orthonormal system in $L_2[a,b]$. The unique solution in $L_2[a,b]$ to (1), subject to the boundary conditions (3)-(4) and initial condition (5) is, in general form:

$$U[x,t] \;=\; \sum_{n=0}^{\infty} c_n \;\; \psi_n[x] \;\; e^{-\lambda_n\, t} \qquad\qquad (8)$$

$$where: \qquad c_n \;=\; \int_a^b r[x] \;\; f[x_0] \;\; \psi_n[x] \;\; dx$$

This Proposition provides the general solution to the regular, self-adjoint S-L problem of deriving U when the process is subject to reflecting barriers. The Proposition demonstrates that having a discrete spectrum permits a representation for the transition probability density in the summation form of (8).[5] However, while useful, (8) is not immediately revealing because time is allowed to vary over $[0,\infty]$. The issue of decomposing U into time dependent and time independent components is addressed in the following section.

II. Density Decomposition Results

By providing an appropriate foundation, Proposition I facilitates the derivation of the general form of U for the regular, self-adjoint S-L problem. This section demonstrates that for this problem U can be decomposed into two components: a limiting equilibrium stationary density which is independent of time and the initial condition; and, a power series of transient terms that are time, boundary and initial condition dependent but with zero net density. In many econometric applications, the assumption of stationarity permits the $\Psi[x]$ distribution to be used directly as the likelihood function. This implicitly assumes that only the limiting behavior of U as $t \rightarrow \infty$ is relevant. The impact of the transient component is ignored. In a sampling context, this can be rationalized by standardizing the variables and assuming the transient components will average out to leave only the asymptotic behavior of a stationary process. Using the S-L approach, theoretical results on the U's associated with different types of boundary restrictions can be derived and the implications for, say, testing theory can be formulated by examining the shape and *iid* behavior of the relevant distributions and proposing appropriate adjustment factors for confidence intervals.

Being in the form of an eigenfunction expansion, (8) cannot be readily applied to the types of closed form distributions typically encountered in econometrics. Further simplification is required. This leads to the following result:[6]

Proposition II: Density Decomposition

Under the conditions required for Proposition I, the transition probability density function for x at time t (U) can be expressed as the sum of a stationary limiting equilibrium distribution that is linearly independent[7] of the boundaries and a power series of transient terms that are boundary and initial condition dependent:

$$U[x,t \mid x_0] = \Psi[x] + T[x,t \mid x_0] \qquad (9)$$

$$where: \quad \Psi[x] = \frac{r[x]^{-1}}{\int_a^b r[x]^{-1}\, dx} \qquad (10)$$

Using the specifications of λ_n , c_n, and ψ_n from Proposition I, the properties of $T[x,t]$ are defined as:

$$T[x,t \mid x_0] = \sum_{n=1}^{\infty} c_n\, e^{-\lambda_n t}\, \psi_n[x] = \frac{1}{r[x]} \sum_{n=1}^{\infty} e^{-\lambda_n t}\, \psi_n[x]\, \psi_n[x_0] \qquad (11)$$

$$with: \quad \int_a^b T[x,t \mid x_0]\, dx = 0 \qquad and \qquad \lim_{t \to \infty} T[x,t \mid x_0] = 0$$

Proposition II permits (9) to be combined with appropriately specified (10)-(11) to analyze the distributional implications of reflecting barriers. The distributional impact of the boundary restrictions enter through $T[x,t]$ (= $T[x,t \mid x_0]$). From the restriction on $T[x,t]$ in (11), the total mass of the transient term is zero. The transient acts to redistribute the mass of the stationary distribution, thereby causing a change in shape. The specific degree and type of alteration depends on the relevant assumptions made about the parameters and initial functional forms. A key feature of the Proposition is that (11) is in the form of a discrete spectrum.[8] Because the power series given in (11) involves powers of $exp[-\lambda_n]$, from Proposition I it follows that for given t the terms in the sum will decrease as $n \to \infty$. This property and the discrete spectrum significantly simplifies the calculation of the transient $T[x,t]$ in practical applications.

To see the implications of Proposition II, consider the variety of boundary independent stationary densities $\Psi[x]$ generated by appropriate choices of $A[x]$ and $B[x]$. A range of results are available in Wong (1964), Borodin and Salminen (2002), Veerstraeten (2004) and Linetsky (2005). A benchmark solution is given by the Brownian motion, constant coefficient case where $A[x] = \mu$ ($\neq$ 0) and $B[x] = ½\sigma^2$. Evaluating (10) gives the solution as (e.g., Veerstraeten 2004) :

$$\Psi[x] = \frac{2\mu}{\sigma^2} \frac{\exp\left\{\frac{2\mu}{\sigma^2} x\right\}}{\exp\left\{\frac{2\mu}{\sigma^2} b\right\} - \exp\left\{\frac{2\mu}{\sigma^2} a\right\}} = \frac{2\mu}{\sigma^2} \frac{\exp\left\{\frac{2\mu}{\sigma^2} (x - a)\right\}}{\exp\left\{\frac{2\mu}{\sigma^2} (b - a)\right\} - 1} \qquad (12)$$

There is no convention for a specific closed form to use for expressing this case. For example, Linetsky (2005) simplifies this solution by setting $\sigma^2 = 1$ and $a = 0$. In either form, $\Psi[x]$ is a scaled exponential density. If $\mu = 0$, the exponential density reduces to a uniform density: $\Psi[x] = 1/(b - a)$. The uniform stationary density is intuitive: if the reflecting boundaries are constant and the process has no drift then as $t \to \infty$ each point in the state space will be equally likely. It follows that the exponential stationary distributions are a consequence of the sample paths drifting to the upper ($A[x] > 0$) or the lower ($A[x] < 0$) boundary and 'bouncing off'. These solutions can be contrasted with Wong (1964) where the stationary exponential density $\Psi[x] = exp[-x]$ corresponding to the

Pearson system $\{d\Psi / dx\} = -\Psi[x]$ is used but the specific interval $I_s = [0, \infty)$ is required due to the density having to integrate to one over the state space.

The simplicity of the closed form stationary density component, $\Psi[x]$, of the transition density in the Brownian motion, constant parameter case does not carry over to the transient component. Following Borodin and Salminen (2002, p.121-2), the simplest constant parameter solution selects the principal solution (delta function initial condition), sets the drift to zero (uniform stationary), $B[x] = 1$ ($\sigma = \surd 2$), and $I_c = [0 \le x \le 1]$ ($\Psi[x] = 1$). This produces the power series of transient terms which using (11) defines the eigenvalues ($\lambda_n = n^2 \pi^2$) and eigenfunctions ($\psi_n = \surd 2 \cos n \pi x$), i.e.:

$$T[x,t \mid x_0] = 2 \left(\sum_{n=1}^{\infty} \exp\left[-n^2 \pi^2 t\right] \cos\left[n\pi x\right] \cos\left[n\pi x_0\right] \right)$$

This 'simplest' solution can be used to illustrate the implications of altering the specification of the S-L problem. In particular, the drift zero, principal solution with $I_c = [0 \le x \le L]$ ($\Psi[x] = 1/L$) and $B[x] = \frac{1}{2}\sigma^2$ produces eigenvalues ($\lambda_n = ((n^2 \pi^2 \sigma^2) / 2L^2)$, eigenfunctions ($\psi_n = \surd(2/L) \cos [(n \pi x) / L]$) and the solution:

$$T[x,t \mid x_0] = \frac{2}{L} \left(\sum_{n=1}^{\infty} \exp\left[-\frac{n^2 \pi^2 \sigma^2}{2L^2} t\right] \cos\left[\frac{n\pi}{L} x\right] \cos\left[\frac{n\pi}{L} x_0\right] \right)$$

Both the interval length and dispersion value act to scale the simple solution. This formulation permits the impact on $T[x,t]$ of increasing the interval length for given t to be assessed. This result can also be used to illustrate the analytical significance of having a process with zero drift.

To see the importance of drift specification, consider the principal solution where $\Psi[x]$ is given by (12) with $I_c = [a , b]$, $A[x] = \mu$ ($\neq 0$) and $B[x] = \frac{1}{2}\sigma^2$. The U for this case has been derived in the context of exchange rates distributions with target rate bands (Svensson 1991; de Yong 1994).

Formal treatments of the Brownian motion, constant parameter solution are available in Linetsky (2005) and, using the alternative Green's function approach, in Veerstraeten (2004):

$$T[x,t] = \frac{\exp\left[\frac{\mu}{\sigma^2}(x - x_0)\right]}{(b - a)} \sum_{n=1}^{\infty} \exp[-\lambda_n t] \frac{\sigma^2 \pi^2}{\lambda_n (b-a)^2} \left[n \cos\left[n\pi \frac{x - a}{b - a}\right] + \frac{\mu (b - a)}{\sigma^2 \pi} \sin\left[n\pi \frac{x - a}{b - a}\right]\right]$$
$$\left[n \cos\left[n\pi \frac{x_0 - a}{b - a}\right] + \frac{\mu (b - a)}{\sigma^2 \pi} \sin\left[n\pi \frac{x_0 - a}{b - a}\right]\right]$$

where the eigenvalues are $\lambda_n = (\mu / 2\sigma^2) + ((\sigma^2 \pi^2 n^2) / 2(b - a)^2)$ and the eigenfunctions retain both the sin and cos terms from the general solution. It is apparent that processes possessing a non-zero drift pose increased analytical complications associated with solving variable coefficient PDE's. This substantial increase in the complexity of the solution for the transient component in the constant coefficient case does not bode well for finding ready to implement solutions in more complicated cases.

This intuition about increased complexity is confirmed by Linetsky (2005) where the Sturm-Liouville problem is solved for the U associated with an Ornstein-Uhlenbeck (OU) process. In this case, the drift is state dependent $A[x] = \kappa (\chi - x)$ with $\kappa > 0$ and χ the long run mean of x ($b > \chi > a$). The infinitesimal variance is constant with $B[x] = ½\sigma^2$. Evaluating (10) for these values gives the solution of the stationary distribution as (e.g., Linetsky 2005, p.447):

$$\Psi[x] = \frac{\sqrt{2\kappa}}{\sigma} \frac{n[z]}{N[\beta] - N[\alpha]}$$

$$z = \frac{\sqrt{2\kappa}}{\sigma}(x - \chi) \qquad \alpha = -\frac{\sqrt{2\kappa}}{\sigma}(\chi - a) \qquad \beta = \frac{\sqrt{2\kappa}}{\sigma}(b - \chi)$$

where $n[\cdot]$ and $N[\cdot]$ are the standard normal density and the cumulative standard normal distribution function, respectively. The process of determining the eigenfunctions is decidedly more complicated (Linetsky 2005, p.447-9), involving functions not commonly encountered in econometrics. More precisely, changing variables to transform the forward equation into Weber-Hermite form permits solutions involving Weber-Hermite parabolic cylinder functions, which are related to Kummer confluent hypergeometric functions available in standard software packages, e.g., Mathematica.[9] The solutions require derivatives of the Kummer functions to be evaluated numerically leading to solutions involving digamma functions. The worked solution for the eigenfunction expansion of U in this case is available in Linetsky (2005, p.449).

III. Beyond Regular Boundaries

Section II demonstrates that, despite having the theoretical advantage of a discrete spectrum, imposing regular reflecting barriers on the state space for the forward equation quickly leads to analytical complexity in actually deriving the eigenfunction expansion for the transition probability density. These disadvantages need to be tempered by considering the alternatives to imposing reflecting boundaries. Consider the well known solution (e.g., Cox and Miller 1965, p.209) for U involving a constant coefficient standard normal variate $Y(t) = (\{x - x_0 - \mu t\} / \sigma)$ over the unbounded state space $I_o = (\infty \leq x \leq \infty)$. In this case the forward equation (1) reduces to: $\frac{1}{2}\{\partial^2 U / \partial Y^2\} = \partial U / \partial t$. By evaluating these derivatives, it can be verified that the principal solution for U is:

$$U[x,t \mid x_0] = \frac{1}{\sigma\sqrt{(2\pi t)}} \exp\left[-\frac{(x - x_0 - \mu t)^2}{2\sigma^2 t} \right]$$

and as $t \to -\infty$ or $t \to +\infty$ then $U \to 0$ and the stochastic process does not possess a non-trivial stationary distribution. In effect, if the process runs long enough then U will evolve to where there is no discernible probability associated with starting from x_0 and achieving a given point x.[10] The absence of a stationary distribution raises a number of practical problems, e.g., unit roots. Imposing regular reflecting boundaries is a certain method of obtaining an stationary distribution and a discrete spectrum (Hansen and Schienkman 1998, p.13). Alternative methods, such as specifying the process to admit natural boundaries where the parameters of the diffusion are zero within the state space, can give rise to continuous spectrum and raise significant analytical complexities. At least since Feller (1952), the search for useful solutions, including those for singular diffusion problems, has produced a number of specific cases of interest. However, without the analytical certainty of the S-L framework, analysis proceeds on a case by case basis.

One possible method of obtaining a stationary distribution without imposing both upper and lower boundaries is to impose only a lower (upper) reflecting barrier and construct the stochastic process such that positive (negative) infinity is non-attracting, e.g., Linetsky (2005); Aït-Sahalia (1999). This can be achieved by using an OU drift term. In contrast, Cox and Miller (1964, p.223-5) use the Brownian motion, constant coefficient forward equation with $x_0 > 0$, $A[x] = \mu < 0$ and $B[x] = \frac{1}{2}\sigma^2$ subject to the lower reflecting barrier at $x = 0$ given in (2) to solve for both the U and the stationary density. The principal solution is solved using the 'method of images' to obtain:

$$U[x,t \mid x_0] = \frac{1}{\sigma\sqrt{2\pi t}}\left\{\exp-\left(\frac{(x - x_0 - \mu t)^2}{2\sigma^2 t}\right) + \exp-\left(\frac{4x_0\mu t - (x - x_0 - \mu t)^2}{2\sigma^2 t}\right)\right\}$$
$$+ \frac{1}{\sigma\sqrt{2\pi t}}\left\{\frac{2\mu}{\sigma^2}\exp\left(\frac{2\mu x}{\sigma^2}\right)\left(1 - N\left[\frac{x + x_0 + \mu t}{\sigma\sqrt{t}}\right]\right)\right\}$$

where $N[x]$ is again the cumulative standard normal distribution function. Observing that $A[\mathrm{x}] = \mu > 0$ again produces $U \to 0$ as $t \to +\infty$, the stationary density for $A[x] = \mu < 0$ follows:

$$\psi[x] = \frac{2|\mu|}{\sigma^2} \exp - \left(\frac{2|\mu|x}{\sigma^2} \right)$$

Though x_0 does not enter the solution, combined with the location of the boundary at $x = 0$, it does implicitly impose the restriction $x > 0$. From Proposition II, $T[x,t \mid x_0]$ can be determined as $U[x,t \mid x_0] - \Psi[x]$.

Following Linetsky (2005), Veerstraeten (2004) and others, the analytical procedure used in section II to determine U involved specifying the parameters of the forward equation and the boundary conditions and then solving for $\Psi[x]$ and $T[x,t]$. Wong (1964) uses a different approach, initially selecting a stationary distribution and then solving for U using the restrictions of the Pearson system to specify the forward equation. In this approach, the functional form of the desired stationary distribution determines the appropriate boundary conditions. While application of this approach has been limited to the restricted class of distributions associated with the Pearson system, it is expedient when a known stationary distribution, such as the standard normal distribution, is of interest. More precisely, let:

$$\Psi[x] = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{x^2}{2} \right], \qquad I_o = (-\infty < x < \infty)$$

In this case, the boundaries of the state space are non-attracting and not regular. Solving the Pearson equation gives: $d\Psi[x]/dx = -x\,\Psi[x]$ and a forward equation of the OU form:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial}{\partial x} xU = \frac{\partial U}{\partial t}$$

The principal solution for this unrestricted equation is:

$$U[x,t \mid x_0] = \frac{1}{\sqrt{2\pi}} \exp\left[\frac{-x^2}{2}\right] \sum_{n=0}^{\infty} \frac{\exp[-nt]}{n!} H_n[x_0] \; H_n[x]$$

where $H_n\,[x]$ are the Hermite polynomials, e.g., Kendall and Stuart (1963, p.155), and the solution for the (discrete spectrum) $T[x,t]$ is given by taking the sum from $n = 1$. Following Wong (1964, p.268) Mehler's formula can be used to express the solution for U as:

$$U[x,t \mid x_0] = \frac{1}{\sqrt{2\pi(1 - e^{-2t})}} \exp\left[\frac{-(x - x_0 e^{-t})^2}{2(1 - e^{-2t})}\right]$$

Given this, as $t \to -\infty$ then $U \to 0$ and as $t \to +\infty$, U achieves the standard normal ergodic distribution.

The ergodic normal distribution is an example where a discrete spectrum is obtained without imposing boundaries on the state space. Another example is given by Wong (1964, p.268-9) where the stochastic process has a state space $I_s = [0 \leq x < \infty)$ and a discrete spectrum involving Laguerre polynomials with a stationary density of the form:

$$\Psi[x] = \frac{x^{\alpha}}{\Gamma[\alpha + 1]} e^{-x} = \frac{1}{\Gamma[\alpha + 1]} \exp\{\alpha \ln[x] - x\}$$

and forward equation:[11]

$$\frac{\partial^2}{\partial x^2}[xU] - \frac{\partial}{\partial x}[(\alpha + 1 - x)]U = \frac{\partial U}{\partial t}$$

where the gamma function $\Gamma[\alpha + 1]$ has $\alpha > -1$. This process is significant in having x dependence on the infinitesimal variance and a solution for U involving Laguerre polynomials that can be solved in closed form. Linetsky (2005) provides results for affine diffusion processes where the coefficients of the forward equation are given by $B[x] = ½\, \sigma \sqrt{(x - \ell)}$ with the shift parameter $\ell < 0$ and $A[x] = \kappa\,(\chi - x)$ with the same parameter restrictions as for the OU process of section II. When subjected to to a lower reflecting barrier (because ∞ is non-attracting) the affine diffusion also has a discrete spectrum. However, a closed form solution is unavailable.[12]

IV. Generalized Pearson Systems

The results in Wong (1964), Linetsky (2005), Veerstraeten (2004) and related studies apply directly to the transition probability densities associated with the unimodal Pearson system. Generalizing this approach to allow more flexibility in the shape of the stationary distribution can be achieved using a higher order exponential density, e.g., Fisher (1921), Cobb et al. (1983), Caudel and Flandoli (1998). Increasing the degree of the polynomial in the exponential comes at the expense of introducing additional parameters resulting in a substantial increase in the analytical complexity of the transition density spectrum. As a consequence, the generalized Pearson distributions typically defy a closed form solution for the transition densities. However, at least since Elliott (1955), it has been recognized that the solution of the associated regular S-L problem will still have a discrete spectrum, even if the specific form of the eigenfunctions and eigenvalues in $T[x,t \mid x_0]$ are not precisely determined (Horsthemke and Lefever 1984, sec. 6.7). Inferences about transient stochastic behavior can be obtained by examining the solution of the deterministic non-linear dynamics. In this process, attention initially focuses on the properties of the higher order exponential distributions.

To this end, assume that the stationary distribution is a fourth degree or "general quartic" exponential:

$$\Psi[x] = K \exp[-\Phi[x]] = K \exp[-(\beta_4 x^4 + \beta_3 x^3 + \beta_2 x^2 + \beta_1 x)]$$

where: K is a constant determined such that the density integrates to one; and, $\beta_4 > 0$.[13] Following Fisher (1921), the class of distributions associated with the general quartic exponential admits both unimodal and bimodal densities and nests the standard normal as a limiting case where $\beta_4 = \beta_3 = \beta_1 = 0$ and $\beta_2 = ½$ with $K = 1/(\sqrt{2\pi})$. The generalized Pearson ODE restriction for the quartic exponential takes the form:

$$\frac{d\Psi[x]}{dx} = \frac{e_3\, x^3 + e_2\, x^2 + e_1\, x + e_0}{g[x]}\, \Psi[x]$$

In this case, the generalization occurs because the degree of the 'shape polynomial' in the numerator has been increased from one to three. It is possible to further generalize to a stationary distribution with a k (> 4) degree exponential. With correct selection of parameters, the quartic exponential density is sufficient to capture the implications of stationary bimodality; a higher degree polynomial is needed if the possible number of stationary modes is greater than two.

The implications of generalizing the Pearson system by increasing the degree of the exponential is apparent from the ODE restriction. In the Pearson system, $g[x] = d_0 + d_1 x + d_2 x^2$ is a polynomial of degree at most two in x that depends on the particular specification of the stationary or transition density desired. The classification of Pearson system of stationary distributions into the various Types I-VII follows the specification of the degree of the polynomial $g[x]$ (Johnson and Kotz 1970, p.9-14). Extending this approach to the generalized Pearson system, when $g[x]$ is a constant there is a family $\mathbb{N}$ of distributions that include the standard normal and the quartic exponential densities, instead of a single distribution defined by the standard normal that is the limiting case of all Pearson distributions types. Permitting g to be a linear transformation of the form $d_1\ x$ restricts the admissible $x\ \varepsilon$ { $0 < x < \infty$). In particular, a stationary distribution of the form:

$$\psi[x] = K_G \exp-[g_0 \ln[x] + g_1 x + g_2 x^2 + g_3 x^3]$$

produces the family $\mathfrak{G}$ of gamma densities that nest the Pearson Type III. Allowing $g[x] = d_2\, x^2$ produces the inverse gamma family nesting the Pearson Type V; and, setting $d_1 = d_2 = 1$and $g[x] = x(1\text{-}x)$ produces the beta family nesting the Pearson Type I. Each of these families requires an appropriate version of the exponential stationary distribution to correspond with the desired $g[x]$ in the generalized Pearson ODE (Cobb et al. 1983).

In specifying the generalized Pearson system, the additional complications introduced by the higher degree polynomial in the numerator of the ODE augments the concern with different solutions of the quadratic polynomial in the denominator that arises with the Pearson system. To avoid complicated solutions for the generalized Pearson ODE involving ratios of polynomials in x, it is expedient to focus attention on the non-linearity in the drift and away from state dependence of the infinitesimal variance. In effect, enhancing precision in the estimation of distributional shape comes at the expense of incorporating state dependence in the variance. This induces a fundamental shift in the conceptual approach to modeling random behavior using diffusion processes. Consider the problem of modeling the drift and diffusion parameters for the short term interest rate process, e.g., Stanton (1997). Following Ait-Sahalia (1996) the preferred approach to empirically determining these parameters has been to fit a flexible, nonlinear functional form for each parameter, such as:

$$A[x;\theta] = \theta_0 + \theta_1 x + \theta_2 x^2 + \frac{\theta_3}{x} \qquad B[x;\beta] = \beta_1 x + x^{\beta_2}$$

The generalized Pearson ODE suggests that such ‘flexibility’ is misleading. Consistent with this observation, Chapman and Pearson (2000) argue against the flexible, non-linear function form approach to capturing nonlinearity in the drift of short-term interest rates due to multicollinearity between the drift and diffusion coefficients. Similarly, Hurn and Lindsay (2002) address the multicollinearity problem by employing orthogonal Legendre polynomials and find estimation of the non-linear drift function depends crucially on “specification of the drift in terms of orthogonal constituents” (p.563). Hence, permitting state dependence of both the drift and volatility imposes significant restrictions on the parameters of the stationary distribution.

Despite being recognized as early as Fisher (1921) as the class of distributions for which the efficiency of the method of moments coincides with maximum likelihood, generalized Pearson distributions such as the quartic exponential density have been mostly ignored in econometrics in favor of processes, such as affine diffusions, that feature state dependence of the infinitesimal variance. Where diffusions from this class are used, as in the "double- well" diffusion process in Ait-Sahalia (1999):

$$dX(t) = (X(t) - X(t)^3)\ dt + dW(t)$$

where $dW(t)$ is a Weiner process, the parametric flexibility needed to fit the non-linearity in the drift is ignored.[14] While the double-well process does have a symmetric bimodal stationary density, the *a priori* restrictions on the non-linear drift term are apparent from the specification of the generalized Pearson ODE. The analytical advantage of setting the infinitesimal variance to a constant is to enhance fitting of the shape polynomial for the stationary distribution. The restrictions imposed by the double well process produce a quartic exponential distribution that is bimodal and symmetric about zero. The parameter restrictions imposed are too severe to be representative of actual economic time series.

V. The Quartic Exponential Distribution

The stationary distribution of the double well process is a special case of the symmetric quartic exponential distribution:

$$\Psi[y] = K_S \exp[-\{\beta_2\ (x - \mu)^2 + \beta_4\ (x - \mu)^4\}] \qquad where \quad \beta_4 \geq 0$$

where μ is the population mean and the symmetry restriction requires $\beta_1 = \beta_3 = 0$. To see why the condition on β_1 is needed, consider change of origin $X = Y - \{\beta_3 / 4\ \beta_4\}$ to remove the cubic term from the general quartic exponential (Matz 1978, p.480):

$$\Psi[y] = K_Q \exp[-\{\kappa (y - \mu_y) + \alpha (y - \mu_y)^2 + \gamma (y - \mu_y)^4\}] \qquad where \quad \gamma \geq 0$$

The substitution of y for x indicates the change of origin which produces the following relations between coefficients for the general and specific cases:

$$\kappa = \frac{8\beta_1\beta_4^2 - 4\beta_2\beta_3\beta_4 + \beta_3^3}{8\beta_4^2} \qquad \alpha = \frac{8\beta_2\beta_4 - 3\beta_3^2}{8\beta_4} \qquad \gamma = \beta_4$$

The symmetry restriction $\kappa = 0$ can only be satisfied if both β_3 and $\beta_1 = 0$. Given the symmetry restriction, the double well process further requires $-\alpha = \gamma = \sigma = 1$. Solving for the modes of Ψ[y] gives $\pm\sqrt{\{|\alpha| / (2\gamma)\}}$ which reduces to $\pm$ 1 for the double well process, as in Ait-Sahlia (1999, Figure 6B, p.1385).

INSERT FIGURE 1 HERE

As illustrated in Figure 1, the selection of a_i in the stationary density $\Psi_i[x] = K_Q\, exp\{ -(.25\, x^4 - .5\, x^2 - a_i\, x) \}$ defines a family of general quartic exponential densities, where a_i is the selected value of κ for that specific density.[15] The coefficient restrictions on the parameters α and γ dictate that these values cannot be determined arbitrarily. For example, given that β_4 is set at .25, then for $a_i = 0$, it follows that $\alpha = \beta_2 = 0.5$. 'Slicing across' the surface in Figure 1 at $a_i = 0$ reveals a stationary distribution that is equal to the double well density. Continuing to slice across as a_i increases in size, the bimodal density becomes progressively more asymmetrically concentrated in positive x values. Though the location of the modes does not change, the amount of density between the modes and around the negative mode falls. Similarly, as a_i decreases in size the bimodal density becomes more asymmetrically concentrated in positive x values. While the stationary density is bimodal over $a_i\ \varepsilon$ {-1,1}, for $|a_i|$ large enough the density becomes so asymmetric that only a unimodal density appears. For the general quartic, asymmetry arises as the amount of the density surrounding each mode (the sub-density) changes with a_i. In this, the individual sub-densities have a symmetric shape. To introduce asymmetry in the sub-densities, the reflecting boundaries at a and b that bound the state space for the regular S-L problem can be used to introduce positive asymmetry in the lower sub-density and negative asymmetry in the upper sub-density.

Solving the forward equation to obtain a closed form for the transition density of a diffusion process with a quartic exponential stationary distribution is confounded by the presence of the cubic non-linearity in the numerator of the generalized Pearson ODE and in the forward equation term: $\frac{\partial}{\partial x}\{\ A[x]\ \ U[x,t]\ \ \}$. Except in the special generalized Pearson cases such as the family $\mathfrak{G}$ of gamma densities, also permitting state variation in $B[x]$ renders the forward equation for higher order exponential densities unsolvable in closed form. To obtain information about $T[x,t \mid x_0]$, attention focuses on solving the non-linear dynamics of the deterministic equation associated with the drift term. For the symmetric quartic exponential, these deterministic dynamics are described by the pitchfork bifurcation ODE:

$$\frac{dx}{dt} = -x^3 + \rho_1\ x + \rho_0$$

where ρ_0 and ρ_1 are the 'normal' and 'splitting' control variables, respectively (e.g., Cobb 1978). While ρ_0 has significant information in a stochastic context, this is not usually the case in the deterministic problem so $\rho_0 = 0$ is assumed. Given this, for $\rho_1 \leq 0$, there is one real equilibrium ($\{dx / dt\} = 0$) solution to this ODE at $x = 0$ where "all initial conditions converge to the same final point exponentially fast with time" (Caudel and Flandoli 1998, p.260). For $\rho_1 > 0$, the solution bifurcates into three equilibrium solutions $x = \{\ 0, \pm\sqrt{\rho_1}\}$, one unstable and two unstable. In this case, the state space is split into two physically distinct regions (at $x = 0$) with the degree of splitting controlled by the size of ρ_1. Even for initial conditions that are 'close', the equilibrium achieved will depend on the sign of the initial condition.

It is well known that the introduction of randomness to the pitchfork ODE changes the properties of the equilibrium solution, e.g., (Arnold 1998, sec.9.2). It is no longer necessary that the state space for the principal solution be determined by the location of the initial condition relative to the bifurcation point. The possibility for randomness to cause some paths to cross over the bifurcation point depends on the size of σ which measures the non-linear signal to white noise ratio. Of the different approaches to introducing randomness (e.g., multiplicative noise), the simplest approach to converting from a deterministic to a stochastic context is to add a Weiner process to the ODE. Augmenting the diffusion equation to allow for σ to control the relative impact of non-linear drift versus random noise produces the "pitchfork bifurcation with additive noise" (Arnold 1998, p.475) which in symmetric form is:

$$dX(t) \;=\; (\rho_1 \; X(t) \;-\; X(t)^3) \;+\; \sigma \; dW(t)$$

While capable of sustaining the common approach in econometrics based on a one-to-one correspondence between invariant Markov forward measures and stationary distributions, the dynamics of the pitchfork process captured by $T[x,t \,|x_0]$ have been "forgotten" (Arnold 1998, p.473).

VI. Conclusion

This paper addresses the problem of analytically solving for the distributional implications of imposing boundary restrictions on a stochastic process. This problem has a number of potential applications in econometrics. The central theoretical result is a proposition on a fundamental decomposition of the transition probability density into: a transient component that is boundary, time t and initial x_0 value dependent; and, a stationary density which is independent of t and x_0. A number of explicit solutions for the transient component are provided. Using the density decomposition result, transition densities can be analytically simulated under a number of parametric and boundary specifications. This task is addressed in a companion paper. Following Wong (1964), Sturm-Liouville methods are employed to solve for the transition densities associated with the Pearson system of stationary distributions. Extensions to generalized Pearson distributions are considered. Implications of the generalized Pearson ODE for the estimation of non-linearity in the drift of a diffusion process is approached by examining properties of quartic exponential stationary density.

Appendix

Preliminaries on solving the Forward Equation:

Due to the widespread application in a wide range of subjects, textbook presentations of the Sturm-Liouville problem possess subtle differences that require some clarification to be applicable to the formulation used in this paper. In particular, to derive the canonical form (6) of the Fokker-Planck equation (1) observe that evaluating the derivatives in (1) gives:

$$B[x]\ \frac{\partial^2 U}{\partial x^2} + \left[2\frac{\partial B}{\partial x} - A[x]\right]\frac{\partial U}{\partial x} + \left[\frac{\partial^2 B}{\partial x^2} - \frac{\partial A}{\partial x}\right] U = \frac{\partial U}{\partial t}$$

This can be rewritten as:

$$\frac{1}{r[x]}\frac{\partial}{\partial x}\left[P[x]\frac{\partial U}{\partial x}\right] + Q[x]U = \frac{\partial U}{\partial t}$$

where:

$$P[x] = B[x]\ r[x] \qquad \frac{1}{r[x]}\ \frac{\partial P}{\partial x} = 2\frac{\partial B}{\partial x} - A[x] \qquad Q[x] = \frac{\partial^2 B}{\partial x^2} - \frac{\partial A}{\partial x}$$

It follows that:

$$\frac{\partial B}{\partial x} = \frac{1}{r[x]}\frac{\partial P}{\partial x} - \frac{1}{r^2}\frac{\partial r}{\partial x}P[x] = 2\frac{\partial B}{\partial x} - A[x] - \frac{B[x]}{r[x]}\frac{\partial r}{\partial x}$$

This provides the solution for the key function $r[x]$:

$$\frac{1}{r[x]}\frac{\partial r}{\partial x} - \frac{1}{B[x]}\frac{\partial B}{\partial x} = -\frac{A[x]}{B[x]} \quad \rightarrow \quad \ln[r] - \ln[k] = -\int^{x} \frac{A[s]}{B[s]}\ ds$$

$$r[x] = B[x]\ \exp\left[-\int^{x} \frac{A[s]}{B[s]}\ ds\right]$$

This $r[x]$ function is used to construct the scale and speed densities commonly found in presentations of solutions to the forward equation, e.g., Karlin and Taylor (1981), Linetsky (2005).

Another specification of the forward equation that is of importance is found in Wong (1964, eq.6-7):

$$\frac{d}{dx}\left[B[x]\ \rho[x]\ \frac{d\theta}{dx}\right] + \lambda\ \rho[x]\ \theta[x] = 0 \qquad \textit{with b.c.} \quad B[x]\ \rho[x]\ \frac{d\theta}{dx} = 0$$

This formulation occurs after separating variables, say with $U[x] = g[x]\, h[t]$. Substituting this result into (1) gives:

$$\frac{\partial^2}{\partial x^2}[B \;\; g \;\; h] \; - \; \frac{\partial}{\partial x}[A \;\; g \;\; h] \; = \; g[x]\frac{\partial h}{\partial t}$$

Using the separation of variables substitution $(1 / h)\{\partial h / \partial t\} = -\lambda$ and redefining $g[x] = \rho\theta$ gives:

$$\frac{d}{dx}\left[\frac{d}{dx}B \;\; g \; - \; A \;\; g\right] \; = \; -\lambda \;\; g \; = \; \frac{d}{dx}\left[\frac{d}{dx}B[x] \;\; \rho[x] \;\; \theta[x] \; - \; A[x] \;\; \rho[x] \;\; \theta[x]\right] \; = \; -\lambda \;\; \rho \;\; \theta$$

Evaluating the derivative inside the bracket and using the condition $\{d/dx\}\; [B\,\rho] - A\rho = 0$ to specify admissible ρ gives:

$$\frac{d}{dx}\left[\theta\frac{d}{dx}(B\rho) \; + \; B\rho \; \frac{d}{dx}\theta \; - \; A\rho\theta\right] \; = \; \frac{d}{dx}\left[B\rho \; \frac{d}{dx}\theta\right] \; = \; -\lambda \;\; \rho[x] \;\; \theta[x]$$

which is equation (6) in Wong (1964). The condition used to define ρ is then used to identify the specification of $B[x]$ and $A[x]$ from the Pearson system. The associated boundary condition follows from observing the $\rho[x]$ will be the ergodic density and making appropriate substitutions into the boundary condition:

$$\frac{\partial}{\partial x}\{B[x] \;\; f[t] \;\; \rho[x] \;\; \theta[x]\} \; - \; A[x] \;\; f[t] \;\; \rho[x] \;\; \theta[x] \; = \; 0 \qquad \Rightarrow \qquad \frac{d}{dx}[B \;\; \rho \;\; \theta] \; - \; A \;\; \rho \;\; \theta \; = \; 0$$

Evaluating the derivative and taking values at the lower (or upper) boundary gives:

$$B[a]\rho[a] \; \frac{d\theta[a]}{dx} \; + \; \theta[a] \; \frac{dB\rho}{dx} - \; A[a]\rho[a]\theta[a] \; = \; 0$$

$$= \; B[a]\rho[a] \; \frac{d\theta[a]}{dx} \; + \; \theta[a]\left[\frac{dB[a]\rho[a]}{dx} \; - \; A[a]\rho[a]\right]$$

Observing the expression in the last bracket is the original condition with the ergodic density serving as U gives the boundary condition stated in Wong (1964, eq.7).

Proof of Proposition II:

(a) ψ_n has exactly n zeroes in $[a,b]$

Hille (1969, p.398, Theorem 8.3.3) and Birkhoff and Rota (1989, p.320, Theorem 5) shows that the eigenfunctions of the Sturm-Liouville system (1') with (3'), (4') and (5) have exactly n zeroes in the interval (a,b). More precisely, since it assumed that $r > 0$, the eigenfunction ψ_n corresponding to the

nth eigenvalue has exactly n zeroes in (a,b).

(b) For $\psi_n = 0$, $\int_a^b \psi_n[x] \; dx = 0$

Proof:

For $\psi_n = 0$ the following applies:

$$\psi_n = \frac{1}{\lambda_n} \frac{d}{dx} \left\{ \frac{d}{dx}[B[x] \; \psi_n] - A[x] \; \psi_n \right\}$$

$$\therefore \int_a^b \psi_n[x] \; dx = \frac{1}{\lambda_n} \left\{ \frac{d}{dx}[B[x] \; \psi_n] \Big|_{x=b} - A[b] \; \psi_n[b] - \frac{d}{dx}[B[x] \; \psi_n] \Big|_{x=a} + B[a] \; \psi_n[a] \right\} = 0$$

Since each $\psi_n[x]$ satisfies the boundary conditions (B.2).

(c) For some k, $\lambda_k = 0$.

Proof:

From Proposition 1:

$$U[x,t] = \sum_{k=0}^{\infty} c_k \; e^{-\lambda_k t} \; \psi_k[x]$$

Since $\int_a^b U[x,t] \; dx = 1$ then:

$$1 = \sum_{k=0}^{\infty} c_k \; e^{-\lambda_k t} \int_a^b \psi_k[x] \; dx$$

But from part (b) this will = 0 (which is a contradiction) unless $\lambda_k = 0$ for some k.

(d) $\lambda_0 = 0$

Proof:

From part (a), ψ_0[x] has no zeroes in (a,b). Therefore, either $\int_a^b \psi_0[x] \; dx > 0$ or $\int_a^b \psi_0[x] \; dx < 0$.

It follows from part (b) that $\lambda_0 = 0$.

(e) $\lambda_n > 0$ for $n \neq 0$. This follows from the strict inequality conditions provided in Proposition 1 and in part (d).

(f) Obtaining the solution for $T[x]$ in Proposition 2.

From part (d) it follows: $$\frac{d}{dx}\left\{ [\ B[x]\ \psi_0[x,t]\ ]_x - A[x]\ \psi_0[x,t] \right\} = 0$$

Integrating this equation from a to x and using the boundary condition gives:

$$[\ B[x]\ \psi_0[x,t]\ ]_x - A[x]\ \psi_0[x,t] = 0$$

This equation can be solved for ψ_0 to get:

$$\psi_0 = A\ [\ B[x]\ ]^{-1} \exp[\int_a^x \frac{A[s]}{B[s]}\ ds\ ] = C\ [\ r[x]\ ]^{-1} \quad \textit{where}:\quad C = \textit{constant}$$

Therefore:

$$\psi_0[x] = \left[\int_a^b r[x]\ C^2\ [\ r[x]\ ]^{-2}\ dx \right]^{-½} C\ [\ r[x]\ ]^{-1} = \frac{[\ r[x]\ ]^{-1}}{\left[\int_a^b r[x]^{-1}\ dx \right]^{½}}$$

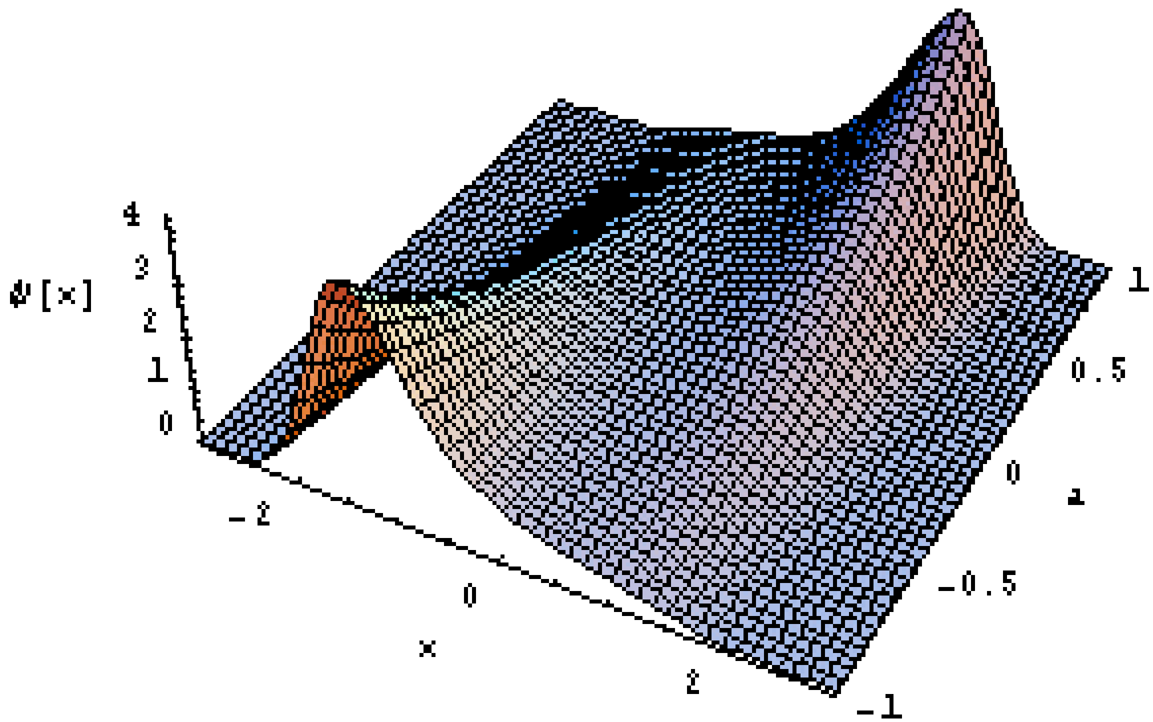
Using the definition in Proposition I and observing that the integral of $f[x]$ over the state space is one it follows:

$$c_0 = \int_a^b \left\{ f[x]\ r[x]\ \frac{r[x]^{-1}}{\left[\int_a^b r[x]\ dx \right]^{½}} \right\} dx = \frac{1}{\left[\int_a^b r[x]\ dx \right]^{½}}$$

$$\therefore \quad c_0\ \psi[x] = \frac{r[x]^{-1}}{\left[\int_a^b [\ r[x]\ ]^{-1}\ dx \right]}$$

(g) The Proof of Proposition 2 now follows from parts (f), (e) and (b).

Figure 1:* **Family of Stationary Densities for $\Psi_i[x] = K_Q\, exp\{ -(.25\, x^4 - .5\, x^2 - a_i\, x) \}$**



* Each of the continuous values for *a* signifies a different stationary density. For example, at $a = 0$ the density is the double well density which symmetric about zero and with modes at ±1.

Bibliography

Aït-Sahalia, Y. (1999), “Transition Densities for Interest Rates and Other Nonlinear Diffusions”, Journal of Finance 54: 1361-95.

Aït-Sahalia, Y. (1996), “Testing continuous-time models of the spot interest rate”, Review of Financial Studies, Vol. 9, pp. 385–426.

Arnold, L. (1998), Random Dynamical Systems, New York: Springer-Verlag.

Ball, C. and A. Roma (1998), “Detecting Mean Reversion Within Reflecting Barriers: Applications to the European Exchange Rate Mechanism”, Applied Mathematical Finance 5: 1-15.

Berg, P. and J. McGregor (1966), Elementary Partial Differential Equations, San Francisco: Holden-Day.

Birkhoff, G. and G. Rota (1989), Ordinary Differential Equations (4th ed.), New York: Wiley.

Borodin, A. and P. Salminen (2002), Handbook of Brownian Motion – Facts and Formulae, Boston, MA: Birkhauser Verlag.

Boyce, W. and R. DiPrima (1977), Elementary Differential Equations and Boundary Value Problems (3rd ed.). New York: Wiley.

Chapman, D. and Pearson, N. (2000), “Is the short rate drift actually nonlinear”, Journal of Finance 55: 355–88.

Cobb, L., P. Koppstein and N. Chen (1983), “Estimation and Moment Recursion Relations for Multimodal Distributions of the Exponential Family”, Journal of the American Statistical Association 78: 124-30.

Cobb, L. (1981), “The Multimodal Exponential Families of Statistical Catastrophe Theory”, in C. Taillie, G. Patil and B. Baldessari (eds.), Statistical Distributions in Scientific Work (vol.4), Dordrecht, Holland: Reidel Press.

Cobb, L. (1978), “Stochastic Catastrophe Models and Multimodal Distributions”, Behavioral Science 23: 360-74.

Cootner, P. (1962), "Stock Prices: Random vs. Systematic changes", Industrial Management Review 3: 24-45.

Cox, D. and H. Miller (1965), The Theory of Stochastic Processes, London: Chapman and Hall.

Crauel, H. and F. Flandoli (1998), "Additive Noise Destroys a Pitchfork Bifurcation", Journal of Dynamics and Differential Equations 10: 259-74.

De Jong, F. (1994), "A univariate analysis of European monetary system exchange rates using a target zone model", Journal of Applied Econometrics 9: 31-45.

Elliott, J. (1955), "Eigenfunction expansions associated with singular differential operators", Transactions of the American Mathematical Society 78: 406-25.

Farnsworth, H. and R. Bass (2003), "The term structure with semi-credible targeting", Journal of Finance 58: 839-65.

Feller, W. (1954), "Diffusion Processes in One Dimension", Transactions of the American Mathematical Society 77: 1-31.

Fisher, R.A. (1921), "On the mathematical foundations of theoretical statistics", Philosophical Transactions of the Royal Society 222A: 309-68.

Gihman, I. and A. Skorohod (1979), The Theorv of Stochastic Processes (vol. II, III), New York: Springer-Verlag.

Greene, W. (1982), "Maximum Likelihood Estimation of Stochastic Frontier Production Models", Journal of Econometrics 18: 285-90.

Hansen, L., J. Scheinkman and N. Touzi (1998), "Spectral Methods for Identifying Scalar Diffusions", Journal of Econometrics 86: 1-32.

Hille, E. (1969), Lectures on Ordinary Differential Equations, London: Addison-Wesley.

Horsthemke, W. and R. Lefever (1984), Noise-Induced Transitions, New York: Springer-Verlag.

Hurn, A. and K. Lindsay (2002), "On the Specification of the Drift and Diffusion Functions for Continuous-time Models of the Spot Interest Rate", Oxford Bulletin of Economics and Statistics 64: 547-64.

Johnson, N. and S. Kotz (1970), Continuous Univariate Distributions-1, New York: Houghton Mifflin.

Karlin, S. and H. Taylor (1981), A Second Course in Stochastic Processes, New York: Academic Press.

Kendall, M. and A. Stuart (1963), The Advanced Theory of Statistics (vol.I), London: Charles Griffen.

Krugman, P. (1991), " Target zones and exchange rate dynamics", Quarterly Journal of Economics 106: 669-82.

Lewis, A. (1998), "Applications of Eigenfunction Expansions in Continuous-Time Finance", Mathematical Finance 8: 349-83.

Linetsky, V. (2005), "On the Transitions Densities for Reflected Diffusions", Advances in Applied Probability 37: 435-60.

Matz, A. (1978), "Maximum Likelihood Parameter Estimtion for the Quartic Exponential Distribution", Technometrics 20: 475-84.

Nicolau, J. (2002), "Stationary Processes that Look Like Random Walks– The Bounded Random Walk Process in Discrete and Continuous Time", Econometric Theory 18: 99-118.

Poitras, G. (1988), "Arbitrage Boundaries, Treasury Bills and Covered Interest Parity", Journal of International Money and Finance 7: 429-45.

Rheinlaender, T. and M. Steinkamp (2004), "A Stochastic Version of Zeeman's Market Model", Studies in Nonlinear Dynamics & Econometrics 8, No. 4, Article 4.

Ricciardi, L. and L. Sacerdote (1987), "On the Probability Densities of an Ornstein-Uhlenbeck Process with a Reflecting Boundary", Journal of Applied Probability 24: 355-69.

Risken, H. (1989), The Fokker-Planck Equation, Methods of Solution and Applications, New York: Springer-Verlag.

Stanton, D. (1997), "A nonparametric model of term structure dynamics and the market price of interest rate risk", Journal of Finance 52: 1569–621.

Svensson, L. (1991), "The term structure of interest rate differentials in a target zone: Theory and Swedish data", Journal of Monetary Economics 28: 87-116.

Veerstraeten, D. (2004), "The Conditional Probability Density Function for a Reflected Brownian Motion", Computational Economics 24: 185-207.

Whittaker, E. and G. Watson (1963), A Course of Modern Analysis (4th ed.), Cambridge, UK: Cambridge University Press.

Wong, E. (1964), "The Construction of a Class of Stationary Markoff Processes", in Proceedings of Symposium on Applied Mathematics R. Bellman (ed.), Providence, RI: American Mathmatical Society.

NOTES

1. A diffusion process is 'regular' if starting from any point in the state space *I*, any other point in *I* can be reached with positive probability (Karlin and Taylor 1981, p.158). This condition is distinct from other definitions of regular that will be introduced: 'regular boundary conditions' and 'regular S-L problem'.

2. The classification of boundary conditions is typically an important issue in the study of solutions to the forward equation. Important types of boundaries include: regular; exit; entrance; and natural. Also important in boundary classification are: the properties of attainable and unattainable; whether the boundary is attracting or non-attracting; and whether the boundary is reflecting or absorbing. In the present context, only regular, attainable, reflecting boundaries are being considered in Sec. II with a few specific extensions to other types of boundaries being incorporated in Sec. III. In general, the specification of boundary conditions is essential is determining whether a given PDE is self-adjoint The presence of the drift term in the boundary condition is required to ensure that the density integrate to one or, in the terminology of Feller (1952), that the boundary condition be norm preserving.

3. In Veerstraeten (2004), the use of Green's functions is implemented by using a transformation that achieves the PDE form: $g\, U_{xx} = U_t$ where the subscript denotes partial differentiation. A Laplace transform is then used to eliminate the time derivative. It is well known that using Laplace transforms to determine closed form solutions is usually restricted to the constant coefficient case because, without constant coefficients, the solution to the transform would involve another differential equation and nothing substantive is achieved by doing the transform. Hence, while Veerstraeten (2004) produces an insightful solution, more general cases require a different solution procedure if the Green's function solution is used to determine the transition probability density.

4. Following Karlin and Taylor (1991, p. 194-5) and Linetsky (2005, p.437) $r[x]$ can be interpreted as $B[x]$ times the 'scale density'.

5. Birkhoff and Rota (1989, p.337) demonstrate that the regular S-L problem has a spectrum that is always discrete and have eigenfunctions that are (trivially) square-integrable. These eigenfunctions will be orthogonal with respect to the weight function $r[x]$.

6. In the following. Proposition, $\Psi[x]$ is proportional to the "speed density" given in Karlin and Taylor (1981, p. 195).

7. This excludes the affect of the normalizing constant: $\int_a^b r[x]^{-1}\, dx$.

8. Hansen et al. (1998, p.12-3) recognize the importance of having solutions with a discrete spectrum and provide a sufficient general condition required for this result: 'finite first moment with the stationary density in natural scale'. This condition will always apply where there are reflecting

barriers. The well-known result that a discrete spectrum is possible with certain singular diffusion problems arising with natural boundaries is also identifed.

9. Whittaker and Watson (1963) is a useful source on Kummer and other transcendental functions.

10. More precisely, the probability $U[x, \infty | x_0]$ is associated with the set of time paths that start from x_0 and achieve an ending in the given volume element dx.as $t \to \infty$.

11. The drift coefficient follows from observing $d\, ln[\Psi] / dx = (\alpha - x)/x$ where the drift is specified as $A[x] - (dB[x]/ dx) = (\alpha - x) \to A[x] = (\alpha - x) + 1$ (Cobb 1978).

12. Following standard convention, a closed-form solution is available if, and only if, at least one solution can be expressed in terms of a bounded number of well-known functions. These well-known functions are defined to be the elementary functions, including the error function, gamma function and the general hypergeometric functions. Solutions which involve infinite series, limits, and continued fractions are not consistent with closed forms.

13. In what follows, except where otherwise stated, it is assumed that $\sigma = 1$. Hence, the condition that K be a constant such that the density integrates to one incorporates $\sigma = 1$ assumption. Allowing $\sigma \neq 1$ will alter either the value of K or the β's from that stated.

14. Ait-Sahalia (1999) also considers a diffusion with non-linear drift ($\alpha_0 + \alpha_1 X(t) + \alpha_2 X(t)^2 + \alpha_3 X(t)^{-1}$) and state dependent infinitesimal variance ($\sigma X(t)^p$). This complicated process could be readily transformed into the family $\mathfrak{C}$ by setting p =1, and changing $1/X$ to $\ln[X]$ in the drift. Conceptual advantage can be gained by adding a cubic term in the drift, e.g., (Cobb 1981, p.76).

15. A number of simplifications were used to produce the 3D image in Figure 1: x has been centered about μ; and, $\sigma = K_Q = 1$. Changing these values will impact the specific size of the parameter values for a given x but will not change the appearance of the density plots.