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Ergodicity and the History of Neoclassical Economic Theory

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ABSTRACT

What is neoclassical economics? What role did the 'ergodicity hypothesis' play in the development of neoclassical economic theory during the 20th century? What is the relevance of ergodicity to the new subject of econophysics? This paper addresses these questions by reviewing the etymology and history of the ergodicity hypothesis from the introduction of the concept in 19th century statistical mechanics until the emergence of econophysics during the 1990's. An explanation of ergodicity is provided that establishes a connection to the fundamental problem of using non-experimental data to verify propositions in economic theory. The relevance of the ergodicity assumption to the emergence of the 'complexity era' in modern economics is also examined.

Keywords: Ergodicity; Ludwig Boltzmann; Econophysics; Neoclassical Economics.

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1. Introduction

At least since Mirowski (1984), it has been recognized that important theoretical elements of neoclassical economics were adapted from mathematical concepts developed in 19th century physics. Much effort by historians of economic thought has been dedicated to exploring the validity and implications of "Mirowski's thesis", e.g., Carlson (1997); De Marchi 1993, especially the connection between the deterministic 'rational mechanics' approach to physics and the subsequent development of neoclassical economic theory (Mirowski 1988; Boumans 1993).[1] This well known literature connecting physics with neoclassical economics is seemingly incongruent with the emergence of econophysics during the last decade of the twentieth century. Econophysics is a "recognized field" (Roehner 2002, p.23) within physics which involves the application of theoretical and empirical methods from physics to the analysis of economic phenomena, e.g., Jovanovic and Schinckus (2012). There are unexplored historical connections between the methods adapted from physics by economists and those currently adopted in econophysics. This paper details the extension of 'experimental' empirical methods central to physics to the 'non-experimental' situation in economics.

As Mirowski (1989b, 1990) is at pains to emphasize, physical theory has evolved considerably from the deterministic approach which underpins neoclassical economics. In detailing historical developments in physics, Mirowski and others are quick to jump from the mechanical determinism of energistics to quantum mechanics to recent developments in chaos theory, overlooking relevance for the evolution of neoclassical economic theory of the initial steps toward modeling the stochastic behavior of physical phenomena by Ludwig Boltzmann (1844-1906), James Maxwell (1831-1879) and Josiah Gibbs (1839-1903). The introduction of the 'ergodicity hypothesis' around the end of the 19th century is relevant to the histories of neoclassical economic theory, 'modern economics' and econophysics. The introduction and subsequent development of stochastic concepts into physics can be historically contrasted with the emergence of econometrics which initially aimed to incorporate stochastic concepts generalizing and empirically testing deterministic neoclassical theory, e.g., Mirowski (1989b). In this process, the ergodicity hypothesis was essential to extending statistical techniques designed for experiments to the non-experimental data of concern in economics.

Early contributors to the econometrics project, such as Frisch, Tinbergen, Wold and Koopmans, featured training in physics or mathematical statistics, subjects where ergodic concepts are employed. Significantly, the initial stochastic generalizations of the deterministic and static equilibrium models of neoclassical economic theory required 'time reversible' ergodic probability models, especially the likelihood functions associated with certain stationary error distributions. In contrast, from the early ergodic models of Boltzmann to the fractals and chaos theory of Mandlebrot, physics has employed a wider variety of stochastic models aimed at capturing key empirical characteristics of different physical problems at hand. These models typically have a mathematical structure that can vary substantively from the constrained optimization techniques that underpin neoclassical economic theory, restricting the straightforward application of these models to economics. Overcoming the difficulties of applying models developed for physical situations to economic phenomena is the central problem confronting econophysics.[2]

Schinckus (2010, p.3816) accurately recognizes that the positivist philosophical foundation of econophysics depends fundamentally on empirical observation: "The empiricist dimension is probably the first positivist feature of econophysics". Following McCauley (2004) and others, this concern with empiricism often focuses on the identification of macro-level statistical regularities that are characterized by the scaling laws identified by Mandelbrot (1997) and Mandelbrot and Hudson (2004) for financial data. Unfortunately, this empirically driven ideal is often confounded by the 'non-repeatable' experiment that characterizes most observed economic and financial data. There is quandary posed by having only a single observed *ex post* time path to estimate the distributional parameters for the ensemble of *ex ante* time paths. In contrast to physical sciences, in the human sciences there is no assurance that *ex post* statistical regularity translates into *ex ante* forecasting accuracy. Resolution of this quandary highlights the usefulness of using the detailing the impact of the 'ergodicity hypothesis' on the histories of neoclassical economic theory, modern economics and econophysics.

2. What is Neoclassical Economics?

Following Colander (2000) and many others, there is considerable confusion about the time line and historical content of 'neoclassical economics'. Aspromourgos (1986) traces the initial introduction of the term to Veblen (1900) where it is used "to characterize the Marshallian version of marginalism". This interpretation is consistent with neoclassical economics having "a positive, basic relationship to the earlier classical theory and some development beyond it" because "Marshall, more than any of the other marginalist founders, sought to present his theory as having a substantial continuity with classical economics". Building on previous contributions by John Hicks, e.g., Hicks (1932), Stigler (1941) popularized the extension of neoclassical economics to include "all the marginalist founders". This interpretation is reflected in Colander (2000, p.133): "In many ways, the two books that tied up the loose ends and captured the essence of neoclassical economics, Hicks's *Value and Capital* (1939) and Samuelson's *Foundations* (1947), were culminating works—they put all the pieces of marginalism together. Important work thereafter was modern." This interpretation is consistent with the emergence of "modern economics" from 'neoclassical economics' being connected to the transition from deterministic to stochastic models.

Unfortunately, this interpretation of neoclassical economics disguises fundamental issues dividing the distinct marginalist approaches of Walras and Marshall, e.g. De Vroey (2012). In addition, it ignores the importance of the "neoclassical synthesis" that appears in the third edition of the influential textboook Economics by Samuelson (1955). Just as the concept of 'neoclassical economics' evolved from Marshallian notions to include writings of all the marginalist founders, the neoclassical systhesis program of integrating Keynesian short run disequilibrium and Walrasian long run equilibrium analysis also evolved to achieve a "new neoclassical synthesis ... that bears little, if any, relation to the old synthesis" (Duarte and De Vroey 2012, p.3). Fundamental issues that separate Marshallian and Walrasian neoclassical economics reappear in the more recent debates in macroeconomics between 'new classical economists', real business cycle economists and the DSGE or new Keynesian economists. Yet, despite considerable variation in the usage of 'neoclassical' to describe different types of theories in economics, it is still possible to identify certain essential elements connecting the seeming disparate approaches that have been afforded this designation.

De Vroey (2012, p.772) captures an essential element: "Equilibrium is as the core of both Marshall's and Walras' reasoning. Both of them adopted the stationary equilibrium conception in which equilibrium is defined as a state of rest." Samuelson (1947) extended this approach to include both static and dynamic theories. While static analysis "described equilibrium as resulting from economic agents solving maximization problems taking prices as parameters ... Samuelson's analysis of dynamics rested on the concept of stationary equilibrium, which holds that equilibrium constitutes a state of rest" (Duarte and De Vroey 2012, p.13). Lucas (1980, p.278) provides further elaboration of this point:

> The underlying idea seems to be taken from physics, as referring to a system 'at rest'. In economics, I suppose such a static equilibrium corresponds to a prediction as to how an economy would behave should external shocks remain fixed over a long period, so that households and firms would adjust to facing the same set of prices over and over again and attune their behavior accordingly.

Lucas attacked "the idea that an economic system is in any sense 'at rest' [as] simply an anachronism", advancing a theoretical system that relied on "the equilibrium discipline, rational expectations and Walrasian microfoundations" (Duarte and De Vroey 2012, p.19).

At best, neoclassical economics is a loose knit collection of theories that adapted and extended the classical notions of the invisible hand and the hedonistic psychology of methodological individualism. Given this, the emergence of econophysics in the mid-1990's suggests another methodological feature. As Roehner (2002, p.20) observes:

> From a methodological perspective, the strong emphasis econophysics has put on the search for regularities is probably one of its most important innovations. In contrast, for most economists a quantitative regularity is considered of no interest unless it can be interpreted in terms of agents' motivation and behavior and has a clearly defined theoretical status.

As such, Koopman's 'measurement without theory' critique of institutional economics can be extended to econophysics.

3. A Brief History of Ergodic Theory

The Encyclopedia of Mathematics (2002) defines ergodic theory as the "metric theory of dynamical systems. The branch of the theory of dynamical systems that studies systems with an invariant measure and related problems." This modern definition implicitly identifies the birth of ergodic theory with proofs of the mean ergodic theorem by von Neumann (1932) and the pointwise ergodic theorem by Birkhoff (1931). These early proofs have had significant impact in a wide range of modern subjects. Of particular relevance to economics, the notions of invariant measure and metric transitivity used in the proofs are fundamental to the measure theoretic foundation of modern probability theory (Doob 1953; Mackey 1974). Building on a seminal contribution to probability theory (Kolmogorov 1933), in the years immediately following it was recognized that the pointwise ergodic theorem generalizes the strong law of large numbers. Similarly, the equality of ensemble and time averages – the essence of the mean ergodic theorem – is necessary to the concept of a stationary stochastic process.[3] Ergodic theory is also the basis for the modern study of random dynamical systems, e.g., Arnold (1988). In contrast, ergodic theory in mathematical physics connects measure theory with the theory of transformation groups. This connection is important in motivating the generalization of harmonic analysis from the real line to locally compact groups.

From the perspective of modern mathematics, statistical physics or systems theory, Birkhoff (1931) and von Neumann (1932) are excellent starting points for a history of ergodic theory. Building on the ergodic theorems, subsequent developments in these and related fields have been dramatic. These contributions mark the solution to a problem in statistical mechanics and thermodynamics that was recognized sixty years earlier when Ludwig Boltzmann (1844-1906) introduced the ergodic hypothesis to permit the theoretical phase space average to be interchanged with the measurable time average. From the histories of both econophysics and neoclassical economics, the selection of the less formally correct and rigorous contributions of Boltzmann are an auspicious beginning. Problems of interest in mathematics are generated by a range of subjects, including physics, chemistry, engineering and biology. The formulation and solution of physical problems in, say, statistical mechanics or particle physics will have mathematical features which are inapplicable or unnecessary in economics. For example, in statistical mechanics, points in the phase space are often multi-dimensional functions representing the mechanical state of the system, leading to a group-theoretic interpretation of the ergodic hypothesis, e.g., Nolte (2010). From the perspective of economics, such complications are largely irrelevant and an alternative history of ergodic theory that captures the etymology and basic physical interpretation is more revealing than a history that focuses on the relevance for mathematics. This arguably more revealing history begins with the formulation by Boltzmann of the problems that von Neumann and Birkhoff were able to solve.

Mirowski (1989a, esp. ch.5) establishes the importance of 19th century physics in the development of the neoclassical economic system advanced by Jevons, Walras and Menger during the marginalist revolution of the 1870's. As such, neoclassical economic theory inherited essential features of mid-19th century physics: deterministic rational mechanics; conservation of energy; and the non-atomistic continuum view of matter that inspired the energetics movement later in the 19th century.[4] Jevons (1877, p.738-9) reflects the entrenched determinist position of the marginalists:

> We may safely accept as a satisfactory scientific hypothesis the doctrine so grandly put forth by Laplace, who asserted that a perfect knowledge of the universe, as it existed at any given moment, would give a perfect knowledge of what was to happen thenceforth and for ever after. Scientific inference is impossible, unless we may regard the present as the outcome of what is past, and the cause of what is to come. To the view of perfect intelligence nothing is uncertain.

It was during the transition from rational to statistical mechanics during the last third of the century that Boltzmann made the contributions that led to the transformation of theoretical physics from the microscopic mechanistic models of Rudolf Clausius (1822-1888) and James Maxwell to the macroscopic probabilistic theories of Josiah Gibbs and Albert Einstein (1879-1955).[5] Coming largely after the start of the marginalist revolution in economics, this fundamental transformation in theoretical physics had little substantive impact on the progression of neoclassical economic theory until the appearance of contributions to empirical testing of demand theory by Tinbergen, Wold and others during the interwar period. Ergodic notions also facilitated contributions to continuous time finance that started in the 1960's and culminated in Black and Scholes (1973) and Merton (1973). In turn, the deterministic mechanics of the energistic approach was well suited to the subsequent axiomatic formalization of neoclassical theory provided in the von Neumann and Morgenstern expected utility approach to modeling uncertainty and the Bourbaki inspired Arrow-Debreu general equilibrium theory, e.g., Weintraub (2002).

Having descended from the deterministic rational mechanics of mid-19th century physics, defining works of modern neoclassical economics, such as Hicks (1939) and Samuelson (1947), do not capture the probabilistic approach to modeling systems initially introduced by Boltzmann and further clarified by Gibbs.[6] Mathematical problems raised by Boltzmann were subsequently solved using tools introduced in a string of later contributions by the likes of the Ehrenfests and Cantor in set theory, Gibbs and Einstein in physics, Lebesque in measure theory, Kolmogorov in probability theory, Weiner and Levy in stochastic processes. Boltzmann was primarily concerned with problems in the kinetic theory of gases, formulating dynamic properties of the stationary Maxwell distribution – the velocity distribution of gas molecules in thermal equilibrium. Starting in 1871, Boltzmann took this analysis one step further to determine the evolution equation for the distribution function. The mathematical implications of this analysis still resonate in many subjects of the modern era. The etymology for "ergodic" begins with an 1884 paper by Boltzmann, though the initial insight to use probabilities to describe a gas system can be found as early as 1857 in a paper by Clausius and in the famous 1860 and 1867 papers by Maxwell.[7]

The Maxwell distribution is defined over the velocity of gas molecules and provides the probability for the relative number of molecules with velocities in a certain range. Using a mechanical model that involved molecular collision, Maxwell (1867) was able to demonstrate that, in thermal equilibrium, this distribution of molecular velocities was a 'stationary' distribution that would not change shape due to ongoing molecular collision. Boltzmann aimed to determine whether the Maxwell distribution would emerge in the limit whatever the initial state of the gas. In order to study the dynamics of the equilibrium distribution over time, Boltzmann introduced the probability distribution of the relative time a gas molecule has a velocity in a certain range while still retaining the notion of probability for velocities of a relative number of gas molecules. Under the ergodic hypothesis, the average behavior of the macroscopic gas system, which can objectively be measured over time using temperature and pressure, can be interchanged with the average value calculated from the ensemble of unobservable and highly complex microscopic molecular motions at a given point in time. In the words of Weiner (1939, p.1): "Both in the older Maxwell theory and in the later theory of Gibbs, it is necessary to make some sort of logical transition between the average behavior of all dynamical systems of a given family or ensemble, and the historical average of a single system."

4. The Boltzmann S-L Method Interpretation of Ergodity

Boltzmann was concerned with demonstrating that the Maxwell distribution emerged in the limit as $t \div 4$ for systems with large numbers of particles. The limiting process for t requires that the system run long enough that the initial conditions do not impact the stationary distribution. At the time, two fundamental criticisms were aimed at this general approach: reversibility and recurrence. In the context of economic time series, reversibility relates to the use of past values of the process to forecast future values.[8] Recurrence relates to the properties of the long run average which involves the ability and length of time for an ergodic process to return to its stationary state. For

Boltzmann, both these criticisms have roots in the difficulty of reconciling the second law of thermodynamics with the ergodicity hypothesis. The situation in economics is less constrained. More generally, using S-L methods, ergodicity requires the transition density of the process to be decomposable into the sum of a stationary density and a mean zero transient term that captures the impact of the initial and boundary conditions of the system on the individual sample paths (Poitras 2011, ch.5). Irreversibility relates to properties of the stationary density and non-recurrence to the behavior of the transient term.

Because the particle movements in a kinetic gas model are contained within an enclosed system, e.g., a vertical glass tube, classical Sturm-Liouville (S-L) methods can be applied to obtain solutions for the transition densities. These results for the distributional implications of imposing regular reflecting boundaries on diffusion processes are representative of the phenomenonological approach to random systems theory which: "studies qualitative changes of the densites of invariant measures of the Markov semigroup generated by random dynamical systems induced by stochastic differential equations" (Crauel et al. 1999, p.27).[9] Because the initial condition of the system is explicitly recognized, ergodicity in these models takes a different form than that associated with the reversible unit shift transformation applied to stationary densities typically adopted in economics. The ergodic transition densities are derived as solutions to the forward differential equation associated with one-dimensional diffusions. The transition densities contain a transient term that is dependent on the initial condition of the system and boundaries imposed on the state space. Irreversibility can be introduced by employing multi-modal stationary densities.

The distributional implications of boundary restrictions, derived by modeling the random variable as a diffusion process subject to reflecting barriers, have been studied for many years, e.g., Feller

(1954). The diffusion process framework is useful because it imposes a functional structure that is sufficient for known partial differential equation (PDE) solution procedures to be used to derive the relevant transition probability densities. Wong (1964) demonstrated that with appropriate specification of parameters in the PDE, the transition densities for popular stationary distributions such as the exponential, uniform, and normal distributions can be derived using S-L methods. Though Boltzmann was not concerned with economic phenomena, the analytical approach employed provides sufficient generality to resolve certain empirical difficulties arising from key stylized facts in non-experimental economic time series. In turn, the framework provides an alternative method to neoclassical economic theory that encompasses the nonlinear dynamics of diffusion processes. In other words, the mathematical framework of classical statistical mechanics permits econophysics to reformulate and clarify the ergodicity assumption thereby obtaining an alternative, more general stochastic approach to economic theory.[10]

The use of the diffusion model to represent the nonlinear dynamics of stochastic processes is found in a wide range of subjects. Restrictions such as the rate of observed genetic mutation in biology or character of heat diffusion in engineering or physics often determine the specific formalization of the diffusion model. Because physical interactions can be complex, mathematical results for diffusion models are pitched at a level of generality sufficient to cover such cases.[11] Such generality is usually not required in economics. As a consequence, where stochastic processes are employed, economic theory typically employs mean-reverting OU processes or geometric Brownian motion. Using theoretical methods from econophysics, it is possible to exploit mathematical properties of bounded state spaces and one dimensional diffusions to overcome certain analytical and empirical problems that can confront such continuous time Markov solutions. The key construct in the S-L method is the ergodic transition probability density function U which is associated with the random (economic) variable x at time t ($U = U[x, t \mid x_0]$) that follows a regular, time homogeneous diffusion process. While it is possible to allow the state space to be an infinite open interval I_o = $(a,b\text{: } \infty \leq a < b \leq \infty)$, a finite closed interval $I_c = [\,a,b\text{: } -\infty < a < b < +\infty\,]$ or the specific interval $I_s = [0 = a < b < \infty)$ are applicable to economic variables.[12] Assuming that U is twice continuously differentiable in x and once in t and vanishes outside the relevant interval, then U obeys the forward equation (e.g., Gihhman and Skorohod 1979, p.102-4):

$$\frac{\partial^2}{\partial x^2}\left\{ B[x]\ U \right\} - \frac{\partial}{\partial x}\{A[x]\ U\} = \frac{\partial U}{\partial t} \qquad (1)$$

where: $B[x]$ ($= \tfrac{1}{2}\sigma^2[x] > 0$) is the one half the infinitesimal variance and $A[x]$ the infinitesimal drift of the process. $B[x]$ is assumed to be twice and $A[x]$ once continuously differentiable in x. Being time homogeneous, this formulation permits state, but not time, variation in the drift and variance parameters.

If the diffusion process is subject to upper and lower reflecting boundaries that are regular and fixed ($-\infty < a < b < \infty$), the "Sturm-Liouville problem" involves solving (1) subject to the separated boundary conditions:[13]

$$\frac{\partial}{\partial x}\{\ B[x]\ U[x,t]\ \}\Big|_{x=a} - A[a]\ U[a,t] = 0 \qquad (3)$$

$$\frac{\partial}{\partial x}\{\ B[x]\ U[x,t]\ \}\Big|_{x=b} - A[b]\ U[b,t] = 0 \qquad (4)$$

And the initial condition:

$$U[x,0] = f[x_0] \qquad where\text{: } \int_a^b f[x_0] = 1 \qquad (5)$$

and $f[x_0]$ is the continuous density function associated with x_0 where $a \leq x_0 \leq b$. When the initial starting value, x_0, is known with certainty, the initial condition becomes the Dirac delta function, $U[x,0] = \delta[x - x_0]$, and the resulting solution for U is referred to as the 'principal solution'. Within the framework of the S-L method, a stochastic process has **the ergodic property** when the transition density satisfies:[14]

$$\lim_{t \to \infty} U[x,t \mid x_0] = \int_a^b f[x_0] \; U[x,t \mid x_0] \; dx_0 = \Psi[x]$$

Important special cases occur for the principal solution ($f[x_0] = \delta[x - x_0]$) and when $f[x_0]$ is from a specific class such as the Pearson distributions. To be ergodic, the time invariant stationary density $\Psi[x]$ is not permitted to 'decompose' the sample space with a finite number of indecomposable sub-densities, each of which is time invariant. Such irreversible processes are not ergodic, even though each of the sub-densities could be restricted to obey the ergodic theorem. To achieve ergodicity, a multi-modal stationary density can be used instead of decomposing the sample space using indecomposable sub-densities with different means.[15] In turn, multi-modal irreversible ergodic processes have the property that the mean calculated from past values of an *ex post* sample path for the process are not sufficiently informative about the modes of the *ex ante* densities to provide accurate predictions.

In order to more accurately capture the *ex ante* properties of economic time series, there are some potentially restrictive features in the S-L framework that can be identified. For example, time homogeneity of the process eliminates the need to explicitly consider the location of t_0.[16] Time homogeneity, as such, is consistent with 'ahistorical' neoclassical economic theorizing which corresponds to a sub-class of U transition densities (U^*) that have a time homogeneous and reversible stationary distribution governing the dynamics of $x(t)$. Significantly, while U is time homogeneous, there are some U consistent with irreversible processes. In order to provide a distinct alternative to stochastic models evolving from neoclassical economics, a relevant issue for econophysics is to determine which concept – time homogeneity or reversibility – is inconsistent with economic processes that capture: ratchet effects in wages; liquidity traps in money markets; structural shifts; and, collapsing conventions in asset markets. In the S-L framework, the initial state of the system (x_0) is known and the ergodic transition density provides information about how a given point x_0 shifts t units along a trajectory. For econometric applications employing the strictly stationary distributions associated with U^*, the location of x_0 is irrelevant while the wider class of transition densities, U, available in econophysics can incorporate x_0 as an initial condition associated with the solution of a partial differential equation.

4. Emergence of the Ergodic Hypothesis in Economics

The recognition by economists of ergodic notions has been uneven and opaque, even though ergodicity is an implicit assumption in the general application to non-experimental data of empirical methods developed for experimental data. In addition, important contributors to early empirical estimation of economic theories were strongly influenced by methods used in the natural sciences, especially physics and mathematical statistics. These contributors include the first two winners of the Nobel prize in Economics – Ragnar Frisch and Jan Tinbergen – as well as the statistician *cum* econometrician Herman Wold.[17] The etymology for "econometrics" begins with Frisch (1926), which appeared the same year Frisch obtained his Ph.D. in mathematical statistics. Together with Irving Fisher, Frisch was responsible for the founding of the Econometric Society in 1930.[18] In the history of neoclassical economic theory, Frisch (1933) provided an important extension of work on commodity supply and demand curves initiated by Henry Schultz (1925, 1928). In addition to work on the measurement of utility, the relevance of Frisch to the evolution of neoclassical theory was described by R.G.D. Allen (1934): "The future progress of economic science depends largely upon the work of investigators, like Professor Frisch, who realise the importance of subjecting the concepts and conclusions of abstract economic theory to the test of statistical determination and verification".

Concern with the implications of estimating empirical relationships for non-experimental data is a common theme in early contributions to econometrics. This concern culminated with Wold (1969) where it is recognized that "experimental versus nonexperimental model building is a dualism that goes to the core of the scientific method." Wold (1969, p.372) summarizes received opinion at the time: "Recognition is due -- and perhaps overdue -- for nonexperimental analysis as an active partner to the method of controlled experiments, and for the integration of both experimental and non-experimental approaches in the scientific method." Building on Wold (1938) which presented the Wold decomposition theorem for stationary time series, Wold and Jureen (1953) explicitly develops the ergodic foundations of demand theory. Dorfman (1953, p.540) describes the contribution to estimation of demand functions: "it follows from the Birkhoff-Khintchine ergodic theorem that the moments (i.e., averages, variances, covariances, etc.) calculated from a time series are consistent estimates of the moments of the population from which that time series is drawn. Classical regression analysis can be applied, but the standard error formulas have to be modified to allow for the increased instability caused by serial correlations."

Wold and Jureen (1953) is significant in explicitly establishing the connection between ergodicity and stationarity for empirically testing neoclassical economic theories. As such, this extends Wold (1949) where the theoretical connection is made only by referencing theorems in Wold (1938). At the time, establishing such a connection seemed arcane against a backdrop of more important issues, such as the simultaneous equations problem raised by Haavelmo (1943). As a consequence, ergodic notions initially entered econometrics opaquely, taking the form of stationarity assumptions and tests for the consistency of parameter estimates. Such notions translated readily to another consuming issue of the time: the estimation of business cycles and macroeconomic models initiated by Tinbergen (1939). Unlike the estimation of commodity demand and supply functions which was central to neoclassical economic theory, macroeconomic models were akin to 'complex systems'. As Wold (1949, p.4) observes:

> a serious obstacle in econometric analysis lies in the complicated structure of economic life, in the fact that if we wish to study the interdependence between two economic factors, we should pay regard to a great many other factors which are more or less correlated with the phenomena under analysis. To demonstrate the complications which may arise it is sufficient to recall J. Tinbergen's studies of business cycles; for the United States, his analysis takes into consideration 70 different economic factors and about 50 relations between them.

The analogy to complex systems theory references an important theme in econophysics (Roehner 2002, p.4; note 2 above). This suggests an unexplored connection with the important contributions to econometrics made by Tinbergen.

Following Boumans (1993), Tinbergen studied mathematics and physics at the University of Leiden and produced a Ph.D. thesis in 1929 "*Minimumproblemen in de natuurkunde en de economie*" (Minimisation problems in Physics and Economics) that was strongly influenced by the physicist Paul Ehrenfest. Yet, despite impressive training in methods of the natural sciences by key figures in the early history of econometrics, the lack of mathematical sophistication by the bulk of economists at the time contributed to unevenness in the diffusion of ergodic notions, e.g., Clark (1947). In addition, there was fierce competition for the intellectual attention space in economics as reflected in Keynes (1939) critical review of the use of regression analysis in Tinbergen (1939) to test business cycle theories. This critical debate in the history of economic thought had two distinct facets. One facet is epistemological, dividing economists into two camps. The epistemology of one camp is set out by Keynes (1939, p.559):

> Prof. Tinbergen is obviously anxious not to claim too much. If only he is allowed to carry on, he is quite ready and happy at the end of it to go a long way towards admitting, with an engaging modesty, that the results probably have no value. The worst of him is that he is much more interested in getting on with the job than in spending time in deciding whether the job is worth getting on with. He so clearly prefers the mazes of arithmetic to the mazes of logic, that I must ask him to forgive the criticisms of one whose tastes in statistical theory have been, beginning many years ago, the other way round.

Keynes questioned the validity of "passing from statistical description to inductive generalisation". This schism between alternative epistemologies persists to the present in the writings of Post Keynesian and other heterdox economists. Though not without adherents, as reflected in almost any measure of academic recognition, this camp has been marginalized within modern economics.

The other facet competing for the intellectual attention space concerned alternative empirical approaches. Though analogies with the natural sciences, in general, and physics, in particular, were common in early econometrics, Koopmans (1947) "measurement without theory" critique of the institutional school of business cycle studies captured essential elements of the phemenological approach used in many natural sciences, including econophysics, e.g., Roehner (2002, p.8). In particular, Koopmans (1947, p.161) uses a celestial mechanics analogy involving Kepler and Newton to critique the lack of theory motivating the empirical methods of Burns and Mitchell (1946): "Burns and Mitchell do not reveal at all in this book what explanations of cyclical fluctuations, if any, they believe to constitute plausible models or hypotheses."[19] The Tinbergen (1939) linear regression approach is identified as more appropriate: "the analysis and interpretation of economic data call for the application of the methods of statistical inference." Also a Nobel prize winner in economics (1975), Koopmans initially had an education in mathematics and theoretical physics, ultimately earning a Ph.D. studying under Tinbergen.

While ergodic notions appear opaquely in implicit assumptions underpinning econometric estimations, the implications of the 'ergodicity hypothesis' for economic theory had limited impact until the contributions on continuous time finance by Merton (1973) and Black and Scholes (1973) where geometric diffusion is used to model the stochastic process for a non-dividend paying common stock price. Earlier instances where ergodicity was identified often involved the use of Markov chains, e.g., Harary and Lipstein (1962) on the dynamics of brand loyalty. Morishima (1965) was able to demonstrate that the Turnpike Theorem and the Relative Stability Theorem from economic growth theory have a "formal parallelism" with the ergodic theorem for finite Markov chains. Other instances involve a subtle generalization of stochastic variables used as an error process in a linear model, e.g., Banerjee and Sengupta (1966). Following Lucas and Prescott (1971, p.), it is conventional to decompose the ensemble of future time paths for the stochastic variable(s) "into transient sets (sets which cannot be entered, and which will be departed with probability ultimately approaching 1) and ergodic sets (sets which once entered cannot be departed, and which contain no transient subsets)".

Extending Samuelson (1968), Samuelson (1976) observed that empirical theory and estimation in economics relies heavily on the use of specific stationary distributions associated with ergodic processes. The theoretical incorporation of ergodicity by Samuelson (1976) involves the addition of a discrete Markov error term into the deterministic cobweb model to demonstrate that estimated forecasts of future values, such as prices, "should be less variable than the actual data". Considerable opaqueness about the definition of ergodicity is reflected in the statement that a "'stable' stochastic process ... eventually forgets its past and therefore in the far future can be expected to approach an ergodic probability distribution" (Samuelson 1976, p.2). The connection between ergodic processes and non-linear dynamics that characterizes present efforts in economics goes unrecognized, e.g., (Samuelson 1976, p.1, 5). While some applications of ergodic processes to theoretical modeling in economics have emerged since Samuelson (1976), e.g., Horst and Wenzelburger (1984); Bullard and Butler (1993); Dixit and Pindyck (1994), econometrics has opaquely produced the bulk of the applications.

Representation of the cobweb model can be used to benchmark the evolution of ergodic notions from Wold (1949, p. 7-9) to Muth (1961, p.330-4) to Samuelson (1978, p.4-7). Ezekiel (1938) traces the development of the cobweb model from 1878, when recurring cycles in commodity production and prices were initially recognized, to contributions by Schultz and Tinbergen in 1930. Ezekiel (1938, p.255) observes: "Many economists have been disturbed by the apparent inconsistency between the persistence of these observed cycles and the tendency towards an equilibrium posited by economic theory." The cobweb model was a challenge to the classical theory (Ezekiel 1938, p.279):

> The equilibrium concept lies at the heart of classic theory. If prices and production do not converge rapidly to an equilibrium, then each industry may recurringly attract to it more labor and investment than it can use to advantage, and may leave that labor and equipment only partly utilized much of the time.

The cobweb model permits a 'neoclassical' explanation for the observed persistance in underemployment of the 1930's (*Ibid.*):

> Even under the conditions of pure competition and static demand and supply, there is thus no "automatic self-regulating mechanism," which can provide full utilization of resources. Unemployment, excess capacity, and the wasteful use of resources may occur even when all the competitive assumptions are fulfilled.

In turn, explanations of the cobweb model illustrate the evolution of ergodic notions in neoclassical economics.

Wold (1949) summarizes the results given in Ezekiel (1938) referring to the sequence of possible time paths for the commodity price in a cobweb model as convergent, cyclical or divergent from the classical (Cournot) equilibrium result. Without providing details of the disturbance that perturbs the equilibrium of supply and demand, Wold (1949, p.8) provides the conventional conclusion about the possible time paths: “Which case will occur depends on the slopes of the demand and supply curves.” The general connection in Haavelmo (1940) between the economic theory being tested and the associated time series properties of the price process goes unnoticed. Nerlove (1958) represents a significant advance in connecting expectations formation and whether the cobweb price equilibrium is stable or unstable. In addition to providing the initial intellectual spark for the rational expectations revolution in economics, the seminal contribution by Muth (1961) did so by opaquely incorporating ergodic notions to arrive at a rational expectations solution to the commodity price.[20] The last section of Muth (1961) demonstrates that a rational expectations solution has better fit to empirical commodity price data than the cobweb model.

Though there is no explicit reference to ‘ergodic’ error process, Muth (1961, p.317) makes the following assumptions:

> 1. The random disturbances are normally distributed. 2. Certainty equivalents exist for the variables to be predicted. 3. The equations of the system, including the expectations formulas, are linear. These assumptions are not quite so strong as may appear at first because any one of them virtually implies the other two.

In turn, when the general error term in the supply equation is serially correlated, the error term is reexpressed as an infinite moving average of identically, independently, normally distributed random variables. This representation is used “to find the weights of the regression functions for prices and price expectations” in terms of the coefficients in the structural equations. In the important case where inventory speculation is permitted and the errors impacting the supply equation are 'independent', Muth demonstrates the "resulting stochastic process [for price and price expectations] is Markovian". At another point, a probability limit (plim) of the slope coefficient in a least squares regression of the expected price estimate on the observed price is evaluated to demonstrate that expectations can be unbiased and the regression coefficient still be less than unity.

Muth (1961) liberally employs ergodic notions without explicitly recognizing the ergodicity hypothesis. An explicit connection to convergence of the cobweb model when the error in the supply equation is "ergodic" is made by Samuelson (1976, p.5): "Given the strong dampening properties of the non-linear system, the conditional probabilities can be shown, under plausible regularity conditions to approach an ergodic probability distribution that is independent of initial p_0". Samuelson (1976, p.5) proceeds to describe the properties of the cobweb model price equilibrium determined by introducing conditional Markov probabilities:

> Heuristically, any single disturbance dies out from the strong dampening; the system, so to speak, eventually forgets its distant past; when continually subjected to independent shocks, it reaches its ergodic Brownian vibration – the natural generalization of non-stochastic equilibrium – when there is a balancing of the shock energy imposed and the frictional energy dissipated by the dampening.

With this contribution, Samuelson completes an analysis of the cobweb model that began with Samuelson (1944), which provides a more general treatment of the deterministic cobweb model than provided by Tinbergen and Schultz. It is fitting that a leading figure of the neoclassical school provides an ergodic solution to a deterministic pricing model by introducing a Markov error term.

4. Phenomenological Method and Neoclassical Economics

In physics, phenomenology lies at the intersection of theory and experiment. Theoretical relationships between empirical observations are modeled without deriving the theory directly from first principles, e.g., Newton's laws of motion. Predictions based on these theoretical relationships are obtained and compared to further experimental data designed to test the predictions. In this fashion, new theories derived from first principles are motivated. Confronted with non-experimental data for important economic variables, such as wage rates, commodity prices, interest rates and the like, modern economics similarly develops theoretical models that aim to fit the 'stylized facts' of those variables. While comparison of 'stylized facts' with predictions of theories initially derived directly from the 'first principles' of constrained maximizing behavior of individuals and firms is recommended practice in neoclassical economics, such theories generally have poor *ex ante* empirical performance. This has given impetus to the inductive approach in mainstream economics, especially econometrics and macroeconomics, an inherently phenomenological approach to theorizing in economics, e.g., Hendry (1995). Given the difficulties in economics of testing model predictions with 'new' experimental data, a variety of mathematical techniques have been adapted to determining theoretical relationships among economic variables that better explain the 'stylized facts'.[21]

The evolution of the 'ergodicity hypothesis' in neoclassical economics ultimately induced the emergence of a 'mainstream economics' a.k.a. 'modern economics' that has been characterized as "centered on dynamics, recursive methods and complexity theory" (Colander et al. 2004). The evolution of economic theory from the deterministic models of neoclassical economics to more modern stochastic models has been incremental and disjointed. The preference for linear models of static equilibrium relationships has restricted the application of frameworks that capture more complex non-linear dynamics, e.g., chaos theory; truncated Levy processes. Yet, important variables in economics have relatively innocuous sample paths compared to some types of variables encountered in physics. There is an impressive range of mathematical and statistical models that, seemingly, could be applied to almost any physical or economic situation. If the process can be verbalized, then a model can be specified. This begs questions such as: are there stochastic models – ergodic or otherwise – that capture the basic 'stylized facts' of observed economic data? Is the random instability in the observed sample paths identified in, say, commodity price time series, consistent with, say, the *ex ante* stochastic bifurcation of an ergodic process, e.g., Chiarella et al. (2008)?

The conventional view of ergodicity in modern economics is reflected by Hendry (1995, p.100): "Whether economic reality is an ergodic process after suitable transformation is a deep issue" which is difficult to analyze rigorously. This opaque interpretation of ergodicity reflects the limited recognition of the role that 'the ergodicity hypothesis' plays in modern economics. In contrast, the ergodic hypothesis in classical statistical mechanics is associated with the more physically transparent kinetic gas model than the often technical and targeted concepts of ergodicity that are identified in modern economics. For Boltzmann, the ergodic hypothesis permitted modeling of the unobserved complex microscopic interactions of individual gas molecules that had to obey the second law of thermodynamics, a concept that has limited application in economics.[22] Despite differences in physical interpretation, economics is also confronted with a similar problem of modeling 'macroscopic' economic variables, such as exchange rates or GNP or interest rates, when it is not possible to derive a theory from known neoclassical principles about the (microscopic) rational behavior of individuals and firms that can consistently predict future empirical observations. At least since Tinbergen (1939), a resolution of such problems has encouraged development of a phenomenological approach to theoretical modeling in modern economics.[23]

Even though the formal solutions proposed were inadequate by standards of modern mathematics, the thermodynamic model introduced by Boltzmann to explain the dynamic properties of the Maxwell distribution is a pedagogically useful starting point to demarcate the histories of modern economics and neoclassical economic theory. To be sure, von Neumann (1932) and Birkhoff (1931) correctly specify ergodicity using measure theory and Lebesque integration – an essential analytical tool unavailable to Boltzmann – but the analysis is too complex to be of much value to all but the most mathematically specialized economists. The physical intuition of the kinetic gas model is lost in the generality of the results. Using Boltzmann as a starting point, the large number of mechanical and complex molecular collisions could correspond to the large number of neoclassical, atomistic competitors and consumers interacting to determine the macroscopic market price.[24] In this context, it is real or nominal variables such as the asset price or the interest rate or the exchange rate, or some combination, that is being measured over time and ergodicity would be associated with the properties of the transition density generating the macroscopic variables. Ergodicity can fail for a number of reasons and there is value in determining the source of the failure.[25]

Halmos (1949, p.1017) is a helpful starting point to sort out the differing notions of ergodicity that arise in range of subjects: "The ergodic theorem is a statement about a space, a function and a transformation". In mathematical terms, ergodicity or 'metric transitivity' is a property of 'indecomposable', measure preserving transformations. Because the transformation acts on points in the space, there is a fundamental connection to the method of measuring relationships such as distance or volume in the space. In von Neumann (1932) and Birkhoff (1931), this is accomplished using the notion of Lebesque measure: the admissible functions are either integrable (Birkhoff) or square integrable (von Neumann). In contrast to, say, statistical mechanics where spaces and functions account for the complex physical interaction of large numbers of particles, neoclassical economic theory can usually specify the space in a mathematically convenient fashion. For example, in the case where there is a single random variable such as a commodity price, then the space is "superfluous" (Mackey 1974, p.182) as the random variable is completely described by the distribution. Multiple random variables can be handled by assuming the random variables are discrete with finite state spaces. In effect, conditions for an 'invariant measure' can often be assumed in economics in order to focus attention on "finding and studying the invariant measures" (Arnold 1998, p.22) where, in the terminology of econometrics, the invariant measure usually corresponds to the (ergodic) stationary error distribution or likelihood function.

The mean ergodic theorem of von Neumann (1932) provides an essential connection to the ergodicity hypothesis in modern economics. It is well known that, in the Hilbert and Banach spaces common to econometric work, the mean ergodic theorem corresponds to the strong law of large numbers. In statistical applications where strictly stationary distributions are assumed, the relevant ergodic transformation, L^*, is the unit shift operator: $L^* \Psi[x(t)] = \Psi[L^* x(t)] = \Psi[x(t+1)]$; $[(L^*)^k] \Psi[x(t)] = \Psi[x(t+k)]$; and $\{(L^*)^{-k}\} \Psi[x(t)] = \Psi[x(t-k)]$ with k being an integer and $\Psi[x]$ the strictly stationary distribution for x that in the strictly stationary case is replicated at each t.[26] Significantly, this reversible transformation is independent of initial time and state. Only the distance between observations is relevant. More significantly,

Because this transformation imposes strict stationarity on $\Psi[x]$, L^* will only work for certain ergodic processes. The ergodic requirement that the transformation be measure preserving is weaker than the strict stationarity of the stochastic process required for L^*. The implications of the reversible ergodic transformation L^* are central to the criticisms of neoclassical economic theory advanced by heterodox economists. e.g., Davidson (1991, p.331): "In an economic world governed entirely by ergodic processes ... economic relationships among variables are timeless, or ahistoric in the sense that the future is merely a statistical reflection of the past".[27] In effect, the use of ergodicity in economics requires: initial and boundary conditions for the state space; and, that the stationary real world distribution for $x(t)$ be sufficiently similar to those for both $x(t+k)$ or $x(t-k)$, i.e., the ergodic transformation L^* is reversible.

Following the seminal contributions by Tinbergen, Wold and Koopmans, iterations and extensions of the GLM to deal with complications arising in empirical estimates dominated subsequent work in econometrics, e.g., Dhrymes (1974) and Theil (1971), leading to application of generalized least squares estimation techniques that encompassed autocorrelated and heteroskedastic error terms. Employing L_2 vector space methods with stationary, i.e., independently, identically distributed (iid), error term distributions ensured these early stochastic models implicitly assumed ergodicity. The generalization of this discrete time estimation approach to the class of ARCH and GARCH error term models by Engle and Granger was of such significance that a Nobel memorial prize in economics was awarded for this contribution, e.g., Engle and Granger (1987). By modeling the evolution of the volatility, this approach permitted a limited degree of non-linearity to be modeled providing a substantively better fit to observed economic time series. Only recently has the ergodicity of the GARCH model and related methods been considered, e.g., Meitz and Saikkonen (2008).

The emergence of ARCH, GARCH and related models was part of a general trend toward the use of inductive methods in economics, often employing discrete, linear time series methods to model transformed economic variables, e.g., Hendry (1995). At least since Dickey and Fuller (1979), it has been found that estimates of univariate time series models for many economic times series reveals evidence of 'non-stationarity'. A number of approaches have emerged to deal with this apparent empirical quandary.[28] In particular, transformation techniques for time series models have received considerable attention. Extension of the Box-Jenkins methodology led to the concept of economic time series being I(0) – stationary in the level – and I(1) – non-stationary in the level but stationary after first differencing. Two I(1) economic variables could be cointegrated if differencing the two series produced an I(0) process, e.g., Hendry (1995). Extending early work on distributed lags, long memory processes have also been employed where the time series is only subject to fractional differencing. Significantly, recent contributions on Markov switching processes and exponential smooth transition autoregressive processes have demonstrated the "possibility that nonlinear ergodic processes can be misinterpreted as unit root nonstationary processes" (Kapetanios and Shin 2011, p.620).

7. Conclusion

Modern economics is a diverse subject that has evolved dramatically for the static, linear models of neoclassical economics.

The introduction of the ergodicity hypothesis by Ludwig Boltzmann provides a point of demarcation between the histories of neoclassical economic theory, modern economics and econophysics. Why begin the history of econophysics with Boltzmann whose concern was only with the kinetic theory of gases and not economic phenomena? This paper demonstrates that the mathematical techniques available in econophysics that are distinct from those derived from neoclassical economics start with the S-L methods pioneered by Boltzmann. From this point, methods available to econophysics differ substantively from the static and deterministic models of rational mechanics that underpin neoclassical economics. As such, the direct use of ergodic processes in econophysics provides a distinct approach to understanding and predicting the stochastic behavior of economic phenomena. For example, whereas economics employs only unimodal processes that ignore the impact of transient terms, econophysics is able to model economic observations that are generated by wider range of ergodic processes. Such models may be more relevant for non-experimental data where the calculation of time averages based on a sufficiently long enough *ex post* sample path can not be expected to provide a statistically reliable estimate of the *ex ante* time or space average for a future time paths that may be observed over a sufficiently distant future calendar time.

To deal with the problem of making statistical inferences from 'non-experimental' data, stochastic generalizations of neoclassical economic theory typically employ stationary densities that are: reversible; unimodal; and, where initial and boundary conditions have no short or long term impact. The possibility of irreversible ergodic processes is not recognized or, it seems, intended. For example, a type of fundamental uncertainty is inherent in bifurcating processes in order to determine the *ex ante* stationary density. A semantic connection can be established between the subjective uncertainty about encountering a future bifurcation point and, say, the possible collapse of an asset price bubble due to a change in Keynesian convention about market valuations. From the early results of Boltzmann in classical statistical mechanics to the fractals of Mandelbrot, econophysics provides a variety of potential stochastic approaches to modeling economic phenomena that may eventually prove to be more insightful than the stochastic models employed in testing the deterministic and static predictions of neoclassical economic theory.

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NOTES

1. The 'Mirowski thesis' has been contentious, for a variety of reasons, e.g., Walker (1991); Varian (1991). Included in these reasons are a tendency to overstate the position and a lack of attention to specific detail. For example, Mirowski identifies the beginnings of neoclassical economics with the energistics approach to physics. From a history of physics perspective, this is not technically correct as energistics did not emerge in full force until after the 1870's when the marginalist revolution commenced. More precisely, the marginalists were more influenced by results from rational mechanics which is a key component of the energistics approach. Cercignani (1998, p.202-9) discusses the connection between Boltzmann and the energetists of that time.

2. To illustrate the methodology used in econophysics papers, consider the 9 econophysical papers appearing in the first issue in the 2013 volume of *Physica A*. The papers can be divided into studies of: complex systems (2); stable Paretian or truncated Levy processes (2); system dynamics (2); and, stochastic volatility (1). A common theme for a number of these articles is reflected in the comment: "Our model generates aggregate dynamics for key economic variables consistent with empirics."

3. A number of different definitions of stationarity for a stochastic process are available, e.g., Ash and Gardner (1975, p.15). In L^2 space which is conventional in empirical analysis in economics, 'wide sense' stationary process has a constant mean and covariance function that depends only on the difference between observations so that the covariance function depends only on t. A 'strictly stationary' process has joint distribution functions that depend only on t. Strict stationarity implies

wide sense stationarity but not the converse. Wide sense stationary processes are also referred to as 'covariance stationary' and 'weak-sense stationary' and 'stationary to order 2'. Working in L^2 space avoids significant complications associated with measurability and continuity of sample functions.

4. In rational mechanics, once the initial positions of the particles of interest, e.g., molecules, are known, the mechanical model fully determines the future evolution of the system. This scientific and philosophical approach is often referred to as Laplacian determinism.

5. Boltzmann and Max Planck were vociferous opponents of energetics. The debate over energetics was part of a larger intellectual debate concerning determinism and reversibility. What Boltzmann, Planck and others had observed in statistical physics was that, even though the behavior of one or two molecules can be completely determined, it is not possible to generalize these mechanics to the describe the macroscopic motion of molecules in large, complex systems, e.g., Brush (1983, esp. ch.II).

6. As such, Boltzmann was part of the larger: "Second Scientific Revolution, associated with the theories of Darwin, Maxwell, Planck, Einstein, Heisenberg and Schrödinger, (which) substituted a world of process and chance whose ultimate philosophical meaning still remains obscure" (Brush (1983, p.79). This revolution superceded the: "First Scientific Revolution, dominated by the physical astronomy of Copernicus, Kepler, Galileo, and Newton, ... in which all changes are cyclic and all motions are in principle determined by causal laws." The irreversibility and indeterminism of the Second Scientific Revolution replaces the reversibility and determinism of the First.

7. There are many interesting sources on these points which provide citations for the historical papers that are being discussed. Cercignani (1988, p.146-50) discusses the role of Maxwell and Boltzmann in the development of the ergodic hypothesis. Maxwell (1867) is identified as "perhaps the strongest statement in favour of the ergodic hypothesis". Brush (1976) has a detailed account of the development of the ergodic hypothesis. Gallavotti (1995) traces the etymology of "ergodic" to the 'ergode' in an 1884 paper by Boltzmann. More precisely, an ergode is shorthand for 'ergomonode' which is a 'monode with given energy' where a 'monode' can be either a single stationary distribution taken as an ensemble or a collection of such stationary distributions with some defined parameterization. The specific use is clear from the context. Boltzmann proved that an ergode is an equilibrium ensemble and, as such, provides a mechanical model consistent with the second law of thermodynamics. It is generally recognized that the modern usage of 'the ergodic hypothesis' originates with Ehrenfest (1911).

8. The connection of the reversibility and recurrence concepts used in this paper with the actual arguments made during the Boltzmann debates is somewhat tenuous. For example, the assumption that the diffusion process is regular deals with the version of the recurrence problem that concerned Boltzmann. The objective of introducing these concepts is pedagogy rather than historical accuracy.

9. The distinction between invariant and ergodic measures is fundamental. Recognizing a number of distinct definitions of ergodicity are available, following Medio (2005, p.70) the Birkhoff-Khinchin ergodic (BK) theorem for invariant measures can be used to demonstrate that ergodic measures are a class of invariant measures. More precisely, the BK theorem permits the limit of the

time average to depend on initial conditions. In effect, the invariant measure is permitted to decompose into invariant 'sub-measures'. The physical interpretation of this restriction is that sample paths starting from a particular initial condition may only be able to access a part of the sample space, no matter how long the process is allowed to run. For an ergodic process, sample paths starting from any admissible initial condition will be able to 'fill the sample space', i.e., if the process is allowed to run long enough, the time average will not depend on the initial condition. Medio [2005, p.73] provides a useful example of an invariant measure that is not ergodic.

10. This is not intended to imply that classical statistical mechanics is the only method available to econophysics. Quite the contrary, there are a range of theoretical approaches employed in physics that have potential application in economics. For example, truncated Levy processes have been suggested as a method of dealing with the infinite variance aspect of 'stable' processes. However, such processes typically are associated with unimodal stationary densities and, as such, represent a distinct approach from that available using the S-L methods of classical statistical mechanics.

11. The phenomenological approach is not without difficulties. For example, the restriction to Markov processes ignores the possibility of invariant measures that are not Markov. In addition, an important analytical construct in bifurcation theory, the Lyapunov exponent, can encounter difficulties with certain invariant Markov measures. Primary concern with the properties of the stationary distribution is not well suited to analysis of the dynamic paths around a bifurcation point. And so it goes.

12. A diffusion process is 'regular' if starting from any point in the state space I, any other point in I can be reached with positive probability (Karlin and Taylor 1981, p.158). This condition is distinct from other definitions of regular that will be introduced: 'regular boundary conditions' and 'regular S-L problem'.

13. The classification of boundary conditions is typically an important issue in the study of solutions to the forward equation. Important types of boundaries include: regular; exit; entrance; and natural. Also important in boundary classification are: the properties of attainable and unattainable; whether the boundary is attracting or non-attracting; and whether the boundary is reflecting or absorbing. In the present context, regular, attainable, reflecting boundaries are usually being considered, with a few specific extensions to other types of boundaries. In general, the specification of boundary conditions is essential in determining whether a given PDE is self-adjoint

14. Heuristically, if the ergodic process runs long enough, then the stationary distribution can be used to estimate the constant mean value. This definition of ergodic is appropriate for the one-dimensional diffusion cases considered in this paper. Other combinations of transformation, space and function will produce different requirements. Various theoretical results are available for the case at hand. For example, the existence of an invariant Markov measure and exponential decay of the autocorrelation function are both assured.

15. An alternative approach to using multimodal densities is to use a mixture of distributions. This approach will typically involve the estimation of more parameters than in the multimodal case. The study of mixing distributions is well developed in physics and related subjects, e.g., Mengersen

(2011).

16. For ease of notation it is assumed that $t_0 = 0$. In practice, solving (1) combined with (3)-(5) requires *a* and *b* to be specified. While *a* and *b* have ready interpretations in physical applications, e.g., the heat flow in an insulated bar, determining these values in economic applications can be more challenging. Some situations, such as the determination of the distribution of an exchange rate subject to control bands (e.g., Ball and Roma 1988), are relatively straight forward. Other situations, such as profit distributions with arbitrage boundaries or output distributions subject to production possibility frontiers, may require the basic S-L framework to be adapted to the specifics of the modeling situation.

17. Wold studied mathematical statistics under Harald Cramer at the University of Stockholm, graduating in 1930.

18. From the Econometric society webpage: "The Econometric Society is the most prestigious learned society in the field of economics, with a world-wide membership. Its main object is to promote studies that aim at a unification of the theoretical-quantitative and empirical-quantitative approach to economic problems and that are penetrated by constructive and rigorous thinking similar to that which has come to dominate in the natural sciences. It operates as a purely scientific organization, without any political, social, financial or nationalistic allegiance or bias."
(http://www.econometricsociety.org/society.asp#history)

19. Similarly, Roehner uses an analogy with Newton and the apple.

20. Sent (2002) details the "contrary tale" of John F. Muth.

21. In this context though not in all contexts, econophysics provides a 'macroscopic' approach. In turn, ergodicity is an assumption that permits the time average from a single observed sample path to (phenomenologically) model the ensemble of sample paths. Given this, econophysics does contain a substantively richer toolkit that encompasses both ergodic and non-ergodic processes. Many works in econophysics implicitly assume ergodicity and develop models based on that assumption.

22. The second law of thermodynamics is the universal law of increasing entropy – a measure of the randomness of molecular motion and the loss of energy to do work. First recognized in the early 19^{th} century, the second law maintains that the entropy of an isolated system, not in equilibrium, will necessarily tend to increase over time. Entropy approaches a maximum value at thermal equilibrium. A number of attempts have been made to apply the entropy of information to problems in economics, with mixed success. In addition to the second law, physics now recognizes the zeroth law of thermodynamics that "any system approaches an equilibrium state" (Reed and Simon [60, p.54]). This implications of the second law for theories in economics was initially explored by Georgescu-Roegen (1971).

23. In this process, the ergodicity hypothesis is required to permit the one observed sample path to be used to estimate the parameters for the *ex ante* distribution of the ensemble paths. In turn, these

parameters are used to predict future values of the economic variable.

24. This interpretation of the microscopic collisions differs from Davidson (1988, p.332): "If there is only one actual economy, and we do not possess, never have possessed and conceptually never will possess an *ensemble* of economics worlds, then even a definition of probability distribution functions is questionable." In this context, points in the phase space at time *t* represent individual realizations of different macroscopic outcomes for the economic system at *t*. This interpretation of the ensembles is closer to Gibbs than Maxwell.

25. Lebowitz (1999) and Volovich (2010) discuss the implications of irreversibility in statistical mechanics. In contrast to an absence of views in mainstream economics, there are heterodox critiques that originate from within economics, e.g., Post Keynesian economists, institutional economists, radical political economists and so on. Because such critiques take motivation from the theories of mainstream economics, these critiques are distinct from econophysics. Following Schinckus (2010, p.3818): "Econophysicists have then allies within economics with whom they should become acquainted."

26. Dhyrmes (1971, p.1-29) discusses the algebra of the lag operator.

27. Critiques of mainstream economics that are rooted in the insights of *The General Theory* recognize the distinction between fundamental uncertainty and objective probability. As a consequence, the definition of ergodic theory in heterodox criticisms of mainstream economics lacks formal precision, e.g., the short term dependence of ergodic processes on initial conditions is not usually recognized. Ergodic theory is implicitly seen as another piece of the mathematical formalism inspired by Hilbert and Bourbaki and captured in the Arrow-Debreu general equilibrium model of mainstream economics.

28. Kapetanios and Shin (2011, p.620) capture the essence of this quandary: "Interest in the interface of nonstationarity and nonlinearity has been increasing in the econometric literature. The motivation for this development may be traced to the perceived possibility that nonlinear ergodic processes can be misinterpreted as unit root nonstationary processes. Furthermore, the inability of standard unit root tests to reject the null hypothesis of unit root for a large number of macroeconomic variables, which are supposed to be stationary according to economic theory, is another reason behind the increased interest."