Ergodicity, Econophysics and
the History of Economic Theory

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ABSTRACT

What role did the 'ergodicity hypothesis' play in the development of economic theory during the 20th century? What is the relevance of ergodicity to the emergence of the new subject of econophysics? This paper addresses these questions by reviewing the etymology and history of the ergodicity hypothesis from the introduction of the concept in 19th century statistical mechanics until the emergence of econophysics during the 1990's. An explanation of ergodicity is provided that establishes a connection to the fundamental problem of using non-experimental data to verify propositions in economic theory. The relevance of the ergodicity assumption in the *ex post / ex ante* quandary confronting important theories in financial economics is also examined.

Keywords: Ergodicity; Ludwig Boltzmann; Econophysics; stationary distribution.

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1. Introduction

At least since Mirowski (1984), it has been recognized that important theoretical elements of neoclassical economics were adapted from mathematical concepts developed in 19th century physics. Much effort by historians of economic thought has been dedicated to exploring the validity and implications of "Mirowski's thesis", e.g., Carlson (1997), especially the connection between the deterministic 'rational mechanics' approach to physics and the subsequent development of neoclassical economic theory (Mirowski 1988).[1] This well known literature connecting physics with neoclassical economics is seemingly incongruent with the emergence of econophysics during the last decade of the twentieth century, a distinct subject which involves the application of theoretical and empirical methods from physics to the analysis of economic phenomena, e.g., Jovanovic and Schinckus (2012). As Mirowski (1989b, 1990) is at pains to emphasize, physical theory has evolved considerably from the deterministic approach which underpins neoclassical economics. In detailing historical developments in physics, Mirowski and others are quick to jump from the mechanical determinism of energistics to quantum mechanics to recent developments in chaos theory, overlooking the relevance for economic theory of the initial steps toward modeling the stochastic behavior of physical phenomena by Ludwig Boltzmann (1844-1906), James Maxwell (1831-1879) and Josiah Gibbs (1839-1903).

The point of demarcation between the histories of neoclassical economic theory and econophysics can be traced to the debate over energistics around the end of the 19th century. While the evolution of physics after energistics involved the introduction and subsequent development of stochastic concepts, fueled by the emergence of econometrics after WWII, economics also incorporated

stochastic concepts aimed at generalizing and empirically testing deterministic neoclassical theory, e.g., Mirowski (1989b). Significantly, stochastic generalization of the deterministic and static equilibrium approach of neoclassical economic theory required the adoption of 'time reversible' probabilistic models, especially the likelihood functions associated with certain stationary distributions. In contrast, from the early ergodic models of Boltzmann to the fractals and chaos theory of Mandlebrot, physics has employed a wider variety of stochastic models aimed at capturing key empirical characteristics of the physical problem at hand. These models typically have a mathematical structure that varies substantively from the constrained optimization techniques that underpin neoclassical economic theory, restricting the straightforward application of these models to economics. Overcoming the difficulties of applying models developed for physical situations to economic phenomena is the central problem confronting econophysics.

Schinckus (2010, p.3816) accurately recognizes that the positivist philosophical foundation of econophysics depends fundamentally on empirical observation: "The empiricist dimension is probably the first positivist feature of econophysics". Following McCauley (2004) and others, this concern with empiricism often focuses on the identification of macro-level statistical regularities that are characterized by the scaling laws identified by Mandelbrot (1997) and Mandelbrot and Hudson (2004) for financial data. Unfortunately, this empirically driven ideal is often confounded by the 'non-repeatable' experiment that characterizes most observed economic and financial data. There is quandary posed by having only a single observed *ex post* time path to estimate the distributional parameters for the ensemble of *ex ante* time paths. In contrast to physical sciences, in the human sciences there is no assurance that *ex post* statistical regularity translates into *ex ante* forecasting accuracy. Divergent approaches to resolving of this quandary highlights the pedagogical usefulness

of demarcating the histories of neoclassical economic theory and econophysics.

To this end, this paper provides an etymology and history of the 'ergodicity hypothesis' in 19th century statistical mechanics. Subsequent use of ergodicity in modern economics is also examined. A classical interpretation of ergodicity is provided that uses Sturm-Liouville (S-L) theory – a mathematical method central to the classical statistical mechanics pioneered by Boltzmann (Nolte 2010). Using S-L methods, the transition probability density of a one-dimensional (ergodic) diffusion process subject to regular upper and lower reflecting barriers can be decomposed into a possibly multimodal limiting stationary density which is independent of time and initial condition, and a power series of time, initial condition and boundary dependent transient terms.[2] In contrast, empirical theory aimed at estimating relationships from neoclassical economics typically ignores the implications of the initial and boundary conditions that generate transient terms and focuses on properties of limiting unimodal stationary densities. To illustrate the implications for economic theory of the expanded class of ergodic processes available to econophysics, properties of the mulitmodal quartic exponential stationary density are considered and used to assess the role of the ergodicity hypothesis in the *ex post / ex ante* quandary confronting important theories in financial economics.

2. A Brief History of Ergodic Theory

The Encyclopedia of Mathematics (2002) defines ergodic theory as the "metric theory of dynamical systems. The branch of the theory of dynamical systems that studies systems with an invariant measure and related problems." This modern definition implicitly identifies the birth of ergodic theory with proofs of the mean ergodic theorem by von Neumann (1932) and the pointwise ergodic theorem by Birkhoff (1931). These early proofs have had significant impact in a wide range of

modern subjects. For example, the notions of invariant measure and metric transitivity used in the proofs are fundamental to the measure theoretic foundation of modern probability theory (Doob 1953; Mackey 1974). Building on a seminal contribution to probability theory (Kolmogorov 1933), in the years immediately following it was recognized that the ergodic theorems generalize the strong law of large numbers. Similarly, the equality of ensemble and time averages – the essence of the mean ergodic theorem – is necessary to the concept of a strictly stationary stochastic process. Ergodic theory is the basis for the modern study of random dynamical systems, e.g., Arnold (1988). In mathematics, ergodic theory connects measure theory with the theory of transformation groups. This connection is important in motivating the generalization of harmonic analysis from the real line to locally compact groups.

From the perspective of modern mathematics, statistical physics or systems theory, Birkhoff (1931) and von Neumann (1932) are excellent starting points for a history of ergodic theory. Building on the ergodic theorems, subsequent developments in these and related fields have been dramatic. These contributions mark the solution to a problem in statistical mechanics and thermodynamics that was recognized sixty years earlier when Ludwig Boltzmann (1844-1906) introduced the ergodic hypothesis to permit the theoretical phase space average to be interchanged with the measurable time average. From the perspective of both econophysics and neoclassical economics, the selection of the less formally correct and rigorous contributions of Boltzmann are a more auspicious beginning for a history of the ergodic hypothesis. Problems of interest in mathematics are generated by a range of subjects, such as physics, chemistry, engineering and biology. The formulation and solution of physical problems in, say, statistical mechanics or particle physics will have mathematical features which are inapplicable or unnecessary in economics. For example, in statistical mechanics, points

in the phase space are often multi-dimensional functions representing the mechanical state of the system, hence the desirability of a group-theoretic interpretation of the ergodic hypothesis. From the perspective of economics, such complications are largely irrelevant and an alternative history of ergodic theory that captures the etymology and basic physical interpretation is more revealing than a history that focuses on the relevance for mathematics. This arguably more revealing history begins with the formulation of the problems that von Neumann and Birkhoff were able to solve.

Mirowski (1989a, esp. ch.5) establishes the importance of 19th century physics in the development of the neoclassical economic system advanced by Jevons, Walras and Menger during the marginalist revolution of the 1870's. As such, neoclassical economic theory inherited essential features of mid-19th century physics: deterministic rational mechanics; conservation of energy; and the non-atomistic continuum view of matter that inspired the energetics movement later in the 19th century.[3] It was during the transition from rational to statistical mechanics during the last third of the century that Boltzmann made the contributions that led to the transformation of theoretical physics from the microscopic mechanistic models of Rudolf Clausius (1822-1888) and James Maxwell to the macroscopic probabilistic theories of Josiah Gibbs and Albert Einstein (1879-1955).[4] Coming largely after the start of the marginalist revolution in economics, this fundamental transformation in theoretical physics had little impact on the progression of neoclassical economic theory until the appearance of contributions on continuous time finance that started in the 1960's and culminated in Black and Scholes (1973). The deterministic mechanics of the energistic approach was well suited to the subsequent axiomatic formalization of neoclassical theory which culminated in the von Neumann and Morgenstern expected utility approach to modeling uncertainty and the Bourbaki inspired Arrow-Debreu general equilibrium theory, e.g., Weintraub (2002).

Having descended from the deterministic rational mechanics of mid-19th century physics, defining works of neoclassical economics, such as Hicks (1939) and Samuelson (1947), do not capture the probabilistic approach to modeling systems initially introduced by Boltzmann and further clarified by Gibbs.[5] Mathematical problems raised by Boltzmann were subsequently solved using tools introduced in a string of later contributions by the likes of the Ehrenfests and Cantor in set theory, Gibbs and Einstein in physics, Lebesque in measure theory, Kolmogorov in probability theory, Weiner and Levy in stochastic processes. Boltzmann was primarily concerned with problems in the kinetic theory of gases, formulating dynamic properties of the stationary Maxwell distribution – the velocity distribution of gas molecules in thermal equilibrium. Starting in 1871, Boltzmann took this analysis one step further to determine the evolution equation for the distribution function. The mathematical implications of this analysis still resonate in many subjects of the modern era. The etymology for "ergodic" begins with an 1884 paper by Boltzmann, though the initial insight to use probabilities to describe a gas system can be found as early as 1857 in a paper by Clausius and in the famous 1860 and 1867 papers by Maxwell.[6]

The Maxwell distribution is defined over the velocity of gas molecules and provides the probability for the relative number of molecules with velocities in a certain range. Using a mechanical model that involved molecular collision, Maxwell (1867) was able to demonstrate that, in thermal equilibrium, this distribution of molecular velocities was a 'stationary' distribution that would not change shape due to ongoing molecular collision. Boltzmann aimed to determine whether the Maxwell distribution would emerge in the limit whatever the initial state of the gas. In order to study the dynamics of the equilibrium distribution over time, Boltzmann introduced the probability distribution of the relative time a gas molecule has a velocity in a certain range while still retaining

the notion of probability for velocities of a relative number of gas molecules. Under the ergodic

hypothesis, the average behavior of the macroscopic gas system, which can objectively be measured

over time using temperature as a macroscopic measure, can be interchanged with the average value

calculated from the ensemble of unobservable and highly complex microscopic molecular motions

and collisions at a given point in time. In the words of Weiner (1939, p.1): "Both in the older

Maxwell theory and in the later theory of Gibbs, it is necessary to make some sort of logical

transition between the average behavior of all dynamical systems of a given family or ensemble, and

the historical average of a single system."

3. Use of the Ergodic Hypothesis in Economics

At least since Samuelson (1976), it has been recognized that empirical theory and estimation in

economics relies heavily on the use of specific stationary distributions associated with the limit of

an ergodic process. As reflected in the evolution of the concept in economics, the specification and

implications of ergodicity have only developed gradually. The early presentation of ergodicity by

Samuelson (1976) involves the addition of a discrete Markov error term into the deterministic

cobweb model to demonstrate that estimated forecasts of future values, such as prices, "should be

less variable than the actual data". Considerable opaqueness about the definition of ergodicity is

reflected in the statement that a "'stable' stochastic process ... eventually forgets its past and

therefore in the far future can be expected to approach an ergodic probability distribution"

(Samuelson 1976, p.2). The connection between ergodic processes and non-linear dynamics that

characterizes present efforts in economics goes unrecognized, e.g., (Samuelson 1976, p.1, 5). While

some applications of ergodic processes to theoretical modeling in economics have emerged since

Samuelson (1976), e.g., Bullard and Butler (1993); Dixit and Pindyck (1994); Horst and

Wenzelburger (2008), econometrics has produced the bulk of the contributions.

Empirical estimation for the deterministic models of neoclassical economics initially proceeded with the addition of a stationary, usually Gaussian, error term to produce a discrete time general linear model (GLM) leading to estimation using ordinary least squares or maximum likelihood techniques. Iterations and extensions of the GLM to deal with complications arising in empirical estimates dominated early work in econometrics, e.g., Dhrymes (1974) and Theil (1971), leading to application of generalized least squares estimation techniques that encompassed autocorrelated and heteroskedastic error terms. Employing L_2 vector space methods with stationary error term distributions ensured these early stochastic models implicitly assumed ergodicity. The generalization of this discrete time estimation approach to the class of ARCH and GARCH error term models by Engle and Granger was of such significance that a Nobel memorial prize in economics was awarded for this contribution, e.g., Engle and Granger (1987). By modeling the evolution of the volatility, this approach permitted a limited degree of non-linearity to be modeled providing a substantively better fit to observed economic time series. Only recently has the ergodicity of the GARCH model and related methods been considered, e.g., Meitz and Saikkonen (2008).

The emergence of ARCH, GARCH and related models was part of a general trend toward the use of inductive methods in economics, often employing discrete, linear time series methods to model transformed economic variables, e.g., Hendry (1995). At least since Dickey and Fuller (1979), it has been recognized that estimates of univariate time series models for many economic times series reveals evidence of 'non-stationarity'. A number of approaches have emerged to deal with this apparent empirical quandary.[7] In particular, transformation techniques for time series models have received considerable attention. Extension of the Box-Jenkins methodology led to the concept of

economic time series being I(0) – stationary in the level – and I(1) – non-stationary in the level but stationary after first differencing. Two I(1) economic variables could be cointegrated if differencing the two series produced an I(0) process, e.g., Hendry (1995). Extending early work on distributed lags, long memory processes have also been employed where the time series is only subject to fractional differencing. Significantly, recent contributions on Markov switching processes and exponential smooth transition autoregressive processes have demonstrated the "possibility that nonlinear ergodic processes can be misinterpreted as unit root nonstationary processes" (Kapetanios and Shin 2011, p.620).

   The conventional view of ergodicity in economics is reflected by Hendry (1995, p.100): "Whether economic reality is an ergodic process after suitable transformation is a deep issue" which is difficult to analyze rigorously. As a consequence, in the limited number of instances where ergodicity is examined in economics a variety of different interpretations appear. In contrast, the ergodic hypothesis in statistical mechanics is associated with the more physically transparent kinetic gas model than the often technical and targeted concepts of ergodicity encountered in modern economics. For Boltzmann, the ergodic hypothesis permitted modeling the unobserved complex microscopic interactions of individual gas molecules that had to obey the second law of thermodynamics, a concept that has obscure application in economics.[8] Despite differences in physical interpretation, economics is also confronted with a similar problem of modeling 'macroscopic' economic variables, such as exchange rates or GNP or interest rates, when a theory that can usefully predict future empirical observations from known first principles about the (microscopic) rational behavior of individuals and firms is unavailable. By construction, such problems can be addressed using a phenomenological approach to theoretical modeling.[9]

Even though the formal solutions proposed were inadequate by standards of modern mathematics, the thermodynamic model introduced by Boltzmann to explain the dynamic properties of the Maxwell distribution is a pedagogically useful starting point to develop the implications of ergodicity in economics. To be sure, von Neumann (1932) and Birkhoff (1931) correctly specify ergodicity using Lebesque integration – an essential analytical tool unavailable to Boltzmann – but the analysis is too complex to be of much value to all but the most mathematically specialized economists. The physical intuition of the kinetic gas model is lost in the generality of the results. Using Boltzmann as a starting point, the large number of mechanical and complex molecular collisions could correspond to the large number of microscopic, 'atomistic' competitors and consumers interacting to determine the macroscopic market price.[10] In this context, it is variables such as the asset price or the interest rate or the exchange rate, or some combination, that is being measured over time and ergodicity would be associated with the properties of the transition density generating the macroscopic variables. Ergodicity can fail for a number of reasons and there is value in determining the source of the failure. In this vein, there are two fundamental difficulties associated with the ergodic hypothesis in Boltzmann's statistical mechanics – reversibility and recurrence – that have a rough similarity to notions available in econophysics but are largely unrecognized in mainstream economics.[11]

Halmos (1949, p.1017) is a helpful starting point to sort out the differing notions of ergodicity that arise in range of subjects: "The ergodic theorem is a statement about a space, a function and a transformation". In mathematical terms, ergodicity or 'metric transitivity' is a property of 'indecomposable', measure preserving transformations. Because the transformation acts on points in the space, there is a fundamental connection to the method of measuring relationships such as

distance or volume in the space. In von Neumann (1932) and Birkhoff (1931), this is accomplished using the notion of Lebesque measure: the admissible functions are either integrable (Birkhoff) or square integrable (von Neumann). In contrast to, say, statistical mechanics where spaces and functions account for the complex physical interaction of large numbers of particles, economic theory can usually specify the space in a mathematically convenient fashion. For example, in the case where there is a single random variable, then the space is "superfluous" (Mackey 1974, p.182) as the random variable is completely described by the distribution. Multiple random variables can be handled by assuming the random variables are discrete with finite state spaces. In effect, conditions for an 'invariant measure' are typically assumed in economics in order to focus attention on "finding and studying the invariant measures" (Arnold 1998, p.22) where, in the terminology of econometrics, the invariant measure usually corresponds to the stationary distribution or likelihood function.

The mean ergodic theorem of von Neumann (1932) provides an essential connection to the ergodicity hypothesis in econometrics. It is well known that, in the Hilbert and Banach spaces common to econometric work, the mean ergodic theorem corresponds to the strong law of large numbers. In statistical applications where strictly stationary distributions are assumed, the relevant ergodic transformation, L^*, is the unit shift operator: $L^* \Psi[x(t)] = \Psi[L^* x(t)] = \Psi[x(t+1)]$; $[(L^*)^k]$ $\Psi[x(t)] = \Psi[x(t+k)]$; and $\{(L^*)^{-k}\} \Psi[x(t)] = \Psi[x(t-k)]$ with k being an integer and $\Psi[x]$ the strictly stationary distribution for x that in the strictly stationary case is replicated at each t.[12] Significantly, this reversible transformation is independent of initial time and state. Only the distance between observations is relevant. Because this transformation imposes strict stationarity on $\Psi[x]$, L^* will only work for certain ergodic processes. The ergodic requirement that the transformation be measure

preserving is weaker than the strict stationarity of the stochastic process required for L^*. The implications of the reversible ergodic transformation L^* are central to the criticisms of neoclassical economic theory advanced by heterodox economists. e.g., Davidson (1991, p.331): "In an economic world governed entirely by ergodic processes ... economic relationships among variables are timeless, or ahistoric in the sense that the future is merely a statistical reflection of the past".[13] In effect, the use of ergodicity in economics requires that the real world distribution for $x(t)$ be sufficiently similar to those for both $x(t+k)$ or $x(t-k)$, i.e., the ergodic transformation L^* is reversible.

4. A Phenomenological Interpretation of Ergodicity

In physics, phenomenology lies at the intersection of theory and experiment. Theoretical relationships between empirical observations are modeled without deriving the theory directly from first principles, e.g., Newton's laws of motion. Predictions based on these theoretical relationships are obtained and compared to further experimental data designed to test the predictions. In this fashion, new theories derived from first principles are motivated. Confronted with non-experimental data for important economic variables, such as wage rates, stock prices, GNP, interest rates and the like, economics similarly develops theoretical models that aim to fit the 'stylized facts' of those variables. While comparison of 'stylized facts' with predictions of theories initially derived directly from the 'first principles' of constrained maximizing behavior of individuals and firms is recommended practice in economics, such theories generally have poor empirical performance. This has given impetus to the inductive approach in econometrics, an inherently phenomenological approach to theorizing in economics, e.g., Hendry (1995). Given the difficulties in economics of testing model predictions with 'new' experimental data, econophysics provides a variety of mathematical techniques that can possibly be adapted to determining mathematical relationships

among economic variables that better explain the 'stylized facts'.[14]

The evolution of economic theory from the deterministic models of neoclassical economics to more modern stochastic models has been incremental and disjointed. The preference for linear models of static equilibrium relationships has restricted the application of frameworks from econophysics that capture more complex non-linear dynamics, e.g., chaos theory; truncated Levy processes. Yet, important variables in economics have relatively innocuous sample paths compared to some types of variables encountered in physics. There is an impressive range of mathematical and statistical models that, seemingly, could be applied to almost any physical or economic situation. If the process can be verbalized, then a model can be specified. This begs questions such as: are there stochastic models – ergodic or otherwise – that capture the basic 'stylized facts' of observed economic data? Is the random instability in the observed sample paths identified in, say, financial time series, consistent with the *ex ante* stochastic bifurcation of an ergodic process, e.g., Chiarella et al. (2008)? In contrast to economics where unimodal distributions are used exclusively, econophysics can employ models where the associated *ex ante* stationary densities are multimodal and irreversible. In this case, the mean calculated from past values of a single, non-experimental *ex post* realization of the process is not necessarily informative about the mean for future values.

Boltzmann was concerned with demonstrating that the Maxwell distribution emerged in the limit as $t \to \infty$ for systems with large numbers of particles. The limiting process for t requires that the system run long enough that the initial conditions do not impact the long run equilibrium distribution. At the time, two fundamental criticisms were aimed at this general approach: reversibility and recurrence. In the context of economic time series, reversibility relates to the use of past values of the process to forecast future values.[15] Recurrence relates to the properties of the

long run average which involves the ability and length of time for an ergodic process to return to its stationary state. For Boltzmann, both these criticisms have roots in the difficulty of reconciling the second law of thermodynamics with the ergodicity hypothesis. The situation in economics is less constrained. More generally, Sturm-Liouville methods can be used to demonstrate that ergodicity requires the transition density of the process to be decomposable into the sum of a stationary density and a mean zero transient term that captures the impact of the initial condition of the system on the individual sample paths. In this context, irreversibility relates to properties of the stationary density and non-recurrence to the behavior of the transient term, e.g., Poitras (2011, ch.5).

Because the particle movements in a kinetic gas model are contained within an enclosed system, e.g., a vertical glass tube, classical Sturm-Liouville (S-L) methods can be applied to obtain solutions for the transition densities. These results for the distributional implications of imposing regular reflecting boundaries on diffusion processes are representative of the phenomenonological approach to random systems theory which: "studies qualitative changes of the densites of invariant measures of the Markov semigroup generated by random dynamical systems induced by stochastic differential equations" (Crauel et al. 1999, p.27).[16] Because the initial condition of the system is explicitly recognized, ergodicity in these models takes a different form than that associated with the reversible unit shift transformation applied to stationary densities typically adopted in economics. The ergodic transition densities are derived as solutions to the forward differential equation associated with one-dimensional diffusions. The transition densities contain a transient term that is dependent on the initial condition of the system and boundaries imposed on the state space. Irreversibility can be introduced by employing multi-modal stationary densities.

The distributional implications of boundary restrictions, derived by modeling the random variable

as a diffusion process subject to reflecting barriers, have been studied for many years, e.g., Feller (1954). The diffusion process framework is useful because it imposes a functional structure that is sufficient for known partial differential equation (PDE) solution procedures to be used to derive the relevant transition probability densities. Wong (1964) demonstrated that with appropriate specification of parameters in the PDE, the transition densities for popular stationary distributions such as the exponential, uniform, and normal distributions can be derived using S-L methods. This paper proposes that the introduction of the S-L framework by Boltzmann provides the historical starting point for econophysics. Though Boltzmann was not concerned with economic phenomena, the analytical approach employed provides sufficient generality to resolve certain empirical difficulties arising from key stylized facts in non-experimental economic time series. In turn, the framework provides an alternative method to neoclassical economic theory that encompasses the nonlinear dynamics of diffusion processes. In other words, the mathematical framework of classical statistical mechanics permits econophysics to reformulate and clarify the ergodicity assumption thereby obtaining an alternative, more general stochastic approach to economic theory.[17]

The use of the diffusion model to represent the nonlinear dynamics of stochastic processes is found in a wide range of subjects. Restrictions such as the rate of observed genetic mutation in biology or character of heat diffusion in engineering or physics often determine the specific formalization of the diffusion model. Because physical interactions can be complex, mathematical results for diffusion models are pitched at a level of generality sufficient to cover such cases.[18] Such generality is usually not required in economics. As a consequence, where stochastic processes are employed, economic theory typically employs mean-reverting OU processes or geometric Brownian motion. Using theoretical methods from econophysics, it is possible to exploit mathematical

properties of bounded state spaces and one dimensional diffusions to overcome certain analytical and empirical problems that can confront such continuous time Markov solutions. The key construct in the S-L method is the ergodic transition probability density function U which is associated with the random (economic) variable x at time t ($U = U[x, t \mid x_0]$) that follows a regular, time homogeneous diffusion process. While it is possible to allow the state space to be an infinite open interval $I_o = (a,b: \infty \leq a < b \leq \infty)$, a finite closed interval $I_c = [a,b: -\infty < a < b < +\infty]$ or the specific interval $I_s = [0 = a < b < \infty)$ are applicable to economic variables.[19] Assuming that U is twice continuously differentiable in x and once in t and vanishes outside the relevant interval, then U obeys the forward equation (e.g., Gihhman and Skorohod 1979, p.102-4):

$$\frac{\partial^2}{\partial x^2} \{ B[x]\ U \} - \frac{\partial}{\partial x} \{A[x]\ U\} = \frac{\partial U}{\partial t} \qquad (1)$$

where: $B[x]$ ($= \frac{1}{2}\sigma^2[x] > 0$) is the one half the infinitesimal variance and $A[x]$ the infinitesimal drift of the process. $B[x]$ is assumed to be twice and $A[x]$ once continuously differentiable in x. Being time homogeneous, this formulation permits state, but not time, variation in the drift and variance parameters.

If the diffusion process is subject to upper and lower reflecting boundaries that are regular and fixed ($-\infty < a < b < \infty$), the "Sturm-Liouville problem" involves solving (1) subject to the separated boundary conditions:[20]

$$\frac{\partial}{\partial x} \{ B[x]\ U[x,t] \}\big|_{x=a} - A[a]\ U[a,t] = 0 \qquad (3)$$

$$\frac{\partial}{\partial x} \{ B[x]\ U[x,t] \}\big|_{x=b} - A[b]\ U[b,t] = 0 \qquad (4)$$

And the initial condition:

$$U[x,0] = f[x_0] \qquad where: \qquad \int_a^b f[x_0] = 1 \qquad (5)$$

and $f[x_0]$ is the continuous density function associated with x_0 where $a \le x_0 \le b$. When the initial

starting value, x_0, is known with certainty, the initial condition becomes the Dirac delta function,

$U[x,0] = \delta[x - x_0]$, and the resulting solution for U is referred to as the 'principal solution'. Within

the framework of the S-L method, a stochastic process has **the ergodic property** when the transition

density satisfies:[21]

$$\lim_{t \to \infty} U[x,t \mid x_0] = \int_a^b f[x_0] \quad U[x,t \mid x_0] \ dx_0 = \Psi[x]$$

Important special cases occur for the principal solution ($f[x_0] = \delta[x - x_0]$) and when $f[x_0]$ is from a

specific class such as the Pearson distributions. To be ergodic, the time invariant stationary density

$\Psi[x]$ is not permitted to 'decompose' the sample space with a finite number of indecomposable sub-

densities, each of which is time invariant. Such irreversible processes are not ergodic, even though

each of the sub-densities could be restricted to obey the ergodic theorem. To achieve ergodicity, a

multi-modal stationary density can be used instead of decomposing the sample space using

indecomposable sub-densities with different means.[22] In turn, multi-modal irreversible ergodic

processes have the property that the mean calculated from past values of an *ex post* sample path for

the process are not sufficiently informative about the modes of the *ex ante* densities to provide

accurate predictions.

In order to more accurately capture the *ex ante* properties of economic time series, there are some

potentially restrictive features in the S-L framework that can be identified. For example, time

homogeneity of the process eliminates the need to explicitly consider the location of t_0.[23] Time

homogeneity, as such, is consistent with 'ahistorical' neoclassical economic theorizing which

corresponds to a sub-class of U transition densities (U^*) that have a time homogeneous and

reversible stationary distribution governing the dynamics of $x(t)$. Significantly, while U is time homogeneous, there are some U consistent with irreversible processes. In order to provide a distinct alternative to stochastic models evolving from neoclassical economics, a relevant issue for econophysics is to determine which concept – time homogeneity or reversibility – is inconsistent with economic processes that capture: ratchet effects in wages; liquidity traps in money markets; structural shifts; and, collapsing conventions in asset markets. In the S-L framework, the initial state of the system (x_0) is known and the ergodic transition density provides information about how a given point x_0 shifts t units along a trajectory. For econometric applications employing the strictly stationary distributions associated with U^*, the location of x_0 is irrelevant while the wider class of transition densities, U, available in econophysics can incorporate x_0 as an initial condition associated with the solution of a partial differential equation.

5. Density Decomposition Results[24]

The theoretical advantage obtained by imposing regular reflecting barriers on the diffusion state space for the forward equation is that an ergodic decomposition of the transition density is assured. The relevance of bounding the state space and imposing regular reflecting boundaries can be illustrated by considering the well known solution (e.g., Cox and Miller 1965, p.209) for U involving a constant coefficient standard normal variate $Y(t) = (\{x - x_0 - \mu t\}/\sigma)$ over the unbounded state space $I_o = (\infty \le x \le \infty)$. In this case the forward equation (1) reduces to: $\frac{1}{2}\{\partial^2 U / \partial Y^2\} = \partial U / \partial t$. By evaluating these derivatives, it can be verified that the principal solution for U is:

$$U[x,t \mid x_0] = \frac{1}{\sigma \sqrt{(2\pi t)}} \exp\left[-\frac{(x - x_0 - \mu t)^2}{2\sigma^2 t} \right]$$

and as $t \to -\infty$ or $t \to +\infty$ then $U \to 0$ and the stochastic process is nonergodic because it does not

possess a non-trivial stationary distribution. The mean ergodic theorem fails: if the process runs long enough, then U will evolve to where there is no discernible probability associated with starting from x_0 and reaching the neighborhood of a given point x. The absence of a stationary distribution is a property of a unit root process. Various confusions surrounding the debate over unit root processes in economics can be resolved in econophysics where a mathematical structure can be employed that ensures an ergodic distribution.

Though it has been recognized in econometrics that imposing regular reflecting boundaries is a certain method of obtaining a stationary distribution and a discrete spectrum (Hansen and Schienkman 1998, p.13), it is difficult for stochastic models aimed at testing neoclassical economic theory to transition to a world where initial and boundary conditions are integral to economic theorizing. Such an approach to economic theory is available in econophysics. Within the S-L framework, econophysics is able to explore alternative stochastic model structures, such as specifying the process to admit natural boundaries where the parameters of the diffusion are zero within the state space, that can give rise to continuous spectrum and raise significant analytical complexities. At least since Feller (1954), the search for useful solutions, including those for singular diffusion problems, has produced a number of specific cases of interest. However, without the analytical certainty of the S-L framework, analysis proceeds on a case by case basis. As such, the historical demarcation between economics and econophysics can be traced to the introduction of S-L methods to physics by Boltzmann.

For a variety of reasons, there is a reluctance to impose boundaries on the 'economic state space'. As a consequence, where stochastic models associated with S-L methods are employed in economics, a stationary distribution is obtained without imposing both upper and lower boundaries.

This can be accomplished by imposing only a lower (upper) reflecting barrier and constructing the stochastic process such that positive (negative) infinity is non-attracting, e.g., Linetsky (2005); Aït-Sahalia (1999). In economics, this is often achieved by using a mean-reverting drift term. In contrast, Cox and Miller (1965, p.223-5) use the Brownian motion, constant coefficient forward equation with $x_0 > 0$, $A[x] = \mu < 0$ and $B[x] = \frac{1}{2}\sigma^2$ subject to the lower reflecting barrier at $x = 0$ given in (2) to solve for both the U and the stationary density. The principal solution is solved using the 'method of images' to obtain:

$$U[x,t \mid x_0] = \frac{1}{\sigma\sqrt{2\pi t}}\left\{\exp-\left(\frac{(x - x_0 - \mu t)^2}{2\sigma^2 t}\right) + \exp-\left(\frac{4x_0\mu t - (x - x_0 - \mu t)^2}{2\sigma^2 t}\right)\right\}$$

$$+ \frac{1}{\sigma\sqrt{2\pi t}}\left\{\frac{2\mu}{\sigma^2}\exp\left(\frac{2\mu x}{\sigma^2}\right)\left(1 - N\left[\frac{x + x_0 + \mu t}{\sigma\sqrt{t}}\right]\right)\right\}$$

where $N[x]$ is again the cumulative standard normal distribution function. Observing that $A[x] = \mu > 0$ again produces $U \to 0$ as $t \to +\infty$, the stationary density for $A[x] = \mu < 0$ has the Maxwell form:

$$\Psi[x] = \frac{2|\mu|}{\sigma^2}\exp-\left(\frac{2|\mu|x}{\sigma^2}\right)$$

Though x_0 does not enter the solution, combined with the location of the boundary at $x = 0$, it does implicitly impose the restriction $x > 0$.

Following Linetsky (2005), Veerstraeten (2004) and others, the analytical procedure used to determine U involves specifying the parameters of the forward equation and the boundary conditions and then solving for the stationary distribution $\Psi[x]$ and transient and boundary related term $T[x,t \mid x_0]$. Wong (1964) uses a different approach, initially selecting a stationary distribution and then solving for U using the restrictions of the Pearson system to specify the forward equation. In this

approach, the functional form of the desired stationary distribution determines the appropriate

boundary conditions. While application of this approach has been limited to the restricted class of

distributions associated with the Pearson system, it is expedient when a known stationary

distribution, such as the standard normal distribution, is of interest. More precisely, let:

$$\Psi[x] \;=\; \frac{1}{\sqrt{2\pi}} \; \exp\!\left[-\frac{x^2}{2}\right], \qquad I_o \;=\; (-\infty < x < \infty)$$

In this case, the boundaries of the state space are non-attracting and not regular. Solving the Pearson

equation gives: $d\Psi[x]/dx = -x\,\Psi[x]$ and a forward equation of the OU form:

$$\frac{\partial^2 U}{\partial x^2} \;+\; \frac{\partial}{\partial x}xU \;=\; \frac{\partial U}{\partial t}$$

Following Wong (1964, p.268) Mehler's formula can be used to express the solution for U as:

$$U[x,t \mid x_0] \;=\; \frac{1}{\sqrt{2\pi(1 - e^{-2t})}} \; \exp\!\left[\frac{-(x - x_0\, e^{-t})^2}{2(1 - e^{-2t})}\right]$$

Given this, as $t \to -\infty$ then $U \to 0$ and as $t \to +\infty$ then U achieves the stationary standard normal

distribution.

6. The Quartic Exponential Distribution

A subtle implication of 'Mirowski's thesis' is that the constrained optimization mathematics used

by neoclassical economics to model the behavior of economic phenomena has had significant

implications for the subsequent development of economic theory. Being derived from the physics

of 19th century rational mechanics, stochastic generalization of the static and deterministic solutions

provided by neoclassical economics has not captured developments in physics that trace back to

Boltzmann. Attention typically focuses on estimating parameters for unimodal stationary

distributions or likelihood functions of appropriately transformed economic variables. For example,

financial economics is fundamentally concerned with the tradeoff between 'risk' and 'return' associated with the unimodal distribution for the rate of change in security or asset prices. In contrast, mathematical methods available in econophysics can deal with bifurcating, multi-modal processes where the concepts of 'risk' and 'return' have less meaning. Despite having potential to explain a number of 'stylized facts' regarding the random instability in the time paths for, say, security prices, the mathematics of bifurcation methods have received little attention in economic theory.

The roots of bifurcation theory can be found in the early solutions to certain deterministic ordinary differential equations. Consider the deterministic dynamics described by the pitchfork bifurcation ODE:

$$\frac{dx}{dt} = -x^3 + \rho_1 x + \rho_0$$

where ρ_0 and ρ_1 are the 'normal' and 'splitting' control variables, respectively (e.g., Cobb 1978, 1981). While ρ_0 has significant information in a stochastic context, this is not usually the case in the deterministic problem so $\rho_0 = 0$ is assumed. Given this, for $\rho_1 \leq 0$, there is one real equilibrium ($\{dx / dt\} = 0$) solution to this ODE at $x = 0$ where "all initial conditions converge to the same final point exponentially fast with time" (Crauel and Flandoli 1998, p.260). For $\rho_1 > 0$, the solution bifurcates into three equilibrium solutions $x = \{0, \pm\sqrt{\rho_1}\}$, one unstable and two stable. In this case, the state space is split into two physically distinct regions (at $x = 0$) with the degree of splitting controlled by the size of ρ_1. Even for initial conditions that are 'close', the equilibrium achieved will depend on the sign of the initial condition. Stochastic bifurcation theory extends this model to incorporate Markovian randomness. In this theory, "invariant measures are the random analogues of deterministic fixed points" (Arnold 1998, p.469). Significantly, ergodicity now requires that the

component densities that bifurcate out of the stationary density at the bifurcation point be invariant measures, e.g., Crauel et al. (1999, sec.3). As such, the ergodic bifurcating process is irreversible in the sense that past sample paths (prior to the bifurcation) cannot reliably be used to generate statistics for the future values of the variable (after the bifurcation).

It is well known that the introduction of randomness to the pitchfork ODE changes the properties of the equilibrium solution, e.g., (Arnold 1998, sec.9.2). It is no longer necessary that the state space for the principal solution be determined by the location of the initial condition relative to the bifurcation point. The possibility for randomness to cause some paths to cross over the bifurcation point depends on the size of volatility of the process, σ, which measures the non-linear signal to white noise ratio. Of the different approaches to introducing randomness (e.g., multiplicative noise), the simplest approach to converting from a deterministic to a stochastic context is to add a Weiner process ($dW(t)$) to the ODE. Augmenting the diffusion equation to allow for σ to control the relative impact of non-linear drift versus random noise produces the "pitchfork bifurcation with additive noise" (Arnold 1998, p.475) which in symmetric form is:

$$dX(t) = (\rho_1\ X(t) - X(t)^3)\ dt + \sigma\ dW(t)$$

In economic applications, e.g., Aït-Sahlia (1999), this diffusion process is referred to as the double well process. While consistent with the common use of diffusion equations in econometrics and other areas of mainstream economics, the dynamics of the pitchfork process captured by $T[x,t\ |x_0]$ have been "forgotten" (Arnold 1998, p.473).

Mainstream economics is married to the transition probability densities associated with unimodal stationary distributions. More flexibility in the shape of the stationary distribution can be achieved using a higher order exponential density, e.g., Fisher (1921), Cobb et al. (1983), Caudel and Flandoli

(1998). Increasing the degree of the polynomial in the exponential comes at the expense of introducing additional parameters resulting in a substantial increase in the analytical complexity, typically defying a closed form solution for the transition densities. However, at least since Elliott (1995), it has been recognized that the solution of the associated regular S-L problem will still have a discrete spectrum, even if the specific form of the eigenfunctions and eigenvalues in $T[x,t \,|x_0]$ are not precisely determined (Horsthemke and Lefever 1984, sec. 6.7) . Inferences about transient stochastic behavior can be obtained by examining the solution of the deterministic non-linear dynamics. In this process, attention initially focuses on the properties of the higher order exponential distributions.

To this end, assume that the stationary distribution is a fourth degree or "general quartic" exponential:

$$\Psi[x] \; = \; K \, \exp[-\Phi[x]] \; = \; K \, \exp[-(\beta_4 \, x^4 \, + \, \beta_3 \, x^3 \, + \, \beta_2 \, x^2 \, + \, \beta_1 \, x)]$$

where: K is a constant determined such that the density integrates to one; and, $\beta_4 > 0$.[25] Following Fisher (1921), the class of distributions associated with the general quartic exponential admits both unimodal and bimodal densities and nests the standard normal as a limiting case where $\beta_4 = \beta_3 = \beta_1 = 0$ and $\beta_2 = \frac{1}{2}$ with $K = 1/(\sqrt{2\pi})$. The stationary distribution of the bifurcating double well process is a special case of the symmetric quartic exponential distribution:

$$\Psi[y] \; = \; K_S \, \exp[-\{\beta_2 \, (x \, - \, \mu)^2 \, + \, \beta_4 \, (x \, - \, \mu)^4\}] \qquad where \quad \beta_4 \, \geq \, 0$$

where μ is the population mean and the symmetry restriction requires $\beta_1 = \beta_3 = 0$. Such multi-modal stationary densities have received scant attention in mainstream economics. To see why the condition on β_1 is needed, consider change of origin $X = Y - \{\beta_3 \,/\, 4\,\beta_4\}$ to remove the cubic term from the general quartic exponential (Matz 1978, p.480):

$$\Psi[y] = K_Q \exp[-\{\kappa \, (y - \mu_y) + \alpha \, (y - \mu_y)^2 + \gamma \, (y - \mu_y)^4\}] \qquad \textit{where} \quad \gamma \geq 0$$

The substitution of y for x indicates the change of origin which produces the following relations

between coefficients for the general and specific cases:

$$\kappa = \frac{8\beta_1\beta_4{}^2 - 4\beta_2\beta_3\beta_4 + \beta_3{}^3}{8\beta_4{}^2} \qquad \alpha = \frac{8\beta_2\beta_4 - 3\beta_3{}^2}{8\beta_4} \qquad \gamma = \beta_4$$

The symmetry restriction $\kappa = 0$ can only be satisfied if both β_3 and $\beta_1 = 0$. Given the symmetry

restriction, the double well process further requires $-\alpha = \gamma = \sigma = 1$. Solving for the modes of $\Psi[y]$

gives $\pm \sqrt{\{|\alpha| / (2\gamma)\}}$ which reduces to ± 1 for the double well process, as in Ait-Sahlia (1999, Figure

6B, p.1385).

INSERT FIGURE 1 HERE

As illustrated in Figure 1, the selection of a_i in the stationary density $\Psi_i[x] = K_Q \, exp\{ -(.25 \, x^4 - .5$

$x^2 - a_i x) \}$ defines a family of general quartic exponential densities, where a_i is the selected value of

κ for that specific density.[26] The coefficient restrictions on the parameters α and γ dictate that these

values cannot be determined arbitrarily. For example, given that β_4 is set at .25, then for $a_i = 0$, it

follows that $\alpha = \beta_2 = 0.5$. 'Slicing across' the surface in Figure 1 at $a_i = 0$ reveals a stationary

distribution that is equal to the double well density. Continuing to slice across as a_i increases in size,

the bimodal density becomes progressively more asymmetrically concentrated in positive x values.

Though the location of the modes does not change, the amount of density between the modes and

around the negative mode decreases. Similarly, as a_i decreases in size the bimodal density becomes

more asymmetrically concentrated in positive x values. While the stationary density is bimodal over

$a_i \, \varepsilon \, \{-1,1\}$, for $|a_i|$ large enough the density becomes so asymmetric that only a unimodal density

appears. For the general quartic, asymmetry arises as the amount of the density surrounding each

mode (the sub-density) changes with a_i. In this, the individual stationary sub-densities have a symmetric shape. To introduce asymmetry in the sub-densities, the reflecting boundaries at a and b that bound the state space for the regular S-L problem can be used to introduce positive asymmetry in the lower sub-density and negative asymmetry in the upper sub-density.

Following Chiarella et al. (2008), the stochastic bifurcation process has a number of features which are consistent with the *ex ante* behavior of a securities market driven by a combination of chartists and fundamentalists. In particular, because the stationary distributions are multi-modal and depend on forward parameters – such as κ, α, γ and a_i in Figure 1 – that are not known on the decision date, the rational expectations models employed in mainstream economics are uninformative. What use is the forecast provided by $E[x(T)]$ when it is known that there are other $x(T)$ values that are more likely to occur? A mean estimate that is close to the bifurcation point would even be unstable. In a multi-modal world, complete fundamental uncertainty – where nothing is known about the evolution of economic variables – is replaced by uncertainty over unknown parameter values that can change due, say, to the collapse of a market convention. The associated difficulty of calculating a mean value forecast or other econometric estimates from past data is compounded by the presence of transients that originate from boundaries and initial conditions. For example, the presence of a recent structural break can be accounted for by appropriate selection of x_0. Of particular relevance to a comparison of econophysics theories with theories arising in mainstream financial economics is the fundamental dependence of investment decisions on x_0 which is not captured by the reversible ergodic processes employed in mainstream economics. The theoretical tools available in econophysics are able to demonstrate this fundamental dependence by exploiting properties of *ex ante* bifurcating ergodic processes to generate *ex post* sample paths that provide a better

approximation to the sample paths of observed financial data.

7. Conclusion

The introduction of the ergodicity hypothesis by Ludwig Boltzmann provides a point of demarcation between the histories of neoclassical economic theory and econophysics. Why begin the history of econophysics with Boltzmann whose concern was only with the kinetic theory of gases and not economic phenomena? This paper demonstrates that the mathematical techniques available in econophysics that are distinct from those derived from neoclassical economics start with the S-L methods pioneered by Boltzmann. From this point, methods available to econophysics differ substantively from the static and deterministic models of rational mechanics that underpin neoclassical economics. As such, the direct use of ergodic processes in econophysics provides a distinct approach to understanding and predicting the behavior of economic phenomena. For example, whereas economics employs only unimodal processes that ignore the impact of transient terms, econophysics is able to model economic observations that are generated by bifurcating ergodic processes. Such models are relevant for non-experimental data where the calculation of time averages based on a sufficiently long enough *ex post* sample path can not be expected to provide a statistically reliable estimate of the *ex ante* time or space average for a future time paths that may be observed over a sufficiently distant future calendar time.

To deal with the problem of making statistical inferences from 'non-experimental' data, stochastic generalizations of neoclassical economic theory typically employ stationary densities that are: reversible; unimodal; and, where initial and boundary conditions have no short or long term impact. The possibility of irreversible ergodic processes is not recognized or, it seems, intended. Significantly, a type of fundamental uncertainty is inherent in bifurcating processes. as illustrated

in the need to select an a_i in Figure 1 in order to determine the *ex ante* stationary density. A semantic connection can be established between the subjective uncertainty about encountering a future bifurcation point and, say, the possible collapse of an asset price bubble due to a change in Keynesian convention about market valuations. From the early results of Boltzmann in classical statistical mechanics to the fractals of Mandelbrot, econophysics provides a variety of potential stochastic approaches to modeling economic phenomena that may eventually prove to be more insightful than the stochastic models employed in testing the deterministic and static predictions of neoclassical economic theory.

Figure 1:* **Family of Stationary Densities for $\Psi_i[x] = K_Q\, exp\{ -(.25\, x^4 - .5\, x^2 - a_i\, x) \}$**
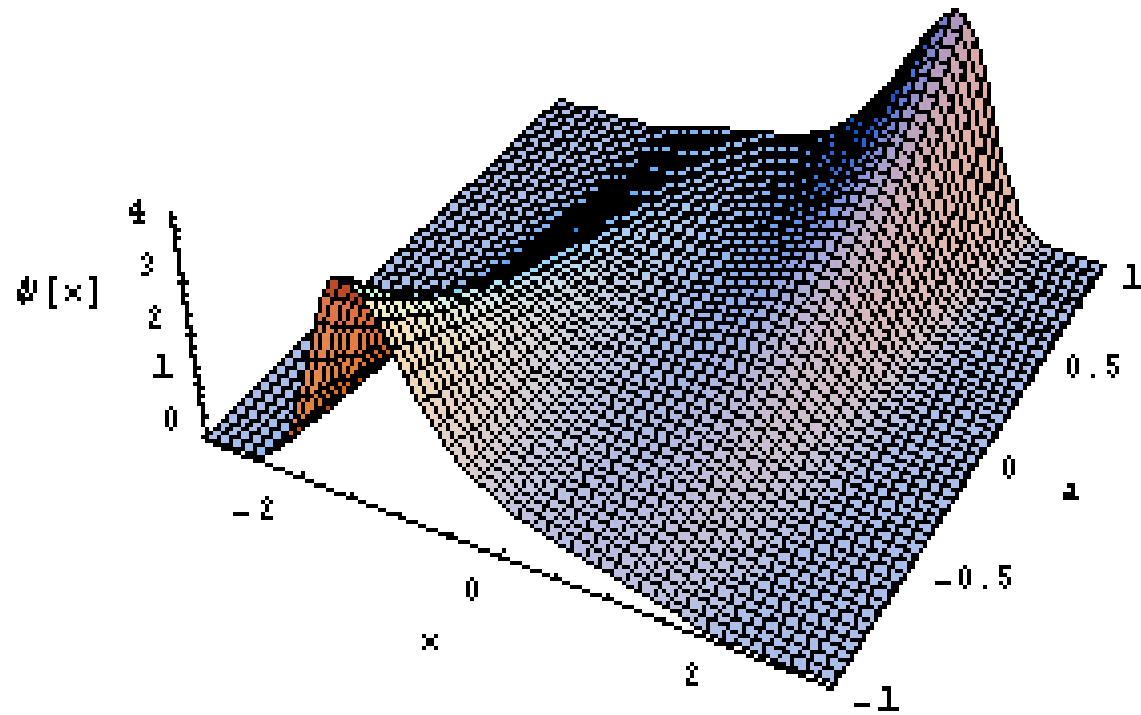
* Each of the continuous values for *a* signifies a different stationary density. For example, at $a = 0$ the density is the double well density which symmetric about zero and with modes at ±1.

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NOTES

1. The 'Mirowski thesis' has been contentious, for a variety of reasons, e.g., Walker (1991); Varian (1991). Included in these reasons are a tendency to overstate the position and a lack of attention to specific detail. For example, Mirowski identifies the beginnings of neoclassical economics with the energistics approach to physics. From a history of physics perspective, this is not technically correct as energistics did not emerge in full force until after the 1870's when the marginalist revolution commenced.   More precisely, the marginalists were more influenced by results from rational mechanics which is a key component of the energistics approach. Cercignani (1998, p.202-9) discusses the connection between Boltzmann and the energetists of that time.

2. Nolte (2010, p.36-7) observes that the attribution of the 'Liouville theorem' to Joseph Liouville (1809-1882) originates with an 1872 paper by Boltzmann. However, there was considerable distance to travel from the 1838 paper by Liouville to the contributions by Boltzmann. "Ironically, by naming it 'Liouville's theorem', Boltzmann obscured his own role in the discovery".

3. In rational mechanics, once the initial positions of the particles of interest, e.g., molecules, are known, the mechanical model fully determines the future evolution of the system. This scientific and philosophical approach is often referred to as Laplacian determinism.

4. Boltzmann and Max Planck were vociferous opponents of energetics. The debate over energetics was part of a larger intellectual debate concerning determinism and reversibility. Jevons (1877, p.738-9) reflects the entrenched determinist position of the marginalists: "We may safely accept as a satisfactory scientific hypothesis the doctrine so grandly put forth by Laplace, who asserted that a perfect knowledge of the universe, as it existed at any given moment, would give a perfect knowledge of what was to happen thenceforth and for ever after. Scientific inference is impossible, unless we may regard the present as the outcome of what is past, and the cause of what is to come. To the view of perfect intelligence nothing is uncertain." What Boltzmann, Planck and others had observed in statistical physics was that, even though the behavior of one or two molecules can be completely determined, it is not possible to generalize these mechanics to the describe the macroscopic motion of molecules in large, complex systems, e.g., Brush (1983, esp. ch.II).

5. As such, Boltzmann was part of the larger: "Second Scientific Revolution, associated with the theories of Darwin, Maxwell, Planck, Einstein, Heisenberg and Schrödinger, (which) substituted a world of process and chance whose ultimate philosophical meaning still remains obscure" (Brush

(1983, p.79). This revolution superceded the: "First Scientific Revolution, dominated by the physical astronomy of Copernicus, Kepler, Galileo, and Newton, ... in which all changes are cyclic and all motions are in principle determined by causal laws." The irreversibility and indeterminism of the Second Scientific Revolution replaces the reversibility and determinism of the First.

6. There are many interesting sources on these points which provide citations for the historical papers that are being discussed. Cercignani (1988, p.146-50) discusses the role of Maxwell and Boltzmann in the development of the ergodic hypothesis. Maxwell (1867) is identified as "perhaps the strongest statement in favour of the ergodic hypothesis". Brush (1976) has a detailed account of the development of the ergodic hypothesis. Gallavotti (1995) traces the etymology of "ergodic" to the 'ergode' in an 1884 paper by Boltzmann. More precisely, an ergode is shorthand for 'ergomonode' which is a 'monode with given energy' where a 'monode' can be either a single stationary distribution taken as an ensemble or a collection of such stationary distributions with some defined parameterization. The specific use is clear from the context. Boltzmann proved that an ergode is an equilibrium ensemble and, as such, provides a mechanical model consistent with the second law of thermodynamics. It is generally recognized that the modern usage of 'the ergodic hypothesis' originates with Ehrenfest (1911).

7. Kapetanios and Shin (2011, p.620) capture the essence of this quandary: "Interest in the interface of nonstationarity and nonlinearity has been increasing in the econometric literature. The motivation for this development may be traced to the perceived possibility that nonlinear ergodic processes can be misinterpreted as unit root nonstationary processes. Furthermore, the inability of standard unit root tests to reject the null hypothesis of unit root for a large number of macroeconomic variables, which are supposed to be stationary according to economic theory, is another reason behind the increased interest."

8. The second law of thermodynamics is the universal law of increasing entropy – a measure of the randomness of molecular motion and the loss of energy to do work. First recognized in the early 19th century, the second law maintains that the entropy of an isolated system, not in equilibrium, will necessarily tend to increase over time. Entropy approaches a maximum value at thermal equilibrium. A number of attempts have been made to apply the entropy of information to problems in economics, with mixed success. In addition to the second law, physics now recognizes the zeroth law of thermodynamics that "any system approaches an equilibrium state" (Reed and Simon [60, p.54]). This implications of the second law for theories in economics was initially explored by Georgescu-Roegen (1971).

9. In this process, the ergodicity hypothesis is required to permit the one observed sample path to be used to estimate the parameters for the *ex ante* distribution of the ensemble paths. In turn, these parameters are used to predict future values of the economic variable.

10. This interpretation of the microscopic collisions differs from Davidson (1988, p.332): "If there is only one actual economy, and we do not possess, never have possessed and conceptually never will possess an *ensemble* of economics worlds, then even a definition of probability distribution functions is questionable." In this context, points in the phase space at time *t* represent individual

realizations of different macroscopic outcomes for the economic system at *t*. This interpretation of the ensembles is closer to Gibbs than Maxwell.

11. Heterodox critiques are associated with views considered to originate from within economics. Such critiques are seen to be made by 'economists', e.g., Post Keynesian economists, institutional economists, radical political economists and so on. Because such critiques take motivation from the theories of mainstream economics, these critiques are distinct from econophysics. Following Schinckus (2010, p.3818): "Econophysicists have then allies within economics with whom they should become acquainted."

12. Dhyrmes (1971, p.1-29) discusses the algebra of the lag operator.

13. Critiques of mainstream economics that are rooted in the insights of *The General Theory* recognize the distinction between fundamental uncertainty and objective probability. As a consequence, the definition of ergodic theory in heterodox criticisms of mainstream economics lacks formal precision, e.g., the short term dependence of ergodic processes on initial conditions is not usually recognized. Ergodic theory is implicitly seen as another piece of the mathematical formalism inspired by Hilbert and Bourbaki and captured in the Arrow-Debreu general equilibrium model of mainstream economics.

14. In this context though not in all contexts, econophysics provides a 'macroscopic' approach. In turn, ergodicity is an assumption that permits the time average from a single observed sample path to (phenomenologically) model the ensemble of sample paths. Given this, econophysics does contain a substantively richer toolkit that encompasses both ergodic and non-ergodic processes. Many works in econophysics implicitly assume ergodicity and develop models based on that assumption.

15. The connection of the reversibility and recurrence concepts used in this paper with the actual arguments made during the Boltzmann debates is somewhat tenuous. For example, the assumption that the diffusion process is regular deals with the version of the recurrence problem that concerned Boltzmann. The objective of introducing these concepts is pedagogy rather than historical accuracy.

16. The distinction between invariant and ergodic measures is fundamental. Recognizing a number of distinct definitions of ergodicity are available, following Medio (2005, p.70) the Birkhoff-Khinchin ergodic (BK) theorem for invariant measures can be used to demonstrate that ergodic measures are a class of invariant measures. More precisely, the BK theorem permits the limit of the time average to depend on initial conditions. In effect, the invariant measure is permitted to decompose into invariant 'sub-measures'. The physical interpretation of this restriction is that sample paths starting from a particular initial condition may only be able to access a part of the sample space, no matter how long the process is allowed to run. For an ergodic process, sample paths starting from any admissible initial condition will be able to 'fill the sample space', i.e., if the process is allowed to run long enough, the time average will not depend on the initial condition. Medio [2005, p.73] provides a useful example of an invariant measure that is not ergodic.

17. This is not intended to imply that classical statistical mechanics is the only method available to econophysics. Quite the contrary, there are a range of theoretical approaches employed in physics that have potential application in economics. For example, truncated Levy processes have been suggested as a method of dealing with the infinite variance aspect of 'stable' processes. However, such processes typically are associated with unimodal stationary densities and, as such, represent a distinct approach from that available using the S-L methods of classical statistical mechanics.

18. The phenomenological approach is not without difficulties. For example, the restriction to Markov processes ignores the possibility of invariant measures that are not Markov. In addition, an important analytical construct in bifurcation theory, the Lyapunov exponent, can encounter difficulties with certain invariant Markov measures. Primary concern with the properties of the stationary distribution is not well suited to analysis of the dynamic paths around a bifurcation point. And so it goes.

19. A diffusion process is 'regular' if starting from any point in the state space *I*, any other point in *I* can be reached with positive probability (Karlin and Taylor 1981, p.158). This condition is distinct from other definitions of regular that will be introduced: 'regular boundary conditions' and 'regular S-L problem'.

20. The classification of boundary conditions is typically an important issue in the study of solutions to the forward equation. Important types of boundaries include: regular; exit; entrance; and natural. Also important in boundary classification are: the properties of attainable and unattainable; whether the boundary is attracting or non-attracting; and whether the boundary is reflecting or absorbing. In the present context, regular, attainable, reflecting boundaries are usually being considered, with a few specific extensions to other types of boundaries. In general, the specification of boundary conditions is essential in determining whether a given PDE is self-adjoint

21. Heuristically, if the ergodic process runs long enough, then the stationary distribution can be used to estimate the constant mean value. This definition of ergodic is appropriate for the one-dimensional diffusion cases considered in this paper. Other combinations of transformation, space and function will produce different requirements. Various theoretical results are available for the case at hand. For example, the existence of an invariant Markov measure and exponential decay of the autocorrelation function are both assured.

22. An alternative approach to using multimodal densities is to use a mixture of distributions. This approach will typically involve the estimation of more parameters than in the multimodal case. The study of mixing distributions is well developed in physics and related subjects, e.g., Mengersen (2011).

23.   For ease of notation it is assumed that $t_0 = 0$. In practice, solving (1) combined with (3)-(5) requires *a* and *b* to be specified. While *a* and *b* have ready interpretations in physical applications, e.g., the heat flow in an insulated bar, determining these values in economic applications can be more challenging. Some situations, such as the determination of the distribution of an exchange rate subject to control bands (e.g., Ball and Roma 1988), are relatively straight forward. Other situations, such as profit distributions with arbitrage boundaries or output distributions subject to production

possibility frontiers, may require the basic S-L framework to be adapted to the specifics of the modeling situation.

24. A more detailed mathematical treatment can be found in de Jong (1994).

25. In what follows, except where otherwise stated, it is assumed that $\sigma = 1$. Hence, the condition that K be a constant such that the density integrates to one incorporates the $\sigma = 1$ assumption. Allowing $\sigma \neq 1$ will scale either the value of K or the β's from that stated.

26. A number of simplifications were used to produce the 3D image in Figure 1: x has been centered about μ; and, $\sigma = K_Q = 1$. Changing these values will impact the specific size of the parameter values for a given x but will not change the general appearance of the density plots.