7/25/92

Distributions for Diffusions Subject to Constant Reflecting Barriers: A Decomposition Result

Submitted by:

John Heaney and Geoffrey Poitras

Faculty of Business Administration
Simon Fraser University
Burnaby B.C.
CANADA V5A 1S6

604-291-4485

ABSTRACT

This paper presents a decomposition result for the transition probability density of a one-dimensional diffusion process subject to upper and lower constant reflecting barriers. The decomposition divides the density into a limiting equilibrium (ergodic) density which is time independent and a power series of time and boundary dependent transient terms. The results are derived using both the classical Sturm-Liouville approach and operator semigroups.

Keywords: Diffusions, Reflecting barriers, Operator semigroups

Acknowledgment:

The comments of the referee significantly improved the paper. This paper was completed while G. Poitras was a visiting Senior Fellow at the National University of Singapore. Financial support was provided, in part, by the SSHRC.

Distributions for Diffusions Subject to Constant Reflecting Barriers: A Decomposition Result

1. Introduction

While the distributional implications of imposing boundary restrictions on diffusion processes have long been recognized (e.g., Feller [7]), direct connections between specific types of boundary restrictions and the associated distributions are available only in special cases (e.g., Giorno, et.al. [8], [9], Ricciardi and Sacerdote [18], Sacerdote [19], Buonocore, et.al. [4], Nobile, et.al. [16]). This is in striking contrast to the numerous specific and general results developed for the unrestricted case. In this vein, applications of stochastic processes often assume that the processes of interest are unrestricted even though there are often legitimate reasons for imposing boundary restrictions. This predisposition toward unrestricted processes can partially be attributed to the complexity of solving the Fokker-Planck equations to identify the distributions associated with bounded processes. With this in mind, this paper presents a density decomposition result which can be used to simplify the evaluation of distributional implications associated with imposing constant reflecting barrier restrictions on diffusions. The decomposition is developed by applying the classical Sturm-Liouville solution to an appropriate specified ODE problem (e.g., Boyce and Di Prima [3]). The result is also derived using the properties of the semigroup of operators associated with the forward equation.

In the following, Section 2 reviews the analytical structure required to develop distributional solutions for diffusion processes subject to constant reflecting barriers. Section 3 provides the general solution to the forward equation applicable to examining the distributional implications of boundary restrictions. The essence of the main result is that the probability density can be decomposed into a limiting equilibrium (ergodic) distribution and a power series of transient terms which are time and boundary dependent. The connection to applied distribution estimation is described heuristically. Section 4 develops the main result in the context of operator semigroups. Section 5 provides an application to the problem of numerically simulating probability densities. Finally, Section 6 provides the proofs of Propositions contained in the paper.

2. Basic Structure

The conditions derived here provide an analysis of a specific class of reflecting boundary problems.

Distributional implications are evaluated by modelling the relevant random variable as a one-dimensional diffusion process subject to constant upper and lower reflecting barriers. Starting from some appropriate initial distribution, the location of the boundaries are found to give rise to transients which act to redistribute the probability mass of the ergodic distribution. The diffusion process framework is used because it imposes a sufficient enough functional structure to develop the theory. However, the results can readily be extended to other types of distributions, such as the Cauchy, which are not usually associated with diffusions. Formally, a diffusion is defined here to be continuous (with probability one), time homogeneous strong Markov process (e.g., Gihman and Skorohod [10], Karlin and Taylor [14]), $\{x(t);\ t \geq t_0\}$, with drift parameter $\beta(x)$ and infinitesimal variance $k(x)$. The associated conditional probability density function will be denoted $U(x,t)$. The usual regularity conditions, (e.g., Arnold [1] p. 40, p. 112) are imposed on β, k and U.

Specifically, sufficient functional structure is imposed such that the probability density function $U(x,t)$ for the stochastic variable x, obeys the forward (Fokker-Planck) equation at time t:

(2.1) $$U_t = [k(x)\,U(\cdot)]_{xx} - [\beta(x)\,U(\cdot)]_x$$ subject to two boundary conditions:

(2.3) $$\frac{\partial}{\partial x}[k(x)U(x,t)]\big|_{x=a} - \beta(a)U(a,t) = 0$$

(2.4) $$\frac{\partial}{\partial x}[k(x)U(x,t)]\big|_{x=b} - \beta(b)U(b,t) = 0$$

and the initial condition:

(2.5) $$U(x,0) = f(x_0)$$

where:

$$\int_a^b f(x_0)dx_0 = 1$$

where $f(x_0)$ is the probability density function associated with x_0
and subscripts denote partial differentiation with respect to the listed variable. This formulation, permits state, but not time, variation in the parameters k and β.

Solving for the $U(\cdot)$ which satisfies (2.1) and (2.3)-(2.5) involves some analytical effort. In the method used here, the forward equation has to be transformed into its "canonical form" which allows that problem to be treated as a regular Sturm-Liouville (S-L) problem (Boyce and Di Prima [3]) where "regular" means the separation between the boundaries is finite and k(x) does not vanish at the boundary. Hence, in the following it is required that k(x) ≠ 0 with a ≤ x ≤ b. While the solution to the one reflecting boundary S-L problem is singular, this case can be sometimes be handled within the two boundary framework by setting the location of $f(x_0)$ sufficiently far from one boundary that it has no "significant" impact on the final result. In any event, admitting both upper and lower boundaries permits application of the following classical result relevant to processes subject to boundaries:

Proposition I:

The regular Sturm-Liouville problem has an infinite sequence of real eigenvalues, $\lambda_0 < \lambda_1 < \lambda_2$ ..... with:

$$\lim_{n \to \infty} \lambda_n = \infty$$

To each eigenvalue there corresponds a unique eigenfunction X_n. The eigenfunctions form a complete orthonormal system in $L_2(a,b)$.

By providing an appropriate basis, Proposition I facilitates the derivation of the decomposition of U(·). The significance of this approach is that the Sturm-Liouville is an ODE problem; the introduction of upper and lower reflecting boundaries immediately transforms the problem of solving an SDE into the more manageable problem of solving an ODE.

The transformation of U(·) into its canonical form as required for direct application of Proposition I involves the introduction of the following functions:

$$r(x) = k(x)\exp\{-\int_a^x \frac{\beta(x')}{k(x')}dx'\}$$

$$\Phi_n(x) \equiv [\int_a^b r(x)\ X_n^2(x)\ dx]^{-1/2}\ X_n(x) \qquad (2.6)$$

As described in Section 6, the r(x) function originates from the derivation of the canonical form while $\Phi_n(x)$ is the result of normalizing the X_n eigenfunctions. Given that expressing the problem in its canonical form permits the application of Proposition I, it follows:

Corollary I.1:

The unique solution in $L_2(a,b)$ to (2.1), subject to the boundary conditions (2.3)-(2.4) and initial condition (2.5) is, in general form:

(2.7) $$U(x,t) = \sum_{n=0}^{\infty} c_n \, \Phi_n(x) \, \exp\{-\lambda_n t\}$$

where: $$c_n = \int_a^b r(x) \, f(x) \, \Phi_n(x) \, dx$$

This corollary provides the general solution to the problem of deriving U when the process is subject to reflecting barriers. However, while useful, (2.7) is not immediately revealing for many practical applications because time is allowed to vary over $[0,\infty]$.

3. Main Results

In physical studies, a central objective of analysis is the ergodic behaviour of a process, i.e., the limiting behaviour of U as $t \to \infty$. However, the ergodic distribution is often of little direct interest in, say, statistical studies. Instead, attention focuses on the properties of $U(\cdot)$ when (2.7) is formulated over a specific time interval such as $\Delta t=1$, the unit time interval. For example, when x is a price, it is the price change distribution taken over some specified interval (daily, monthly, annual) which is of interest. Given this, properties of the distribution are typically "identified" by examining the behaviour of functions of the associated random variables, taken over $t \in [0,T]$ where the sample set can be either discrete or continuous. This often involves using a likelihood function derived from a series of one-step ahead distributions (e.g., Lo [15], Feigen [6]). Given this, it is possible to develop relevant results for the $U(\cdot)$'s associated with different types of boundary restrictions. Testable restrictions can be formulated about the shape and iid behaviour of the relevant distributions.

As stated, (2.7) cannot be readily applied to the types of distributions typically encountered in empirical studies. Further, simplification is required. This leads to the following fundamental result:

Proposition II: Density Decomposition

The transition probability density function for x at time t (U(x,t)) can be expressed as the sum of a limiting equilibrium density which is independent of time and a power series of transient terms which are boundary and time dependent, i.e.:

(3.1) $$U(x,t) = R(x) + K(x,t)$$

where:

(3.2) $$R(x) = \frac{r(x)^{-1}}{\int_a^b r(x)^{-1}\, dx}$$

(3.3) $$\int_a^b K(x,t)dx = 0$$

(3.4) $$\lim_{t\to\infty} K(x,t) = 0$$

with:

(3.5) $$K(x,t) \equiv \sum_{n=1}^{\infty} c_n \exp[-\lambda_n t]\ \Phi_n(x)$$

If the normalizing constant referred to in the Proposition:

$$\int_a^b r(x)^{-1} dx$$

is ignored, then the ergodic distribution can also be said to be independent of the boundaries. In many cases, Proposition II provides sufficient structure to analyze the distributional implications of reflecting barriers. In particular, because the series given in (3.5) involves powers of $\exp\{-\lambda_n\}$, the terms in the sum will decrease monotonically as $n \to \infty$ permitting truncation of the series past some point.

To illustrate some specific cases, consider the variety of limiting equilibrium distributions R(x) generated by appropriate choices of $\beta(x)$, k(x) and [a,b]. For example, if $\beta(x) = \mu$ and $k(x) = \frac{1}{2}\sigma^2$ then R(x) is exponential. If $\mu = 0$, the exponential reduces to a uniform over ($-\infty < a < b < \infty$). A normal distribution requires $k(x) = \frac{1}{2}\sigma^2$ and $\beta = -\mu(x - x_0)$. A lognormal requires $k(x) = \frac{1}{2}\sigma^2 x$ and $\beta = -\mu(\ln(x) - \ln(x_0))$. Other cases follow appropriately. If the process is unrestricted and constant parameter, the one-step-ahead ($\Delta t = 1$) distribution will replicate at each $t \in [0,T]$ and the appropriately specified process will be iid. With boundary restrictions, the distributional impact enters through K(x,t). From (3.3) the total mass of the transient term is zero. The transient acts to redistribute the mass of the equilibrium distribution, thereby causing a change in shape. The specific degree and type of alteration depends on the relevant assumptions made about the parameters, initial functional forms and selection of t.

To see heuristically how the transients and ergodic density interact, consider a system constrained by upper and lower boundaries which has been unperturbed for a sufficiently long time such that U(·) has settled down to R(x). If the boundaries are then moved, this will induce transients which act to redistribute probability mass at each point in time changing the observed shape of U(·). In the long run, the transients die away and U(·) returns to the original (except for a scaling factor) ergodic distribution. Using the eigenvalues derived from the S-L problem it is possible to numerically simulate the changing shape of U(·) over time. With appropriate selection of $f(x_0)$, it is possible to extend this approach to examining processes, such as the α-stable, which have upper moments for the unrestricted process which vanish as $t \rightarrow \infty$ but still have defined eigenvalues. By imposing boundaries on the paths of these processes, it is possible to evaluate the distributional impact of imposing boundary restrictions.

4. The Semigroup Connection

The objective of this Section is demonstrate that Proposition II can also be developed using the semigroup properties of differential operators to derive a specific representation for $f,U \in L_2(S,\mu)$, defined over the bounded state space S ($x \in S$) with some implied measure μ. This formulation permits the representation of U(x,t) by specifying a complete orthonormal basis provided by the eigenfunctions of the D operator associated with the forward equation:

$$D \equiv \frac{\partial}{\partial x^2} k(x) - \frac{\partial}{\partial x} \beta(x)$$

Significantly, in applying the operator semigroup approach, D does not necessarily have to be a differential operator in order to form the eigenfunction expansion. However, the completeness of the eigenfunctions is essential. Given the initial conditions (2.5) and the boundary conditions (2.3)-(2.4) which restrict the domain of U, $Dom(U) \subseteq [a,b]$, in Corollary I.1 the solution to U(x,t) was given by:

$$(4.1) \qquad U(x,t) = \sum_{n=0}^{\infty} c_n \; \Phi_n(x) \; \exp\{-\lambda_n \; t\}$$

where:

$$c_n = \int_a^b r(x) f(x) \Phi_n(x) dx$$

In terms of the operator D, in (4.1) Φ_n is the function which results from normalizing the eigenfunctions of D; r(x) is a weighting function depending directly on μ.

Given this, T(t) can be defined as an integral operator with kernel G(t). In the presence of boundary restrictions at a and b, the integral operation is specified:

$$(4.2) \qquad U(x,t) = G(t) \circ f(x) \equiv \int_a^b G(t,x,x') \; r(x') \; f(x') \; dx'$$

Significantly, G obeys the semigroup property: G(t) ∘ G(s) = G(t+s). Further, <u>because G is self-adjoint</u> (e.g., Reed and Simon [17]), G has the decomposition:

$$(4.3) \qquad G(t) = G^0(t) + G^1(t)$$

where:

$$G^0(t,x,x') = \Phi_0(x) \; \exp\{-\lambda_0 t\} \; \Phi_0(x')$$

$$= \Phi_0(x) \; \Phi_0(x') \qquad \textit{since} \quad \lambda_0 = 0$$

$$G^1(t,x,x') = \sum_{n=1}^{\infty} \Phi_n(x) \; \exp\{-\lambda_n t\} \; \Phi_n(x')$$

It follows that the decomposition is orthogonal, i.e., $G^1(t) \circ G^0(t) = 0$. This leads to the following:

<u>Proposition III:</u> Decomposition of T

The semigroup of bounded linear operators {T(t)} which generate a solution to (4.1) subject to the initial condition U(x,0) = $f(x_0)$ and other well-posed boundary and initial conditions, has a unique orthogonal decomposition:

$$T(t) = T^0(t) + T^1(t)$$

where: $T^0(t) \; T^1(t) = 0$

$$(4.4) \qquad \frac{d\,T^0(t)}{dt} = 0$$

$$(4.5) \qquad \frac{d\,T^1(t)}{dt} = D$$

$$(4.6) \qquad \lim_{t \to \infty} T^1(t) = 0$$

Conditions (4.4)-(4.6) provide fundamental information about the properties of T(t). Given the strict contraction semigroup property for T(t), Proposition III can be used to show that there is a unique fixed point of T which is T^0. Formally, when G is self-adjoint the λ=0 eigenvalue is the invariant of the semigroup. In turn, this invariant admits the (normalized) inverse of the density derived with respect to μ. This invariant is directly related to R(x) in (3.1).

Similar to the implications of Proposition II, (4.4)-(4.6) demonstrate that T is composed of a time invariant operator (4.4) and a transient operator (4.6), i.e., the transient operator dies out over time. In a statistical context, interpreting the invariant of the semigroup as a density leads directly to specifying U(x,t) as a transition probability density. In this case, T^0 is the ergodic density while T^1 acts to redistribute the ergodic density at a given point in time. When the discrete spectrum of the operator is generated by assuming a combination of bounded Dom(U) and self-adjointness of D, these transient terms arise from the boundary restrictions being imposed on the underlying process. (Of statistical importance, the presence of boundaries on Dom(U) typically results in a loss of the iid property.) It follows that there may be significant analytical advantages to expressing practical applications in terms of bounded Dom(U). In particular, Proposition III provides a method for extending results developed for unrestricted stochastic processes to the restricted process case.

5. Applications of the Decomposition

In financial economics, the function T(x,t=1) is of specific interest because data is typically sampled at fixed frequencies. Statistics are formed from functions of sums of the resulting random variables. In this case, the t in the exponential term in (20) is set equal to 1 and (22) is not required. In addition, because stationarity is required for the validity of many statistical procedures, the distribution of the change rather than the level is often of interest, e.g., for many financial variables such as stock prices, exchange rates and interest rates. The transformation of (18) from levels into changes follows from observing that, by definition:

$$x = x_0 + \Delta x$$
$$where: \quad a \le x, x_0 \le b$$

$$Hence: \quad -(b-a) \le \Delta x \le (b-a)$$

This leads to the following:

Corollary II.1: Distribution for Changes

Given the initial distribution $f(x_0)$, the probability distribution for the change in x between 0 and t, $V(\Delta x,t)$, is given by:

$$V(\Delta x,t) = \int_a^{b-\Delta x} f(x_0) U(x_0+\Delta x,t \mid x_0) dx_0 \qquad (23a)$$

when:

$$0 \le \Delta x \le (b-a)$$

$$V(\Delta x,t) = \int_{a-\Delta x}^{b} f(x_0) U(x_0+\Delta x,t \mid x_0) dx_0 \qquad (23b)$$

when:

$$-(b-a) \le \Delta x \le 0$$

$$= 0$$

when:

$$|\Delta x| > (b-a)$$

Given:

$$U(x_0+\Delta x,t \mid x_0) = \sum_{n=0}^{\infty} r(x_0) \phi_n(x_0) \phi_n(x_0+\Delta x) \exp\{-\lambda_n t\}$$

In the absence of boundaries, the distribution for Δx associated with a stochastic process defined by $\beta = 0$ and k equal to a constant is the familiar $\Delta x \sim N(0,\sigma^2 t)$.

Corollary II.1 demonstrates that the distribution of changes involves integration of $f(x_0)$ and $U(\cdot)$ over the appropriate regions. Given this, from (20) both the level and change distributions have three undetermined components which drive the solution: c_n, λ_n, and Φ_n. In turn, c_n depends on r(x), f(x) and Φ_n while r(x) and Φ_n depend fundamentally on the selection of β and k. (The analytical simplification provided by assuming $f(x) = \delta(x - x_0)$ referred to previously should now be apparent.) Analysis of the affect of various configurations for β(x), k(x) and $f(x_0)$ on $U(\cdot)$ and $V(\cdot)$ can be examined by simulating these distributions, based on (18)-(23). The resulting procedure requires the eigenvalues, and the associated eigenfunctions, to be derived based on the specific parametric configuration selected. By this method, the restricted and unrestricted distributions for either the level or the change in x can be derived and compared directly. In this comparison, the restricted distribution will depend on $f(x_0)$ while the unrestricted distribution will not.

Figures 1-3 present results for three different sets of parametric configurations. In all three Figures the parameters for the distribution of the unrestricted process are specified to be zero-drift (β=0), unit variance (σ=1) process over Δt =1. Hence, the differences between the two distributions plotted in Figures 1-3 are due to solely to the presence of the boundaries. Solving for the exact restricted distribution involves deriving a sufficient approximation to T(x;t=1).[1] Given the equivalence of the underlying unrestricted processes, the primary factors to consider in examining the simulations are: the width of the boundaries; the dispersion and location of $f(x_0)$ within the boundaries; and, the effects of one-sided boundaries. In particular, Figure 1 specifies $f(x_0)$ to be a uniform distribution contained between the upper and lower boundaries, [a,b].[2] Figure 2 restricts $f(x_0)$ to be uniform over [c,d] where $a < c < d < b$. Figure 3 considers the affect of a one-sided boundary.

The results presented have a number of important implications. Figure 1 confirms that introducing boundary restrictions on a stochastic process is a feasible method for modelling many empirically observed distributions, especially those for speculative price changes. Figure 2 illustrates that boundary restrictions can have a distributional impact even when placed a significant distance above or below currently admissible levels. Hence, boundaries do not have to be 'close' in order to affect the distribution. Finally, Figure 3 demonstrates that when only one boundary is effective, the resulting distribution is asymmetric. This result has important application to the estimation of both production and arbitrage profit functions. In contrast to previous ad hoc treatments of the distributional effects of one-sided boundary restrictions, the methods used here permit direct analysis of specific parametric situations.

6. <u>Proofs of the Propositions</u>

To derive Corollary I.1 it is required that the forward equation (2.1) be transformed into its canonical form, i.e., in a form which is consistent with the typical presentation of the S-L problem (e.g., Boyce and Di Prima [3] chp. 10). Specifically, it has to be shown that (2.1) can be rewritten as:

(5.1) $$\frac{1}{r(x)}[\, p(x)\ U_x]_x \ +\ q(x)\ U \ =\ U_t$$

where:

$$r(x) \ \equiv\ k(x)\ \exp\{-\int^{x} \frac{\beta(x')}{k(x')}\, dx'\}$$

$$p(x) \ \equiv\ k(x)\ r(x)$$

$$q(x) \ \equiv\ k_{xx} \ -\ \beta_x$$

This can be accomplished by evaluating the derivatives in (2.1) and (5.1), equating coefficients and solving for r(x), p(x) and q(x) in terms of k(x) and β(x). Karlin and Taylor ([14] p. 194-5) use a similar approach to address the backward equation where r(x) is referred to as the "scale density" and R(x) is proportional to the "speed density". However, the forward equation is better suited to addressing the reflecting barrier problem.

To express (5.1) as a regular S-L problem now requires a separation of variables solution to the canonical form of the general type:

(5.2) $$U(x,t) \ =\ \exp\{-\lambda t\}\ X(x)$$

Substituting this condition into (5.1) and the appropriately adjusted boundary conditions (2.3) and (2.4) gives the formulation:

(5.3) $$\frac{1}{r(x)}[k(x)r(x)X_x(x)]_x \ +\ [q(x) \ +\ \lambda]X(x) \ =\ 0$$

subject to:

(5.4) $$\frac{\partial}{\partial x}[k(x)\ X(x)]\big|_{x=b} \ -\ \beta(b)\ X(b) \ =\ 0$$

(5.5) $$\frac{\partial}{\partial x}[k(x)\ X(x)]\big|_{x=a} \ -\ \beta(a)\ X(a) \ =\ 0$$

Given the assumptions on k(x), β(x) required for diffusions and the finiteness of [a,b], it follows that r(x) and q(x) are continuous and that (5.3)-(5.5) form a regular S-L problem. The solution to this problem which is given in Proposition I is well-known, e.g., Hille ([12] chp. 8), Birkoff and Rota ([2] chp. 10). It is readily verified that the eigenfunctions are orthonormal to r(x) and that the $\Phi_n(x)$'s in (2.6) are the normalized form of the X_n's. Corollary I.1 now follows from observing that, because the X_n are complete in $L_2(a,b)$ then for $g(x) \in L_2(a,b)$ it is possible to express g in terms of the orthonormal basis as:

$$g(x) = \sum_{n=0}^{\infty} a_n \Phi_n(x)$$

where:

$$a_n = \int_a^b r(x)\ g(x)\ \Phi_n(x)\ dx$$

Using the solution for U(x,t) in (5.2) and comparing c_n with a_n gives the result in Corollary I.1.

To derive Proposition II involves applying a result in Hille [12] where it is shown that the eigenfunctions Z_n of the S-L system:

$$[S(x)y(x)]_x + [Q(x) + \lambda]y(x) = 0$$

with boundary conditions:

$$\alpha\ y_x\ |_{x=a} - \beta y(a) = 0$$

$$\gamma y_x\ |_{x=b} - \sigma y(b) = 0$$

have exactly n zeroes in the interval [a,b]. The connection with the canonical system given in (5.3)-(5.5) is given by:

$$S = \frac{p}{r} = k$$

$$Q = k_{xx} - \frac{1}{2}\beta_x - \frac{1}{4}\frac{\beta^2}{k}$$

$$y = r^{1/2}\ U$$

Thus, because it is assumed that r > 0, each X_n and, consequently, Φ_n specified in (2.6) must have exactly n zeroes in [a,b] for n ϵ [0, ∞). Deriving Proposition II now requires exploiting the properties of the eigenvalues.

For $\lambda_n \neq 0$, it follows from (2.1) and (5.2) that:

$$\lambda_n X_n = \frac{d}{dx}\{[k(x)\ X_n]_x - \beta(x)\ X_n\}$$

$\therefore$, *for* $\lambda \neq 0$:

$$\int_a^b X_n(x)\ dx = \frac{1}{\lambda_n}\{\{[kX_n]_x\ |_{x=b} - \beta(b)X_n(b)\} - \{[kX_n]_x\ |_{x=a} - \beta(a)X_n(a)\}\}$$

$$= 0$$

The integral equals zero because each X_n satisfies the boundary conditions. However, because X_0 has no zeroes on [a,b], it follows that the value of $\lambda_0 = 0$ in order to avoid violating this integral condition. The need for one of the $\lambda = 0$ and:

$$\int_a^b X_n\ dx \neq 0$$

is verified by integrating U(·) in (2.7) from a to b. Because this will involve integrating the Φ_n (and hence the X_n), some λ must equal zero in order to avoid a contradiction, i.e., U must integrate to 1 over [a,b]. Finally, because $\lambda_0 = 0$, Proposition I requires that $\lambda_n > 0$ for n > 0.

It remains to derive R(x). From $\lambda_0 = 0$, it follows:

$$\frac{d}{dx}\{[k(x)\ X_0]_x - \beta(x)\ X_0\} = 0$$

Integrating from a to x and using the boundary condition gives:

$$[k(x)\ X_0]_x - \beta(x)\ X_0 = 0$$

This has the solution:

$$X_0 = \gamma\ [k(x)]^{-1} \exp\{\int^{x} \frac{\beta(x')}{k(x')}\, dx'\} \qquad \textit{where } \gamma \textit{ is a constant}$$

$$= \gamma\ [r(x)]^{-1}$$

$\therefore$

$$\Phi_0(x) = [\int_a^b r(x)\ \gamma^2\ [r(x)]^{-2}\ dx]^{-\frac{1}{2}}\ \gamma [r(x)]^{-1}$$

$$= \frac{r^{-1}}{[\int_a^b r^{-1} dx]^{\frac{1}{2}}}$$

Now using the definition for c_0 from Corollary I.1:

$$c_0 = \int_a^b f(x)\ r(x)\ \Phi_0\ dx$$

$$= \frac{1}{[\int_a^b [r(x)]^{-1}\ dx]^{\frac{1}{2}}}$$

Hence:

$$c_0\ \Phi_0(x) = R(x) = \frac{r^{-1}}{\int_a^b r^{-1}\ dx}$$

The specification for K(x,t) in the decomposition follows immediately from the specification of U(·) in Corollary I.1.

To derive the decomposition result using operator semigroups as stated in Proposition III, recall that from (4.2) in the text it follows:

$$G(t) \circ G(s) = \int_a^b G(t,x,x'')G(s,x',x'')r(x'')\ dx''$$

$$= \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} \int_a^b \Phi_n(x)\exp[-\lambda_n t]\Phi_n(x'')\Phi_{n'}(x'')\exp[-\lambda_{n'} s]\Phi_{n'}(x')r(x'')dx''$$

Observing:

$$\int_a^b \Phi_n(x'')\Phi_{n'}(x'')r(x'')\ dx'' = \delta_{n,n'}$$

$$\begin{aligned} \textit{where}:\quad & \delta_{n,n'} = 0 \qquad \textit{if} \quad n \neq n' \\ & \delta_{n,n'} = 1 \qquad \textit{if} \quad n = n' \end{aligned}$$

Given this, the semigroup property for G is specified:

$$G(t) \circ G(s) = \sum_{n=0}^{\infty} \Phi_n(x)\ \exp[-\lambda_{n'} s]\ \Phi_{n'}(x')\ \delta_{n,n'}$$

$$= \sum_{n=0}^{\infty} \Phi_n(x)\ \exp[-\lambda_n(t+s)]\ \Phi_n(x')$$

$$= G(t+s)$$

The decomposition of G(t) given in (4.3) can now be derived as:

$$G(t) = G^0(t) + G^1(t)$$

where:

$$G^0(t) = \Phi_0(x)\ \exp[-\lambda_0(x')]\ \Phi_0(x') = \Phi_0(x)\ \Phi_0(x') \qquad \textit{since}\ \lambda_0 = 0$$

$$G^1(t,x,x') = \sum_{n=1}^{\infty} \Phi_n(x)\ \exp[-\lambda_n t]\ \Phi_n(x')$$

Hence, $G^0(t) \circ G^1(t) = 0$ from the orthonormality of the Φ_n.

Given the properties of G, Proposition III is derived by observing that (4.3) implies orthogonality of the components of T, i.e., $T^0(t)\ T^1(t) = 0$ follows from the orthogonality of the components of G. T^0 has the representation $\Phi_0(x)\ \Phi_0(x')$ which is time independent, except for a scaling factor. Hence, T^1 carries all the time dependence, i.e., (4.5) and (4.6) in the text. (4.1) follows since $\lambda_n > 0$ when $n > 0$. To derive the unique fixed point of T(t), observe:

$$T(t)\ \Phi_0(t) = T^0\ \Phi_0 = \Phi_0$$

where: $$T^1(t)\ \Phi_0 = 0$$

Thus Φ_0 is a fixed point of T (and T^0). This fixed point is unique since the $\{\Phi_n\}$ are complete.

References

[1] Arnold, L. (1974) Stochastic Differential Equations. New York: Wiley.

[2] Birkhoff, G. and G. Rota (1969) Ordinary Differential Equations (2nd ed.). New York: Wiley.

[3] Boyce, W. and R. DiPrima (1977) Elementary Differential Equations and Boundary Value Problems (3rd ed.). New York: Wiley.

[4] Buonocore, A., V. Giorno, A. Nobile, L. Ricciardi (1990) On the Two-Boundary First Crossing-Time Problem for Diffusion Process. J. Appl. Prob. 27, 102-114.

[5] Cox, D. and H. Miller (1965) The Theory of Stochastic Processes. London: Chapman and Hall.

[6] Feigen, P. (1976) Maximum Likelihood Estimation for Continuous-Time Stochastic Processes. Adv. Appl. Prob. 8, 712-36.

[7] Feller, W. (1952) Two Singular Diffusion Problems. Annals of Mathematics 54, 173-82.

[8] Giorno, V., A. Nobile, L. Ricciardi (1989) A Symmetry-Based Constructive Approach to Probability Densities for One-Dimensional Diffusion Processes. J. Appl. Prob. 27, 707-21.

[9] ---------------------------------- (1990) On the Asymptotic Behaviour of First-Passage-Time Densities for One-Dimensional Diffusion Processes and Varying Boundaries. Adv. Appl. Prob 22, 883-914.

[10] Gihman, I. and A. Skorohod (1979) The Theory of Stochastic Processes (vol. II, III). New York: Springer-Verlag.

[12] Hille, E. (1969) Lectures on Ordinary Differential Equations. London: Addison-Wesley.

[14] Karlin, S. and H. Taylor. (1981) A Second Course in Stochastic Processes. New York: Academic Press.

[15] Lo, A. (1988) Maximum Likelihood Estimation of Generalized Ito Processes with Discretely Sampled Data. Econometric Theory 4, 231-47.

[16] Nobile, A., L. Ricciardi, and L. Sacerdote (1985) Exponential trends of First-Passage-Time Densities for a Class of Diffusion Processes with Steady-State Distribution. J. Appl. Prob. 22, 611-18.

[17] Reed, M. and B. Simon (1980) Methods of Modern Mathematical Physics New York: Academic Press.

[18] Ricciardi, L. and L. Sacerdote (1987) On the Probability Densities of an Ornstein-Uhlenbeck Process with a Reflecting Boundary. J. Appl. Prob. 24, 355-369.

[19] Sacerdote, L. (1990) On the Solution of the Fokker-Planck Equation for a Feller Process. Adv. Appl. Prob. 22, 101-110.

NOTES

1. Given the analytical solution for the relevant eigenfunctions and eigenvalues, solving for the boundary restricted distribution involves solving for T(x;t=1) using (20). Because this involves an infinite sum, it is not possible to solve for an exact solution. Instead, the simulations take the first ten terms in the sum. This accounts for almost all of the value of the sum. From Proposition I, the exponential term will go to 0 as $n \to \infty$.

2. The selection of the uniform distribution for f(·) was based on the result that the ergodic distribution for a bounded process with no drift is uniform over the interval between the boundaries.